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中　華　民　國　92　年　3　月　19　日

Abstract

Let R be a prime ring with no nonzero nil one-sided ideals, d a nonzero derivation on R, and $f(X_1,\cdots,X_t)$ a multilinear polynomial not central-valued on R. Suppose that $d(f(x_1,\cdots,x_t))$ is either invertible or nilpotent for all $x_1,\cdots,x_t$ in some nonzero ideal of R. We prove that R is either a division ring or the ring of 2x2 matrices over a division ring . This theorem is a simultaneous generalization of a number of results proved earlier.

Let R be a noncommutative prime ring and d a nonzero derivation on R . A classical theorem of Posner asserts that the subset $\{\,[x^d,x] \mid x\in R\,\}$ is not contained in the center of R. Under the additional assumption that char $R\neq 2$ and $d^3\neq 0$, we prove that the additive subgroup of R generated by the subset $\{\,[x^d,x] \mid x\in R\,\}$ contains a noncentral Lie ideal of R.

Let R be a prime ring of characteristic not 2 or 3, L a noncentral Lie ideal of R, and d a nonzero derivation on R We prove that the additive subgroup of R generated by the subset $\{\,[x^d,x] \mid x\in L\,\}$contains a noncentral Lie ideal of R.

Key Words : Prime ring, nil, derivation, multilinear polynomial, central-valued, invertible, nilpotent, division ring, Posner, noncentral Lie ideal, characteristic.

摘要

設 R 為一質環，且不具非零之詣零單邊理想，d 為 R 上之一非零導算，$f(X_{1,\dots,}X_t)$ 為一多重線性之多項式，且在 R 上非恆中心值。假設對 R 中某一非零理想中之元素 $x_1,\cdots,x_n$，$d(f(x_1,\cdots,x_n))$ 恆為可逆或冪零。我們證明，R 必為一可除環或某一可除環上之 2 階方陣環，本定理為一些先前已證得成果之共同推廣。

設 R 為一非可換之質環，d 為 R 上之一非零導算，一個古典的 Posner 定理說 R 的子集 $\{[x^d,x] \mid x\in R\}$ 必不能完全包含於 R 之中心。在 char $R\neq 2$ 及 $d^3\neq 0$ 的進一步假設下，我們證明，由 R 的子集 $\{[x^d,x] \mid x\in R\}$ 所生成之加法子群，必包含 R 的一個非中心 Lie 理想。

設 R 為特徵數非 2 或 3 之一質環，L 為 R 中一非中心 Lie 理想，d 為 R 上之一非零導算。我們證明，由 R 的子集 $\{[x^d,x] \mid x\in L\}$ 所生成之加法子群，必包含 R 的一個非中心 Lie 理想。

關鍵字詞：質環，詣零，導算，多重線性多項式，具中心值，可逆，冪零，可除環，Posner，非中心 Lie 理想，特徵數。

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Derivations with Invertible or Nilpotent Values on a Multilinear Polynomial

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Abstract. Let R be a prime ring with no non-zero nil one-sided ideals, d a non-zero derivation on R, and $f(X_1, \ldots, X_t)$ a multilinear polynomial not central-valued on R. Suppose $d(f(x_1, \ldots, x_t))$ is either invertible or nilpotent for all $x_1, \ldots, x_t$ in some non-zero ideal of R. Then it is proved that R is either a division ring or the ring of 2×2 matrices over a division ring. This theorem is a simultaneous generalization of a number of results proved earlier.

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Keywords: derivation, prime ring, multilinear polynomial, Lie ideal

This paper continues a line of investigation in the literature concerning derivations having values satisfying certain properties. Bergen, Herstein, and Lanski [3] classified those semiprime rings R possessing a non-zero derivation d such that $d(x)$ is either 0 or invertible for all $x \in R$. They proved that R is either a division ring or the ring of 2×2 matrices over a division ring. Later, Bergen and Carini [2] obtained the same conclusion assuming that $d(x)$ is 0 or invertible merely for all x in some non-central Lie ideal of R. Recently, T.K. Lee [8] extended this result by studying the more general situation when $d(f(x_1, \ldots, x_t))$ is either 0 or invertible for all $x_1, \ldots, x_t$ in R, where $f(X_1, \ldots, X_t)$ is a multilinear polynomial not central-valued on R.

As to derivations having nilpotent values, Felzenszwalb and Lanski [6] proved that, if R is a prime ring with no non-zero nil one-sided ideals and d is a derivation such that $d(x)$ is nilpotent for all x in some non-zero ideal of

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R, then $d = 0$. The extensions of this theorem to Lie ideals were obtained by Carini and Giambruno [4] in the case of char $R \neq 2$, and by Lanski [7] in the case of arbitrary characteristic. In a recent paper [10], a full generalization in this vein was proved by the second-named author. She showed that, if $d(f(x_1, \ldots, x_t))$ is nilpotent for all $x_1, \ldots, x_t$ in some non-zero ideal of R, where $f(X_1, \ldots, X_t)$ is a multilinear polynomial not central-valued on R, then $d = 0$.

On the other hand, Bergen [1] proved a result concerning a derivation with invertible or nilpotent values. It was shown that, if R is a ring with no non-zero nil one-sided ideals and d is a non-zero derivation on R such that $d(x)$ is invertible or nilpotent for all x in R, then R is a division ring or the ring of 2×2 matrices over a division ring. In the present paper, we will consider the situation when $d(f(x_1, \ldots, x_t))$ is invertible or nilpotent for all $x_1, \ldots, x_t$ in some non-zero ideal of a prime ring R, where $f(X_1, \ldots, X_t)$ is a multilinear polynomial not central-valued on R. More precisely, we will prove the following:

Theorem. *Let K be a commutative ring with unity and R a unital prime K-algebra with no non-zero nil one-sided ideals. Let $f(X_1, \ldots, X_t)$ be a multilinear polynomial over K with some coefficients invertible in K that is not central-valued on R. Suppose d is a non-zero derivation on R such that $d(f(x_1, \ldots, x_t))$ is invertible or nilpotent for all $x_1, \ldots, x_t$ in some non-zero ideal I of R. Then R is a division ring or the ring of 2×2 matrices over a division ring.*

Proof. First, we show that R contains units. If $d(f(x_1, \ldots, x_t))$ is nilpotent for all x_i in I, by Theorem 3 in [10], $f(X_1, \ldots, X_t)$ is central-valued on R, contrary to our hypothesis. Hence, for some $r_1, \ldots, r_t$ in R, $d(f(r_1, \ldots, r_t))$ is invertible and so R contains units. Next, we show that R is a simple ring. Let J be a proper ideal of R, i.e., $J \neq R$. For $x_1, \ldots, x_t \in J^2 \cap I \subseteq I$, $d(f(x_1, \ldots, x_t))$ is invertible or nilpotent. Since each x_i is in J^2, we have $f(x_1, \ldots, x_t) \in J^2$ and so $d(f(x_1, \ldots, x_t)) \in J$ is not invertible. Hence, $d(f(x_1, \ldots, x_t))$ is nilpotent for all $x_1, \ldots, x_t$ in the ideal $J^2 \cap I$ of R. Since $f(X_1, \ldots, X_t)$ is not central-valued on R, it follows again from Theorem 3 in [10] that $J^2 \cap I = 0$. Since R is prime and $I \neq 0$, we conclude that $J = 0$. Thus, R is a simple ring with unity, and so is a primitive ring. Consequently, R acts densely as a ring of linear transformations on a vector space ${}_DV$ over some division ring D. We will assume V is infinite-dimensional over D and eventually arrive at a contradiction. We proceed through a series of reductions, almost all of which rely heavily on the Jacobson density theorem.

Since $f(X_1, \ldots, X_t)$ is multilinear with some coefficients invertible in K, we may write

$$f(X_1, \ldots, X_t) = \alpha_1 X_1 \cdots X_t + \sum_{\sigma \neq 1} \alpha_\sigma X_{\sigma(1)} \cdots X_{\sigma(t)},$$

where α_1 is invertible in K and the sum is taken over all permutations

σ in the symmetric group S_t other than the identity permutation 1. Set $f^d(X_1, \ldots, X_t)$ to be the polynomial obtained from $f(X_1, \ldots, X_t)$ by replacing each coefficient α_σ with $d(\alpha_\sigma \cdot 1)$. Suppose $v, w \in V$ and $r \in R$ with $\operatorname{rank} r > t$ and $vr = wr = 0$. We consider $vd(r)$ and $wd(r)$. If $vd(r)$ and $wd(r)$ are D-independent, choose $v_1, \ldots, v_{t-1} \in V$ such that $vd(r), wd(r), v_1r, \ldots, v_{t-1}r$ are D-independent. By the density of R on ${}_DV$, there exist $s_1, \ldots, s_t \in R$ such that $wd(r)s_i = 0$ for all i, $vd(r)s_1 = v_1$, $vd(r)s_i = 0$ for $i = 2, \ldots, t$, $v_{i-1}rs_i = v_i$ for $i = 2, \ldots, t-1$, $v_{t-1}rs_t = \alpha_1^{-1}v$, and $v_{j-1}rs_i = 0$ for $j \neq i$. Then we have

$$\begin{aligned}
& wd(f(rs_1, \ldots, rs_t)) \\
= {} & w\left(f^d(rs_1, \ldots, rs_t) + \textstyle\sum_{i=1}^t f(rs_1, \ldots, d(rs_i), \ldots, rs_t)\right) \\
= {} & w \textstyle\sum_{i=1}^t f(rs_1, \ldots, d(rs_i), \ldots, rs_t) \\
= {} & w \textstyle\sum_{i=1}^t (f(rs_1, \ldots, d(r)s_i, \ldots, rs_t) + f(rs_1, \ldots, rd(s_i), \ldots, rs_t)) \\
= {} & 0
\end{aligned}$$

and

$$\begin{aligned}
& vd(f(rs_1, \ldots, rs_t)) \\
= {} & v\ \left(f^d(rs_1, \ldots, rs_t) + \textstyle\sum_{i=1}^t f(rs_1, \ldots, d(rs_i), \ldots, rs_t)\right) \\
= {} & v \textstyle\sum_{i=1}^t f(rs_1, \ldots, d(rs_i), \ldots, rs_t) \\
= {} & v \textstyle\sum_{i=1}^t (f(rs_1, \ldots, d(r)s_i, \ldots, rs_t) + f(rs_1, \ldots, rd(s_i), \ldots, rs_t)) \\
= {} & v \textstyle\sum_{i=1}^t f(rs_1, \ldots, d(r)s_i, \ldots, rs_t) \\
= {} & vf(d(r)s_1, rs_2, \ldots, rs_t) \\
= {} & v(\alpha_1 d(r)s_1rs_2 \cdots rs_t) \\
= {} & v.
\end{aligned}$$

Thus, $d(f(rs_1, \ldots, rs_t))$ can be neither invertible nor nilpotent. This contradiction shows that $vd(r)$ and $wd(r)$ must be D-dependent.

Suppose v and w are D-independent and $r_1, \ldots, r_t \in R$ are such that the rank of $f(r_1, \ldots, r_t)$ is bigger than t and $vf(r_1, \ldots, r_t) = wf(r_1, \ldots, r_t) = 0$. Then $vd(f(r_1, \ldots, r_t))$ and $wd(f(r_1, \ldots, r_t))$ are D-dependent, and so for some $\alpha, \beta \in D$ not both zero,

$$\alpha vd(f(r_1, \ldots, r_t)) + \beta wd(f(r_1, \ldots, r_t)) = 0.$$

Thus, $(\alpha v + \beta w)d(f(r_1, \ldots, r_t)) = 0$, while $\alpha v + \beta w \neq 0$ so $d(f(r_1, \ldots, r_t))$ is not invertible. Therefore, $d(f(r_1, \ldots, r_t))$ is nilpotent.

Let $v \in V$ and $r \in R$ with $\operatorname{rank} r > t(t+2)$ and $vr = 0$. Suppose $vd(r) \neq 0$. We choose $v_1, \ldots, v_t$, $w_{01}, \ldots, w_{0t}$, $w_{11}, \ldots, w_{1t}, \ldots, w_{t1}, \ldots, w_{tt} \in V$ such that the $t(t+2)+1$ vectors $vd(r)$, v_jr, $w_{ij}r$ are D-independent. Then there exist $s_1, \ldots, s_t \in R$ such that $vd(r)s_1 = v_2$, $vd(r)s_j = 0$ for $j = 2, \ldots, t$, $v_1rs_1 = 0$, $v_jrs_j = v_{j+1}$ for $j = 2, \ldots, t-1$, $v_trs_t = \alpha_1^{-1}v$, $v_irs_j = 0$

for $i \neq j$, $w_{ij}rs_j = w_{i,j+1}$ for all i and $j = 1, \ldots, t-1$, $w_{it}rs_t = \alpha_1^{-1}w_{i1}$ for all i, and $w_{ij}rs_k = 0$ for $j \neq k$ and all i. Then $w_{i1}f(rs_1, \ldots, rs_t) = w_{i1}$ for each $i = 0, 1, \ldots, t$ and so $f(rs_1, \ldots, rs_t)$ has rank bigger than t. Since $vf(rs_1, \ldots, rs_t) = v_1 f(rs_1, \ldots, rs_t) = 0$ and v, v_1 are D-independent, $d(f(rs_1, \ldots, rs_t))$ is nilpotent. On the other hand, $vd(f(rs_1, \ldots, rs_t)) = v$, so $d(f(rs_1, \ldots, rs_t))$ cannot be nilpotent, a contradiction. Hence, $vr = 0$ implies $vd(r) = 0$ for $r \in R$ with $\operatorname{rank} r > t(t+2)$.

We claim that $vr = 0$ implies $vd(r) = 0$ for all $r \in R$. If $\operatorname{rank} r > t(t+2)$, we are done already. So assume $\operatorname{rank} r \leq t(t+2)$. Since $\dim_D V = \infty$ and $\dim_D Vr < \infty$, it follows that $\operatorname{Ker} r$, the kernel of r, is infinite-dimensional, and so there exist $v_1, \ldots, v_{t(t+2)+1} \in \operatorname{Ker} r$ such that $v, v_1, \ldots, v_{t(t+2)+1}$ are D-independent. Choose $s \in R$ such that $vs = 0$ and $v_i s = v_i$ for all $i = 1, \ldots, t(t+2)+1$. Then $\operatorname{rank} s > t(t+2)$, and so $vs = 0$ yields $vd(s) = 0$. Note that $v(r-s) = 0$ and $v_i(r-s) = -v_i$ for all $i = 1, \ldots, t(t+2)+1$. Thus, we have $\operatorname{rank}(r-s) > t(t+2)$, and hence, $vd(r-s) = 0$. Therefore, $vd(r) = v(d(r-s) + d(s)) = 0$ as well.

Next, we claim that vr and $vd(r)$ are D-dependent for $v \in V$ and $r \in R$. Assume on the contrary that vr and $vd(r)$ are D-independent for some $v \in V$ and $r \in R$. Then there exists $s \in R$ such that $vrs = 0$ and $vd(r)s \neq 0$. From $v(rs) = (vr)s = 0$, it follows that $vd(rs) = vrd(s) = 0$ and so $vd(r)s = v(d(rs) - rd(s)) = 0$, a contradiction. Therefore, vr and $vd(r)$ are D-dependent for $v \in V$ and $r \in R$, which results in $vd(r) = \alpha vr$ for some $\alpha \in D$.

For a fixed $v \in V$, we have $vd(r_1) = \alpha_1 vr_1$ and $vd(r_2) = \alpha_2 vr_2$ for $r_1, r_2 \in R$. If vr_1 and vr_2 are D-independent, the identity $vd(r_1 + r_2) = vd(r_1) + vd(r_2)$ gives $\beta v(r_1 + r_2) = \alpha_1 vr_1 + \alpha_2 vr_2$ for some $\beta \in D$ and so $\alpha_1 = \alpha_2 = \beta$. If vr_1 and vr_2 are non-zero but D-dependent on each other, take $r_3 \in R$ so that vr_3 is D-independent on vr_1 or vr_2. Then $\alpha_1 = \alpha_2 = \alpha_3$, where $\alpha_3 \in D$ satisfies $vd(r_3) = \alpha_3 vr_3$. In other words, for each $v \in V$, there exists $\alpha_v \in D$ such that $vd(r) = \alpha_v vr$ for all $r \in R$. Fix an element $r \in R$ with $\operatorname{rank} r > 1$. For $u, v \in V$, we have $ud(r) = \alpha_u ur$ and $vd(r) = \alpha_v vr$ for some $\alpha_u, \alpha_v \in D$. If ur and vr are D-independent, then $\alpha_u = \alpha_v = \alpha_{u+v}$, where $\alpha_{u+v} \in D$ satisfies $(u+v)d(r) = \alpha_{u+v}(u+v)r$. If ur and vr are non-zero but D-dependent on each other, then $\alpha_u = \alpha_v = \alpha_w$ for some $w \in V$ such that wr is D-independent on ur or vr. Hence, there exists $\alpha \in D$ such that $vd(r) = \alpha vr$ for all $v \in V$ and $r \in R$. Note that $\alpha \neq 0$; otherwise, $Vd(R) = 0$ and so $d = 0$, a contradiction. For any $v \in V$ and $r, s \in R$, we have $\alpha vrs = vd(rs) = vd(r)s + vrd(s) = 2\alpha vrs$, and hence, $\alpha vrs = 0$. Thus, $\alpha VR^2 = 0$ and so $R^2 = 0$, contradicting the primeness of R. Therefore, V must be finite-dimensional over D, say, $\dim_D V = n$.

Thus, R is isomorphic to the ring $M_n(D)$ of $n \times n$ matrices over D. In other words, each x in R can be expressed uniquely as $\sum_{i,j} \xi_{ij} e_{ij}$, where $\xi_{ij} \in D$ and $\{e_{ij} \mid i, j = 1, \ldots, n\}$ is a set of matrix units. It is well known (see, for instance, [5]) that there exist a derivation δ on D and an element $a = \sum_{i,j} \alpha_{ij} e_{ij}$ in R such that $d = \overline{\delta} + \operatorname{ad}_a$, where $\overline{\delta}$ is the derivation on R

induced by δ, $\overline{\delta}(\sum_{i,j}\xi_{ij}e_{ij}) = \sum_{i,j}\delta(\xi_{ij})e_{ij}$, and ad_a is the inner derivation defined by a, i.e., $\mathrm{ad}_a(x) = [a, x] = ax - xa$ for $x \in R$.

Assume $n > 2$. We claim first that a is a diagonal matrix, namely, $\alpha_{kh} = 0$ for $h \neq k$. Since $f(X_1, \ldots, X_t)$ is not central-valued on R, by the Lemma in [9] and Lemma 2 in [8], there exist $a_1, \ldots, a_t \in R$ such that $f(a_1, \ldots, a_t) = \alpha e_{pq} \neq 0$ for some $\alpha \in D$ and $p \neq q$. For distinct h, k, let σ be a permutation in S_t such that $\sigma(p) = h$ and $\sigma(q) = k$. Let φ be the automorphism of R given by $(\sum_{i,j}\xi_{ij}e_{ij})^\varphi = \sum_{i,j}\xi_{ij}e_{\sigma(i)\sigma(j)}$. Then $f(a_1^\varphi, \ldots, a_t^\varphi) = f(a_1, \ldots, a_t)^\varphi = \alpha e_{hk}$ and $d(\alpha e_{hk}) = d(f(a_1^\varphi, \ldots, a_t^\varphi))$ is invertible or nilpotent. Now the rank of $d(\alpha e_{hk}) = \delta(\alpha)e_{hk} + [a, \alpha e_{hk}] = (\delta(\alpha) + a\alpha)e_{hk} - \alpha e_{hk}a$ is at most 2, so it is not invertible. Hence, $d(\alpha e_{hk})$ is nilpotent. Since $d(\alpha e_{hk}) = \delta(\alpha)e_{hk} + [\sum_{i,j}\alpha_{ij}e_{ij}, \alpha e_{hk}] = \delta(\alpha)e_{hk} + \sum_i \alpha_{ih}\alpha e_{ik} - \sum_j \alpha\alpha_{kj}e_{hj}$, we have $e_{kk}d(\alpha e_{hk}) = (\alpha_{kh}\alpha)e_{kk}$, and for any m, $e_{kk}d(\alpha e_{hk})^m = (\alpha_{kh}\alpha)^m e_{kk}$. Thus, $(\alpha_{kh}\alpha)^m = 0$ for some m and so $\alpha_{kh} = 0$.

Now we show that $a = \sum_{i=1}^n \alpha_{ii}e_{ii}$ is a scalar matrix, i.e., $\alpha_{jj} = \alpha_{11}$ for all j. For $j \neq 1$, set $c = 1 + e_{1j}$ and $b = c^{-1}ac$, and consider the derivation $d' = \overline{\delta} + \mathrm{ad}_b$ on R. For $x_1, \ldots, x_t \in R$, we have

$$\begin{aligned}
& d'(f(x_1, \ldots, x_t)) \\
= \ & \overline{\delta}(f(x_1, \ldots, x_t)) + [c^{-1}ac, f(x_1, \ldots, x_t)] \\
= \ & \overline{\delta}(c^{-1}f(cx_1c^{-1}, \ldots, cx_tc^{-1})c) + [c^{-1}ac,\ c^{-1}f(cx_1c^{-1}, \ldots, cx_tc^{-1})c] \\
= \ & c^{-1}\overline{\delta}(f(cx_1c^{-1}, \ldots, cx_tc^{-1}))c + c^{-1}[a,\ f(cx_1c^{-1}, \ldots, cx_tc^{-1})]c \\
= \ & c^{-1}d(f(cx_1c^{-1}, \ldots, cx_tc^{-1}))c,
\end{aligned}$$

which is invertible or nilpotent for all $x_1, \ldots, x_t \in R$. As we have seen in the preceding paragraph, b must be a diagonal matrix. However, $b = (1 - e_{1j})(\sum_i \alpha_{ii}e_{ii})(1 + e_{1j}) = \sum_i \alpha_{ii}e_{ii} + (\alpha_{11} - \alpha_{jj})e_{1j}$, so $\alpha_{jj} = \alpha_{11}$, and hence, $a = \alpha_{11} \in D$. Thus, d is induced by the derivation $\delta' = \delta + \mathrm{ad}_a$ on D. Without loss of generality, we may assume $d = \overline{\delta}$ by replacing δ with δ'. We will proceed to show that δ is an inner derivation.

Let $R' = \{\sum_{ij}\xi_{ij}e_{ij} \mid \xi_{in} = \xi_{nj} = 0\ \forall\, i, j\}$. Then R' is a subring of R that is isomorphic to $M_{n-1}(D)$. Moreover, R' is d-invariant, i.e., $d(R') \subseteq R'$. For all $x_i \in R'$, $d(f(x_1, \ldots, x_t)) \in R'$ is not invertible in R, and hence, is nilpotent. By Theorem 3 in [10], $f(X_1, \ldots, X_t)$ is central-valued on R' and, *a fortiori*, is central-valued on D. Consequently, D is finite-dimensional over its center Z. Thus, in order to show that δ is inner, it suffices to prove $\delta(Z) = 0$. Assume on the contrary that $\delta(\beta) \neq 0$ for some $\beta \in Z$. Recall that there exist $a_1, \ldots, a_t \in R$ such that $f(a_1, \ldots, a_t) = \alpha e_{pq} \neq 0$ for some $\alpha \in D$ and $p \neq q$. Set $c = 1 + \beta e_{qp}$. Then $d(c^{-1}(\alpha e_{pq})c) = d(f(c^{-1}a_1c, \ldots, c^{-1}a_tc))$ is invertible or nilpotent. However,

$$\begin{aligned}
& d(c^{-1}(\alpha e_{pq})c) \\
= \ & d((1 - \beta e_{qp})(\alpha e_{pq})(1 + \beta e_{qp})) \\
= \ & \delta(\beta\alpha)e_{pp} + \delta(\alpha)e_{pq} - \delta(\beta^2\alpha)e_{qp} - \delta(\beta\alpha)e_{qq}
\end{aligned}$$

lies in a subring that is isomorphic to $M_2(D)$. Hence, $d(c^{-1}(\alpha e_{pq})c)$ is nilpotent and, in fact, square-zero. Inspecting the (p,q)-entry of $d(c^{-1}(\alpha e_{pq})c)^2$, we obtain $\delta(\beta)[\alpha, \delta(\alpha)] = 0$ and so $[\alpha, \delta(\alpha)] = 0$. And the (p,p)-entry gives $\delta(\beta)^2\alpha^2 = 0$, a contradiction. Thus, δ is an inner derivation on D and so we may assume $d = \mathrm{ad}_a$ for some $a \in D$.

For $\gamma \in D$, set $c = 1 + \gamma e_{12}$ and $b = c^{-1}ac$. Consider the derivation $d' = \mathrm{ad}_b$ on R. For $x_1, \ldots, x_t \in R$, we have

$$\begin{aligned} & d'(f(x_1, \ldots, x_t)) \\ = \; & [c^{-1}ac,\, f(x_1, \ldots, x_t)] \\ = \; & c^{-1}[a,\, f(cx_1c^{-1}, \ldots, cx_tc^{-1})]c \\ = \; & c^{-1}d(f(cx_1c^{-1}, \ldots, cx_tc^{-1}))c, \end{aligned}$$

which is invertible or nilpotent for all $x_1, \ldots, x_t \in R$. As we have seen before, $b = (1 - \gamma e_{12})a(1 + \gamma e_{12}) = a + [a, \gamma]e_{12}$ must be a scalar matrix, so $a\gamma = \gamma a$. Hence, $a \in Z$ and so $d = 0$, a contradiction. Thus, $n \leq 2$, i.e., R is a division ring D or the ring $M_2(D)$ of 2×2 matrices over a division ring D. This completes the proof of the theorem. □

We conclude this paper with a remark. In the case where $R = M_2(D)$ with $\mathrm{char}\, D \neq 2$ and $f(X_1, \ldots, X_t)$ is not central-valued on D, the derivation d must be inner. This can be proved via an argument exactly the same as that in the proof of the Main Theorem in [8].

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ON CERTAIN SUBGROUPS OF PRIME RINGS WITH DERIVATIONS

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ABSTRACT

Let R be a noncommutative prime ring and let d be a nonzero derivation on R. A classical theorem of Posner asserts that the subset $\{[x^d, x] \mid x \in R\}$ is not contained in the center of R. Under the additional assumption that char $R \neq 2$ and $d^3 \neq 0$, we show that the additive subgroup of R generated by the subset $\{[x^d, x] \mid x \in R\}$ contains a noncentral Lie ideal of R.

Let R be a noncommutative prime ring and let d be a nonzero derivation on R. In [6] Posner proved that the subset $\{[x^d, x] \mid x \in R\}$ is not contained in the center of R. Under the additional assumption that the characteristic of R is not 2, Brešar and Vukman [2] showed that the subring S of R generated by $\{[x^d, x] \mid x \in R\}$ contains a nonzero right ideal of R and a nonzero left ideal of R. Recently, the first-named author [3] extended their result by showing that the subring S mentioned above contains a nonzero two-sided ideal of R. In this note we prove that the additive subgroup of R generated by $\{[x^d, x] \mid x \in R\}$ contains a noncentral Lie ideal of R provided $d^3 \neq 0$.

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In what follows, R always denotes a ring. For an additive subgroup A of R, let $\bar{A}$ be the subring of R generated by A. For additive subgroups A and B of R, let $[A,B]$ be the subgroup of R generated by the elements of the form $[a,b]$ with $a,b \in R$. We shall frequently use the following identities in R:

$$[xy,z]+[yz,x]+[zx,y]=0$$

and

$$[xyz,u]+[yzu,x]+[zux,y]+[uxy,z]=0$$

for all $x,y,z,u \in R$.

We begin with recalling an elementary result:

Lemma 1. [1, Lemma 9.1.2] *If A is an additive subgroup of R, then* $[\bar{A},R]=[A,R]$.

Next we investigate the additive subgroup $[R^d,R]$, the additive subgroup of R generated by elements of the form $[x^d,y]$ with $x,y \in R$. The following result is of independent interest.

Proposition 2. *Let R be a noncommutative prime ring of characteristic not 2 and let d be a nonzero derivation on R. Then the additive subgroup $[R^d,R]$ of R contains a noncentral Lie ideal of R.*

Proof. Let $H=[R^d,R]$. Then $[x^d,y]\in H$ for all $x,y\in R$. Replacing x by xz^d, we have $[x^dz^d,y]+[xz^{d^2},y]\in H$. Since $[x^dz^d,y]\in[\overline{R^d},R]=[R^d,R]=H$ by Lemma 1, it follows that $[xz^{d^2},y]\in H$ for all $x,y,z\in R$. That is, $[RR^{d^2},R]\subseteq H$. Similarly, we have $[R^{d^2}R,R]\subseteq H$. For $x,y,z,u\in R$, we have $[xy^{d^2}z,u]=-[y^{d^2}zu,x]-[zux,y^{d^2}]-[uxy^{d^2},z]\in [R^{d^2}R,R]+[R,R^{d^2}]+[RR^{d^2},R]\subseteq H$. That is, $[RR^{d^2}R,R]\subseteq H$. Since char $R\neq 2$ and $d\neq 0$, we have $d^2\neq 0$ by [6, Theorem 1] and so $RR^{d^2}R$ is a nonzero ideal of R. Thus H contains the Lie ideal $[RR^{d^2}R,R]$ which is noncentral in R because R is not commutative.

We are now ready to prove our main result.

Theorem 3. *Let R be a noncommutative prime ring of characteristic not 2 and let d be a derivation on R such that $d^3\neq 0$. Then the additive subgroup G of R generated by $\{[x^d,x]\mid x\in R\}$ contains a noncentral Lie ideal of R.*

Proof. For $x\in R$, we have $[x^d,x]\in G$. Linearizing this relation on x, we get

$$[x^d,y]+[y^d,x]\in G \tag{1}$$

for all $x,y\in R$. Replacing y by yz, we have

$$[x^d, yz] + [(yz)^d, x] \in G.$$

Since $[(yz)^d, x] = [yz, x]^d - [yz, x^d] = [yz, x]^d + [x^d, yz]$, it follows that

$$2[x^d, yz] + [yz, x]^d \in G. \tag{2}$$

This together with

$$2[y^d, zx] + [zx, y]^d \in G$$

and

$$2[z^d, xy] + [xy, z]^d \in G,$$

yields

$$2\{[x^d, yz] + [y^d, zx] + [z^d, xy]\} \in G, \tag{3}$$

since $[yz, x] + [zx, y] + [xy, z] = 0$. On the other hand,

$$2\{[x^d, yz] + [(yz)^d, x]\} \in G, \tag{4}$$

so the difference of (3) and (4) yields

$$2\{[y^d, zx] + [z^d, xy] - [(yz)^d, x]\} \in G. \tag{5}$$

Since

$$\begin{aligned}
&[y^d, zx] + [z^d, xy] - [(yz)^d, x] \\
&\quad = [y^d, zx] + [z^d, xy] + [x, (yz)^d] \\
&\quad = [y^d, zx] + [x, y^d z] + [z^d, xy] + [x, yz^d] \\
&\quad = -[z, xy^d] - [y, z^d x] \\
&\quad = [xy^d, z] + [z^d x, y],
\end{aligned}$$

we have

$$2\{[xy^d, z] + [z^d x, y]\} \in G \tag{6}$$

for all $x, y, z \in R$. Setting $x = u^d, y = v$ and $z = w$ in (6), we get

$$2\{[u^d v^d, w] + [w^d u^d, v]\} \in G \tag{7}$$

for all $u, v, w \in R$. Similarly, we have

$$2\{[v^d w^d, u] + [u^d v^d, w]\} \in G \tag{8}$$

and

$$2\{[w^d u^d, v] + [v^d w^d, u]\} \in G. \tag{9}$$

Substracting (9) from the sum of (7) and (8), we get $4[u^dv^d, w] \in G$ for all $u, v, w \in R$. That is, $4[(R^d)^2, R] \subseteq G$, and so $4[V, R] \subseteq G$ where $V = \overline{(R^d)^2}$ is the subring of R generated by elements of the form x^dy^d with $x, y \in R$.

Assume first that there exist $a, b \in R$ such that $c = (a^db^d)^d \neq 0$. Consider the subring UcU where $U = \overline{R^d}$ is the subring of R generated by elements of the form x^d with $x \in R$. Each element in UcU is a sum of elements of the form $x_1^d \dots x_r^d c y_1^d \dots y_s^d$ with $x_i, y_i \in R$. If $r + s$ is odd, then

$$x_1^d \dots x_r^d c y_1^d \dots y_s^d$$
$$= x_1^d \dots x_r^d (a^db^d)^d y_1^d \dots y_s^d \in \overline{(R^d)^2} = V;$$

and if $r + s$ is even, then

$$x_1^d \dots x_r^d c y_1^d \dots y_s^d$$
$$= x_1^d \dots x_r^d (a^{d^2}) b^d y_1^d \dots y_s^d + x_1^d \dots x_r^d a^d (b^{d^2}) y_1^d \dots y_s^d \in \overline{(R^d)^2} = V.$$

Hence, $UcU \subseteq V$. Since $d^3 \neq 0$, by a theorem of Herstein [5, Theorem 1], U contains a nonzero ideal I of R. Thus $4[IcI, R] \subseteq 4[UcU, R] \subseteq 4[V, R] \subseteq G$, and $4[IcI, R]$ is a noncentral Lie ideal of R.

Now assume that $(x^dy^d)^d = 0$ for all $x, y \in R$. That is,

$$x^{d^2}y^d + x^dy^{d^2} = 0 \tag{10}$$

for all $x, y \in R$. Then

$$(xy)^{d^3} = x^{d^3}y + 3(x^{d^2}y^d + x^dy^{d^2}) + xy^{d^3} = x^{d^3}y + xy^{d^3}$$

for all $x, y \in R$ and so d^3 is a derivation on R. Also it follows from $x^{d^2}y^d = -x^dy^{d^2}$ that

$$x^{d^3}y^d = -x^{d^2}y^{d^2} = x^dy^{d^3} \tag{11}$$

for all $x, y \in R$. Replacing x by x^d and y by y^d respectively in (1), we get

$$[x^{d^2}, y^d] + [y^{d^2}, x^d] \in G.$$

Since $[y^{d^2}, x^d] = y^{d^2}x^d - x^dy^{d^2} = -y^dx^{d^2} + x^{d^2}y^d = [x^{d^2}, y^d]$, we have

$$2[x^{d^2}, y^d] \in G \tag{12}$$

for all $x, y \in R$. On the other hand, replacing x by x^{d^2} in (1), we get

$$[x^{d^3}, y] + [y^d, x^{d^2}] \in G.$$

and so

$$2\{[x^{d^3}, y] - [x^{d^2}, y^d]\} \in G. \tag{13}$$

The sum of (12) and (13) yields

$$2[x^{d^3}, y] \in G$$

for all $x, y \in R$. In other words, $2[R^{d^3}, R] \subseteq G$. Now d^3 is a nonzero derivation on R, so by Proposition 2 there exists a noncentral Lie ideal L of R such that $L \subseteq [R^{d^3}, R]$. Hence $2L$ is a noncentral Lie ideal contained in G. This completes the proof of the theorem.

Note that the conclusion of the Theorem need not be true in the case of characteristic 2. For instance, let F be a field with char $F = 2$ and let $R = M_n(F)$ be the algebra of n by n matrices over F. Suppose that $n > 2$; then any noncentral Lie ideal of R contains $[R, R]$ by a theorem of Herstein [4, Theorem 1.5]. Let a be any noncentral element in R and let d be the inner derivation on R induced by a, namely $x^d = [a, x]$ for all $x \in R$. Since char $R = 2$, we have $[x^d, x] = x^d x + x x^d = (x^2)^d$ and so R^d contains the additive subgroup G of R generated by $\{[x^d, x] | x \in R\}$. Now d is a linear transformation on R whose kernel contains the subalgebra $F[a]$ generated by a over F which is of dimension at least 2 over F. So R^d, the image of d, has dimension at most $n^2 - 2$ over F. Therefore neither R^d nor the subgroup G contains any noncentral Lie ideal of R, because $[R, R]$ has dimension $n^2 - 1$ over F.

We conclude with an open question: Is it possible to generalize the Theorem by dropping the assumption that $d^2 \neq 0$?

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A NOTE ON DERIVATIONS, ADDITIVE SUBGROUPS, AND LIE IDEALS OF PRIME RINGS

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ABSTRACT

Let R be a prime ring of characteristic not 2 or 3 and L a noncentral Lie ideal of R. If D is a nonzero derivation of R, then the additive subgroup of R generated by the subset $\{[x^D, x] \mid x \in L\}$ contains a noncentral Lie ideal of R.

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1 INTRODUCTION

A well-known Posner's Theorem[24, Theorem 2] asserts that the subset $S = \{[x^d, x] \mid x \in R\}$ does not lie in the center of a noncommutative prime ring R, provided that d is a nonzero derivation of R. However, analyzing the structure of S, say in the case when R is a matrix algebra over a field, one might suspect if it is possible to say more about the size of S than just that S does not lie in the center of R.

It seems that the first results from the size point of view concerning the relationship between derivations and subsets of rings were due to Herstein.[16,17] He proved (modulo some technical restrictions) that the subring of ring R generated by R^d, where d is a nonzero derivation of R, contains a nonzero two-sided ideal of R. Later Brešar and Vukman[7] and Chebotar[9] obtained similar results for the subring of a prime ring R generated by the set S. As it was recently proved by Chebotar and P.-H. Lee[10] and Chuang and T.-K. Lee[12], the additive subgroup of R generated by S is "large" itself, namely, it contains a noncentral Lie ideal of R.

The relationship between derivations and sizes of related Lie ideals of prime rings was first studied by Bergen, Herstein and Kerr.[5] They proved (modulo some technical restrictions) that the subring of a prime ring R generated by L^d, where d is a nonzero derivation of R and L is a noncentral Lie ideal of R, contains a nonzero two-sided ideal of R. Posner's theorem was extended to the case of Lie ideals of prime rings by P.-H. Lee and T.-K. Lee[20] and Lanski.[19] Recently, Carini and De Filippis[8] described prime rings R such that for a nonzero derivation d of R, a noncentral Lie ideal L of R and a positive integer n, the subset $\{[x^d, x]^n \mid x \in L\}$ lies in the center of R.

Our main aim is the following Theorem 1 which simultaneously generalizes some results from[10,12,13,19,20].

Theorem 1. *Let R be a prime ring of characteristic not* 2 *or* 3 *and L a noncentral Lie ideal of R. If D is a nonzero derivation of R, then the additive subgroup of R generated by the subset $\{[x^D, x] \mid x \in L\}$ contains a noncentral Lie ideal of R.*

We will prove first an obvious corollary of Theorem 1 as Theorem 2 which will play a key role in the proof of Theorem 1.

Theorem 2. *Let R be a prime ring of characteristic not* 2 *or* 3 *and L a noncentral Lie ideal of R. If D is a nonzero derivation of R, then the additive subgroup $[L^D, L]$ of R generated by the subset $\{[x^D, y] \mid x, y \in L\}$ contains a noncentral Lie ideal of R.*

Since the subring generated by a noncentral Lie ideal of a prime ring contains a nonzero two-sided ideal of the ring[15, Lemma 1.3], our next result generalizes the results from[7,9].

Corollary 3. *Let R be a prime ring of characteristic not 2 or 3 and L a noncentral Lie ideal of R. If D is a nonzero derivation of R, then the subring of R generated by the subset $\{[x^D, x] \mid x \in L\}$ contains a nonzero two-sided ideal of R.*

Generally speaking, when dealing with Lie ideals it is more natural to consider Lie derivations instead of ordinary ones. It seems that the main reason for considering just ordinary derivations sometimes was the hope that for some Lie subrings of prime rings Lie derivations could be described by means of ordinary derivations. As a matter of fact it was a problem of Herstein posed in his 1961 AMS Hour Talk[14].

This problem has been carried forward mainly by Martindale and some of his students.[18,22,25,26] Recently it was solved by Brešar[6] for prime rings, by Beidar and Chebotar[4] for Lie ideals of prime rings, by Swain[25] for skew-symmetric elements, and by Beidar, Brešar, Chebotar and Martindale[2] for Lie ideals of skew-symmetric elements provided that the rings involved were of sufficiently high dimension. We refer the reader to the paper[3] where it was shown that this restriction could be removed and thereby Herstein's problem was solved in full generality. A result from[3] enables us to work with Lie derivations and to prove the following.

Corollary 4. *Let R be a prime ring of characteristic not 2 or 3 and L a noncentral Lie ideal of R. If δ is a noncentral Lie derivation of L, then the additive subgroup generated by the subset $\{[x^\delta, x] \mid x \in L\}$ contains a noncentral Lie ideal of R.*

2 PROOF OF THEOREM 2

Our first aim is to reduce the problem to the case of inner derivations. Recall that a derivation $d = ad(a)$ for $a \in R$ is called the inner derivation of R induced by a if $x^d = [x, a] = xa - ax$ for all $x \in R$. For additive subgroups A and B of R let $[A, B]$ be the additive subgroup of R generated by all commutators $[a, b]$, $a \in A$, $b \in B$.

Lemma 5. *Let R be a prime ring of characteristic not 2 and L a noncentral Lie ideal of R. Let D be a nonzero derivation of R. Then there exists a nonzero inner derivation $d = \mathrm{ad}(a)$ of R induced by $a \in L$ such that*

$$L^d \subseteq [L^D, L] \quad and \quad [L^d, L] \subseteq [L^D, L]. \tag{1}$$

Proof. For any $x, y, z \in L$ we have

$$[[x^D, y^D], z] = \{[[x^D, z], y^D] + [x^D, [y^D, z]]\} \in [L^D, L]. \tag{2}$$

Since

$$[[x^{D^2}, y], z] = [[x^D, y]^D, z] - [[x^D, y^D], z],$$

it follows from (2) that

$$[[x^{D^2}, y], z] \in [L^D, L] \tag{3}$$

for all $x, y, z \in L$. Taking $x \in [[L, L], [L, L]]$ we have that $x^{D^2} \in L^D$. In light of[19, Lemma 2] $[[L, L], [L, L]]$ is a noncentral Lie ideal of R, so there exists $b \in [[L, L], [L, L]]$ such that b^{D^2} is not a central element in R by[20, Theorem 4].

Define $d = \mathrm{ad}(a)$ to be the nonzero inner derivation of R induced by $a = b^{D^2} \in L$. Clearly $y^d = [y, b^{D^2}] \in [L^D, L]$ for all $y \in L$ and it follows from (3) that

$$[y^d, z] \in [L^D, L] \quad \text{for all } y, z \in L.$$

Thus we complete the proof.

Our next result is somehow an approximation of Theorem 2.

Lemma 6. *Let R be a prime ring of characteristic not 2, I a nonzero ideal of R and L a noncentral Lie ideal of R. Let d be a nonzero derivation of R satisfying $R^d \subseteq L$. Then any additive subgroup G of R containing both L^d and $[I^d, L]$ contains a noncentral Lie ideal of R.*

Proof. Clearly

$$[y^d, u] = \{[y, u]^d - [y, u^d]\} \in G \tag{4}$$

for all $y \in L, u \in I$. It follows from (4) that for all $u, v, w \in I$ we have

$$[[v^{d^2}, w], u] = \{[[v^d, w]^d, u] + [[u, v^d], w^d] + [[w^d, u], v^d]\} \in G.$$

By[20, Theorem 1] there exists $b \in I$ such that b^{d^2} is not a central element of R. Define $h = \mathrm{ad}(b^{d^2})$ to be the nonzero derivation of I induced by b^{d^2};

then $[I^h, I] \subseteq G$. By[10, Proposition 2] $[I^h, I]$ contains a noncentral Lie ideal of I. Hence by[5, Lemma 1] there exists a nonzero ideal J of I such that $[J, I] \subseteq [I^h, I] \subseteq G$. Thus G contains the noncentral Lie ideal $[IJI, I]$ of R.

We will consider two cases separately according to $d^3 \neq 0$ or $d^3 = 0$. In the first case the proof relies heavily on Chuang's theorem[11].

Lemma 7. *Let R be a prime ring of characteristic not 2 and L a noncentral Lie ideal of R. If d is a derivation of R such that $R^d \subseteq L$ and $d^3 \neq 0$, then $[L^d, L]$ contains a noncentral Lie ideal of R.*

Proof. Note that

$$[[x^{d^2}, y], z] = \{[[x^d, y]^d, z] + [[y^d, z], x^d] + [[z, x^d], y^d]\} \in [L^d, L] \tag{5}$$

for all $x, y, z \in L$. It follows from (5) that for $x, y \in L$ and $u \in R$ we have

$$[[u, x^{d^3}], y] = \{[[u, x^{d^2}]^d, y] - [[u^d, x^{d^2}], y]\} \in [L^d, L]. \tag{6}$$

By[11] there exists $b \in L$ such that b^{d^3} is not a central element of R. Since $b^{d^3} \in L^d \subseteq L$, the nonzero derivation $h = \mathrm{ad}(b^{d^3})$ of R satisfies $R^h \subseteq L$, $L^h \subseteq [L^d, L]$ and $[R^h, L] \subseteq [L^d, L]$. Then it follows from Lemma 6 that $[L^d, L]$ contains a noncentral Lie ideal of R.

The following result is due to Chuang and T.-K. Lee[12]. We include here its proof for the sake of completeness. We refer the reader to[1] for the definition and basic properties of Martindale symmetric quotient rings.

Lemma 8. *Let R be a prime ring of characteristic not 2 with Martindale symmetric quotient ring Q. Let I be a nonzero two-sided ideal of R and $0 \neq a \in Q$. Then the additive subgroup T of Q generated by the subset $\{xay + yax \mid x, y \in I\}$ contains a nonzero two-sided ideal of R.*

Proof. Choose a nonzero two-sided ideal J of R such that $J + aJ \subseteq I$. Then

$$\begin{aligned} 2xayaz &= [(xay)az + za(xay)] - [(zax)ay + ya(zax)] \\ &\quad + [(yaz)ax + xa(yaz)] \in T \end{aligned}$$

for all $x, y, z \in J$. That is, $2JaJaJ \subseteq T$. Note that $2JaJaJ$ is a nonzero two-sided ideal of R. Hence, the lemma is proved.

Now the case when $d^3 = 0$ can be settled.

Lemma 9. *Let R be a prime* ring *of characteristic not 2 or 3 and L a noncentral Lie ideal of R. Let $d = \mathrm{ad}(a)$ with $a \in L$ be a nonzero derivation of R satisfying $d^3 = 0$. Then any additive subgroup G of R containing both L^d and $[L^d, L]$ contains a noncentral Lie ideal of R.*

Proof. First note that $[[[x,[a,x]],a],u] \in [L^d, L]$ for all $x \in R$, $u \in L$. Since $d^3 = 0$, it follows from[23, Corollary 1] that there exists $\lambda \in C$, where C is the center of the Martindale symmetric quotient ring of R, such that $(a-\lambda)^2 = 0$. Let $b = a - \lambda$; then $[[[x,[b,x]],b],u] \in [L^d, L]$, or equivalently $2[[xbx,b],u] \in [L^d, L]$ for all $x \in R$, $u \in L$. Hence $[[x(2b)y + y(2b)x, b], u] \in [L^d, L]$ for all $x, y \in R, u \in L$, and it follows from Lemma 8 that $[[v,a],u] = [[v,b],u] \in [L^d, L] \subseteq G$ for all $v \in I$, $u \in L$, where I is a nonzero two-sided ideal of R contained in the additive subgroup generated by the subset $\{x(2a)y + y(2a)x \mid x, y \in R\}$. Thus the additive subgroup G of R contains both L^d and $[I^d, L]$ and so by Lemma 6 contains a noncentral Lie ideal of R.

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Lemma 5 there exists a nonzero inner derivation $d = \mathrm{ad}(a)$ with $a \in L$ such that $L^d \subseteq [L^D, L]$ and $[L^d, L] \subseteq [L^D, L]$. If $d^3 = 0$, it follows from Lemma 9 that the additive subgroup $[L^D, L]$ of R contains a noncentral Lie ideal of R. On the other hand, if $d^3 \neq 0$, it follows from Lemma 7 that $[L^d, L]$, and *a fortiori* $[L^D, L]$, contains a noncentral Lie ideal of R. Therefore the proof of Theorem 2 is complete.

3 PROOF OF THEOREM 1

We begin with the following auxiliary result.

Lemma 10. *Let R be a prime ring and L a noncentral Lie ideal of R. Let d be a nonzero derivation of R and H the additive subgroup of R generated by the subset $\{[x^d, y] + [y^d, x] \mid x, y \in L\}$. Then*

(i) $2\{[[x, y^d], z] + [[z^d, x], y]\} \in H$ *for all* $x, y, z \in L$,
(ii) $4[x, [y^d, z^d]] \in H$ *for all* $x, y, z \in [L, L]$.

Proof. By our definition

$$[x^d, y] + [y^d, x] \in H \tag{7}$$

for all $x, y \in L$. Replacing y by $[y, z]$, we have

$$[x^d, [y, z]] + [[y, z]^d, x] \in H, \tag{8}$$

for all $x, y, z \in L$. From $[[y,z]^d, x] = [[y,z],x]^d - [[y,z],x^d] = [[y,z],x]^d + [x^d,[y,z]]$, it follows that

$$2[x^d, [y, z]] + [[y, z], x]^d \in H.$$

This relation together with similar ones

$$2[y^d,[z,x]]+[[z,x],y]^d\in H$$

and

$$2[z^d,[x,y]]+[[x,y],z]^d\in H$$

yield

$$2\{[x^d,[y,z]]+[y^d,[z,x]]+[z^d,[x,y]]\}\in H \tag{9}$$

for all $x,y,z\in L$, since $[[y,z],x]+[[z,x],y]+[[x,y],z]=0$. Comparing (9) with (8) we obtain

$$2\{[y^d,[z,x]]+[z^d,[x,y]]-[[y,z]^d,x]\}\in H \tag{10}$$

for all $x,y,z\in L$. Observing that

$$\begin{aligned}&[y^d,[z,x]]+[z^d,[x,y]]-[[y,z]^d,x]\\ &\quad=[y^d,[z,x]]+[z^d,[x,y]]+[x,[y,z]^d]\\ &\quad=[y^d,[z,x]]+[x,[y^d,z]]+[z^d,[x,y]]+[x,[y,z^d]]\\ &\quad=-[z,[x,y^d]]-[y,[z^d,x]]=[[x,y^d],z]+[[z^d,x],y],\end{aligned}$$

we arrive at

$$2\{[[x,y^d],z]+[[z^d,x],y]\}\in H \tag{11}$$

for all $x,y,z\in L$, that is (i) is proved.

For $u,v,w\in[L,L]$, setting $x=u^d, y=v$ and $z=w$ in (11), we get

$$2\{[[u^d,v^d],w]+[[w^d,u^d],v]\}\in H. \tag{12}$$

Similarly,

$$2\{[[v^d,w^d],u]+[[u^d,v^d],w]\}\in H \tag{13}$$

and

$$2\{[[w^d,u^d],v]+[[v^d,w^d],u]\}\in H \tag{14}$$

for all $u, v, w \in [L, L]$. Subtracting (14) from the sum of (12) and (13), we get $4[[u^d, v^d], w] \in H$ for all $u, v, w \in [L, L]$. The proof is thereby complete.

Next we consider two cases separately according as $[L_0^d, L_0^d]^d$ is central or not, where $L_0 = [L, L]$.

Lemma 11. *Let R be a prime ring of characteristic not 2 or 3 and L a noncentral Lie ideal of R. Let d be a nonzero derivation of R and H the additive subgroup of R generated by the subset* $\{[x^d, y] + [y^d, x] \mid x, y \in L\}$. *If there exist* $a, b \in [L, L]$ *such that* $[a^d, b^d]^d$ *is not a central element of R, then H contains a noncentral Lie ideal of R.*

Proof. By Lemma 10(i) we have $2\{[[x, [a^d, b^d]^d], y] + [[y^d, x], [a^d, b^d]]\} \in H$ for all $x, y \in [L, L]$. By Lemma 10(ii) we have $4[[y^d, x], [a^d, b^d]] \in H$ and so $4[[x, [a^d, b^d]^d], y] \in H$ for all $x, y \in [L, L]$. Let $h = \mathrm{ad}(4[a^d, b^d]^d)$; then h is a nonzero derivation of R. Since $[L, L]$ is a noncentral Lie ideal of R and $[[L, L]^h, [L, L]] \subseteq H$, it follows from Theorem 2 that $[[L, L]^h, [L, L]]$, and *a fortiori* H, contains a noncentral Lie ideal of R.

To prove the following Lemma 12 we will need the concept of X-inner derivations (see[1] for details).

Lemma 12. *Let R be a prime ring of characteristic not 2 or 3 and L a noncentral Lie ideal of R. Let d be a nonzero derivation of R and H the additive subgroup of R generated by the subset* $\{[x^d, y] + [y^d, x] \mid x, y \in L\}$. *Suppose that* $[x^d, y^d]^d$ *is a central element of R for all* $x, y \in [L, L]$. *Then H contains a noncentral Lie ideal of R.*

Proof. First note that $L_0 = [L, L]$ is a noncentral Lie ideal of R. By assumption we have $[[x^{d^2}, y^d] + [x^d, y^{d^2}], z] = 0$ for all $x, y, z \in L_0$. If d is not an X-inner derivation, it follows from[21, Theorem 1] that $[[x, y] + [u, v], z] = 0$ for all $x, y, z, u, v \in [R, R]$, and so $[[x, y], z] = 0$ for all $x, y, z \in [R, R]$ which would yield the commutativity of the ring R, a contradiction. Hence d is an X-inner derivation and we have

$$[x^d, y^d]^{d^2} = [x^{d^3}, y^d] + 2[x^{d^2}, y^{d^2}] + [x^d, y^{d^3}] = 0 \tag{15}$$

for all $x, y \in L_0$. Set $L_1 = [[L_0, L_0], [L_0, L_0]]$; then L_1 is also a noncentral Lie ideal of R. Note that $(L_1)^{d^i} \subseteq L_0$ for all $i = 1, 2, 3$. Since $[x^d, y] \equiv [x, y^d]$ (mod H) for all $x, y \in L$ we have

$$[x^{d^3}, y^d] \equiv [x^{d^2}, y^{d^2}] \equiv [x^d, y^{d^3}] \pmod H \tag{16}$$

for all $x, y \in L_1$. It follows from (15) and (16) that

$$4[x^{d^3}, y^d] \equiv 4[x^{d^2}, y^{d^2}] \equiv 4[x^d, y^{d^3}] \equiv 0 \pmod H \tag{17}$$

and so by (17)

$$4[x^{d^4}, y] \equiv 4[x^{d^3}, y^d] \equiv 0 \pmod{H} \tag{18}$$

for all $x, y \in L_1$. Since $[x^d, y]^{d^3} = [x^{d^4}, y] + 3[x^{d^3}, y^d] + 3[x^{d^2}, y^{d^2}] + [x^d, y^{d^3}]$, it follows from (17) and (18) that

$$4[x^d, y]^{d^3} \equiv 0 \pmod{H} \tag{19}$$

for all $x, y \in L_1$. By Theorem 2 there is a noncentral Lie ideal L_2 of R which is contained in the additive subgroup of R generated by $\{4[x^d, y] \mid x, y \in L_1\}$. Thus $L_2 \subseteq L_1$ and $(L_2)^{d^3} \subseteq H$. Hence,

$$[x, y]^{d^3} \equiv 0 \pmod{H} \tag{20}$$

for all $x, y \in L_2$. Clearly $[x, y]^{d^3} = [x^{d^3}, y] + 3[x^{d^2}, y^d] + 3[x^d, y^{d^2}] + [x, y^{d^3}]$. Observing that $[x^{d^3}, y] \equiv [x^{d^2}, y^d] \equiv [x^d, y^{d^2}] \equiv [x, y^{d^3}] \pmod{H}$ for all $x, y \in L_2$ we obtain by (20) that $8[x^{d^3}, y] \equiv 8[x^{d^2}, y^d] \equiv 8[x^d, y^{d^2}] \equiv 0 \pmod{H}$ for all $x, y \in L_2$. Since $[x^d, y]^{d^2} = [x^{d^3}, y] + 2[x^{d^2}, y^d] + [x^d, y^{d^2}]$ we get

$$8[x^d, y]^{d^2} \equiv 0 \pmod{H} \tag{21}$$

for all $x, y \in L_2$. By Theorem 2 there is a noncentral Lie ideal L_3 of R which is contained in the additive subgroup of R generated by $\{8[x^d, y] \mid x, y \in L_2\}$. Thus $L_3 \subseteq L_2$ and $(L_3)^{d^2} \subseteq H$. Hence,

$$[x, y]^{d^2} \equiv 0 \pmod{H} \tag{22}$$

for all $x, y \in L_3$. Clearly $[x, y]^{d^2} = [x^{d^2}, y] + 2[x^d, y^d] + [x, y^{d^2}]$. Observing that $[x^{d^2}, y] \equiv [x^d, y^d] \equiv [x, y^{d^2}] \pmod{H}$ for all $x, y \in L_3$, we get by (22) that $4[x^{d^2}, y] \equiv 4[x^d, y^d] \equiv 0 \pmod{H}$ for all $x, y \in L_3$. Since $[x^d, y]^d = [x^{d^2}, y^d] + [x^d, y^d]$, we get

$$4[x^d, y]^d \equiv 0 \pmod{H} \tag{23}$$

for all $x, y \in L_3$. By Theorem 2 there is a noncentral Lie ideal L_4 of R which is contained in the additive subgroup of R generated by $\{4[x^d, y] \mid x, y \in L_3\}$. Thus $L_4 \subseteq L_3$ and $(L_4)^d \subseteq H$. Hence,

$$[x, y]^d \equiv 0 \pmod{H} \tag{24}$$

for all $x, y \in L_4$. Since $[x^d, y] \equiv [x, y^d] \pmod{H}$ for all $x, y \in L_4$ it follows from (24) that $2[x^d, y] \equiv 0 \pmod{H}$, that is, $2[x^d, y] \in H$ for all $x, y \in L_4$. Thus Theorem 2 completes the proof.

It is then obvious that Theorem 1 follows immediately from Lemmas 11 and 12.

Proof of Corollary 4. Note that $[L, L]$ is a noncentral Lie ideal of R. Let C be the center of the Martindale symmetric quotient ring of R. Denote by $\langle L \rangle$ the subring of R generated by L. By[3, Corollary 1.4] there exists a derivation $d: \langle L \rangle \to \langle L \rangle C + C$ such that $x^d = x^\delta$ for all $x \in [L, L]$. Clearly, $\langle [L, L] \rangle^d \subseteq \langle [L, L] \rangle$. Note that $\langle [L, L] \rangle$ is both a noncentral subring and a Lie ideal of R, so by[15, Lemma 1.3] it contains a nonzero ideal I of R. It follows from Theorem 1 that the additive subgroup of R generated by subset $\{[x^d, x] \mid x \in [L, L]\}$ contains a noncentral Lie ideal of $\langle [L, L] \rangle$. Therefore it contains a noncentral Lie ideal of R. The proof is then complete.

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