行政院國家科學委員會補助專題研究計畫成果報告

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計畫類別：☑個別型計畫　　□整合型計畫
計畫編號：NSC 90－2115－M－002－023
執行期間：90 年 8 月 01 日至 91 年 7 月 31 日

計畫主持人：張秋俊　台灣大學數學系
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行政院國家科學委員會專題研究計畫成果報告

Emden-Fowler 方程之研究

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一、中文摘要

考慮廣義 Emden-Fowler 方程 $y''+g(x)y^{\gamma}=0$, $\gamma=2n-1$, n>1 整數，$g(x)$>0。在此論文裡，我們探討在適當條件下滿足 y(a)=0 之有界正解之存在性、唯一性及一般（正）解之非震盪性（漸近表現）結果。

關鍵詞：非線性方程，正解，唯一性，
非震盪性，漸近表現。

Abstract

Consider the generalized Emden-Fowler equation

$$(*)\ y''+g(x)y^{\gamma}=0 \quad \text{in} \quad [0,\infty),$$

where $\gamma=2n-1$, with n>1 an integer and $g(x)$>0. In this article we will give results about existence and uniqueness of bounded positive solution and nonoscillation-behavior of solutions.

Keywords: nonlinear equation,
exitence, uniqueness,
nonoscillation, asymptotic
behavior.

二、本年進度

此為一年期之計畫。所得結果尚滿意。進一步欲了解文中 $G_y(x)$ 恆大於 0 之條件。

三、成果自評

對整個方程之存在性、唯一性、非震盪性及漸近表現有全盤的了解，理論漸趨完整。

On Emden-Fowler equation

By

CHIU-CHUN CHANG（張秋俊）

Abstract. Consider the generalized Emden-Fowler equation

(*) $y'' + g(x)y^{\gamma} = 0$ in $[0,\infty]$, where $\gamma = 2n-1$, with $n > 1$ an integer and $g(x) > 0$. In this article we will review some results about existence, uniqueness of bounded positive solution and add some new results about nonoscillation and asymptotic behaviour of solutions. So that the whole theory will be fairly complete.

AMS subjection classification : 34C15

Key words : nonlinear equation, existence, uniqueness, nonoscillation, asymptotic behavior.

1. Introduction

Consider the generalized Emden-Fowler equation

(1) $y'' + g(x)y^{\gamma} = 0 \quad , x \in (0,\infty)$

where $\gamma = 2n-1 > 1$ and $g(x)$ is assumed to be positive and continuous on $[0,\infty)$. We also assume that the solutions of (1) are continuable to the entire non-negative real axis; say under the condition that $g(x)$ is locally of bounded variation [5].

In this article we will first review some results about existence and uniqueness of bounded positive solution and give new results about nonoscillation of the solutions as well as the asymptotic behavior of positive solutions. So that the whole theory will be fairly complete.

2. Existence

We seek bounded positive solution y of (1) with $y(0) = 0$, so that the solution will positive and increasing and tends to a positive constant. As in [2], we built such a solution by first consider similar solution U_m on $\left[\frac{1}{m}, m\right]$ and take their limit. The existence of U_m is quaranted by previous theory [9]. The uniform limit $U_\infty = y$ (on compact intervals) will be a bounded positive solution of (1) under the integral condition.

(2) $\int_0^\infty g(x)x^{\frac{\gamma+1}{2}}\,dx < \infty.$

Theorem 1. Under condition (2), there exists a bounded positive solution of (1) with $y(0) = 0$.

proof. The essential part of the proof is to show that the limit U_∞ is bounded.

From the equation

(3) $U_m(m) = U_m(a) + \int_0^m (t-a)g(t)U_m^{\gamma}(t)\,dt, 0 < a < n,$

we have

$$U_m(m) \le U_m(a) + \int_a^m tg(t)\,U_m^{\gamma}(t)\,dt,\ 0 \le a < n,$$

$$\le U_m(a) + U_m(m)\int_a^m tg(t)\,U_m^{\gamma-1}(t)\,dt$$

$$\le U_m(a) + U_m(m)\cdot C\cdot \int_a^\infty tg(t)\cdot t^{\frac{\gamma-1}{2}}\,dt$$

$$= U_m(a) + U_m(m)\cdot C\cdot \int_a^\infty g(t)t^{\frac{\gamma+1}{2}}\,dt$$

(2) implies $U_m(m) \le U_m(a) + \in U_m(m)$

for a large ($\in$ fixed number less than 1).

Hence U_∞ is a bounded solution.

3. Uniqueness

In considering the uniqueness problem of bounded positive solution of (1), we first state the following Pohozaev identity [4].

(3) $$G_y(s) = (s-a)y'^2(s) - y(s)y'(s) + \frac{2}{\gamma+1}(s-a)g(s)y^{\gamma+1}(s)$$

$$= \frac{2}{\gamma+1}\int_a^s \left[\frac{\gamma+3}{2} + \frac{(t-a)g'(t)}{g(t)}\right] g(t)y^{\gamma+1}(t)\,dt$$

$$= \frac{2}{\gamma+1}\int_a^s Q(t)g(t)y^{\gamma+1}(t)dt.$$

we assume that

(4) $Q(t) = \dfrac{\gamma+3}{2} + \dfrac{(t-a)g'(t)}{g(t)}$ is eventually of one sign and hence it is non-positive in accordance with (2). We further assume that there exists a bounded positive solution y of (1) with $y(a) = 0$ so that the corresponding $G_y(s)$ is positive in (a, ∞).

Theorem 2. If $Q(t)$ has at most one zero in (a, ∞), then $G_y(s)$ is positive in (a, ∞) for any bounded solution y.

proof. It is clear from the facts that $Q(a)>0$, $Q(x)\le 0$ eventually, y is bounded and that

$(s-a)y'(s)$ tends to zero.

We can now state the following uniqueness theorem.

Theorem 3. Under condition (4) there exist at most one bounded positive solution of (1) with $y(a)=0$.

Proof. The essential part of the proof is the following comparison formula [3]. Let y, y_1 be two positive solutions of (1) with $y_1(a)=y(a)=0$ and $G_y(s)>0$ in (a,∞), then

(5) $G_{y_1}(x)>(<)\left(\frac{y_1}{y}\right)^{\gamma+1} G_y(x)$ in (a,∞) for cases $y_1'(a)>(<)\, y'(a)$.

For the proof, we consider

(6) $L(t)=G_{y_1}(t)-\left(\frac{y_1}{y}\right)^{\gamma+1}(t)\, G_y(t)$.

Then $L(a)=0$ and

(7) $L'(t)=-(\gamma+1)\left(\frac{y_1}{y}\right)^{\gamma+1}\left[\frac{y_1'}{y_1}-\frac{y'}{y}\right]G_y(t)$.

$L(t)>0\,(<0)$ as long as $\frac{y_1}{y}$ is decreasing (increasing) which is true initially (near a) as can be seen from the identity

(8) $(y_1'y-y'y_1)(x)=\int_a^x y_1 y g(s)(y^{\gamma-1}-y_1^{\gamma-1})\, ds$.

If there is a point x_0 such that

(9) $\frac{y_1'}{y_1}(x_0)=\frac{y'}{y}(x_0)$, then from (8) we have

(10) $y^{\gamma-1}(x_0)\geq(\leq) y_1^{\gamma-1}(x_0)$.

But $L(x_0)>0$ implies

(11) $(x_0-a)y_1'^2(x_0)-y_1'(x_0)y(x_0)+\frac{2}{\gamma+1}(x_0-a)g(x_0)y_1^{\gamma+1}(x_0)$

$$>(<)\left(\frac{y_1}{y}\right)^{\gamma+1}(x_0)\left\{(x_0-a)y'^2(x_0)-y'(x_0)y(x_0)+\frac{2}{\gamma+1}(x_0-a)g(x_0)y^{\gamma+1}(x_0)\right\}.$$

or,

$$(11)\quad (x_0-a)y_1'^2(x_0)-y_1'(x_0)y_1(x_0)>\text{ or }<\left(\frac{y_1}{y}\right)^{\gamma-1}(x_0)\{(x_0-a)y_1'^2(x_0)-y_1'(x_0)y_1'(x_0)\}$$

As is well known that $(x_0-a)y_1'^2(x_0)-y_1'(x_0)y_1(x_0)<0$, this leads to a contradiction with (10).

Therefore, we have $\left(\frac{y_1}{y}\right)'<0(>0)$ all the way in (a,∞) for cases $y_1'(a)>(<)y'(a)$.

Hence $L(t)$ is increasing (decreasing) in (a,∞). But since $G_{y_1}(\infty)=0=G_y(\infty)$ and $L(\infty)=0$, a contradiction follows. Hence we have proved the uniqueness theorem.

4. nonoscillation

To guarantee the nonoscillation property of equation (1), we assume (2) and (4) for each fixed a.

Theorem (4). Under condition (2) and (4), The equation (1) in nonoscillatory.

proof. Let y be a solution of (1) with infinite many zeros $a_1, a_2, \ldots a_n \ldots$, in increasing order. On $[a_v,\infty]$ there exists a unique bounded positive solution y_v with $y_v(a_v)=0$, and $G_{y_v}(x)>0$ on (a_v,∞).

As in the proof of Theorem 3, by considering the function

$$L(t)=G_y(t)-\left(\frac{y}{y_v}\right)^{\gamma+1}(t)G_{y_v}(t)\quad\text{in } (a_v,\infty),$$

we can show that the case $y'(a_v)\le y_v'(a_v)$ must be excluded for otherwise $\frac{y}{y_v}$ will be increasing all the way violating the oscillation property of y.

Since $Q \le 0$ eventually, we may assume $g' < 0$ and $y'(a_m) < y'(a_v)$ for $m > v$ because as is well known that $y'^2(s) + \frac{2}{\gamma+1} g(s) y^{\gamma+1}(s)$ is decreasing.

$Q \le 0$ eventually implies $G_y(x)$ is eventually decrensing and positive. Hence we have

(12) $a_v y_v'^2(a_v) \le a_v y'^2(a_v) \le C, v = 1,2,\ldots$.

From the equation

$$y_v(x) = y_v(a_v) + \int_{a_v}^{x} (s - a_v) g(s) \cdot y_v^{\gamma}(s) ds + (x - a_v) y_v'(x),$$

we have

(13) $y_v(2a_v) = 0 + \int_{a_v}^{2a_v} (s - a_v) g(s) \cdot y_v^{\gamma}(s) ds + a_v y_v'(2a_v)$, and

$$\int_{a_v}^{2a_v} (s - a_v) g(s) \cdot y_v^{\gamma}(s) ds \le y_v'(2a_v) \int_{a_v}^{2a_v} (s - a_v) g(s) [y_v'(a_v)(s - a_v)]^{\gamma-1} ds$$

$$= y_v(2a_v) \int_{a_v}^{2a_v} (s - a)^{\frac{\gamma+1}{2}} g(s) \left[y_v'(a_v)(s - a_v)^{\frac{1}{2}} \right]^{\gamma-1} ds$$

$$\le y_v(2a_v)) \left[y_v'(a_v)(2a_v - a_v)^{\frac{1}{2}} \right]^{\gamma-1} \int_{a_v}^{2a_v} (s - a)^{\frac{\gamma+1}{2}} g(s)\, ds .$$

(12), (13) implies

(14) $1 \le \left[y_v'(a_v) a_v^{\frac{1}{2}} \right]^{\gamma-1} \int_{a_v}^{\infty} g(s) s^{\frac{\gamma+1}{2}} ds + \frac{1}{2}$

$\le C^{\gamma-1} \cdot \in + \frac{1}{2}$ for v large.

This will leads to a contradiction if $\in$ is sufficiently small. Therefore equation (1), is nonoscillatory.

5. Asymptotic behavior of positive solutions

As to the asymptotic behavior of positive (increasing) solution, we assume that it is not of the order x because it is quite known to be the case [8]

$\int^{\infty} gs^{\gamma} ds < \infty$.

So, the cases we consider below satisfy

$\int^{\infty} gs^{\gamma} ds = \infty$ and $\int^{\infty} gs^{\frac{\gamma+1}{2}} ds < \infty$.

Let $w = \dfrac{sy'(s)}{y(s)}$, then w satisfies

(15) $w'(s) = \dfrac{w(1-w)}{s} - sgy^{\gamma-1}, 0 < w(s) < 1$.

By integration,

(16) $\int_a^x sgy^{\gamma-1} ds - \int_a^x \dfrac{w(1-w)}{s} ds = w(a) - w(x) = 1 - w(x) > 0,$

if $y(a) = 0$.

First, we show that for unbounded positive solution y,

we must have $\lim_{x\to\infty} \int_a^x sgy^{\gamma-1} ds = \infty$.

For from the equation

$$y(x) = y(a) + \int_a^x sgy^{\gamma} ds + xy'(x)$$

$$\leq y(a) + y(x)\int_a^x sgy^{\gamma-1} ds + xy'(x)$$

$$\leq y(a) + \in y(x) + xy'(x), \text{ if } \int_a^x sgy^{\gamma-1} ds \text{ is bounded}$$

and a is sufficiently large, then $1 \leq \dfrac{y(a)}{y(x)} + \in + \dfrac{xy'}{y}$.

This would leads to case $y(x) \sim x$. [8]

Hence (16) means tsat

$\int^x sgy^{\gamma-1}\text{ds}$ *md* $\int^x \dfrac{w(1-w)}{s} ds$ is of the same infinite order.

For (15), we have another expression

(17) $w'(x) = \frac{1}{y^2(x)}\{y'y - (x-a)y'^2 - (x-a)gy^{\gamma+1}$

$$= -\frac{1}{y^2(x)}\int_a^x \left\{(\gamma-1)\left(w-\frac{1}{2}\right)+Q(s)\right\}gy^{\gamma+1}ds.$$

Since $Q \le 0$ eventually, this equation shows that there is no local minimum of w with value less than $\frac{1}{2}$. If $w \le \frac{1}{2}$ eventually, then from the equation.

(18) $y(x) = y(a) + \int_a^x (s-a)gy^\gamma ds + xy'(x),$ we have

(19) $y(x) \le y(a) + y(x)\int_a^\infty gt^{\frac{\gamma+1}{2}}dt + y.w(x)$

$\le y(a) + \in y(x) + \frac{1}{2}y(x)$, This implies y is a bounded solution.

Therefore, we conclude that w must eventually large than $\frac{1}{2}$. Have we have.

(20) $\int_a^x sgy^{\gamma-1}ds > \int_a^x \frac{w(1-w)}{s}ds > \frac{1}{2}\int_a^x \frac{1-w}{s}ds$, a large, or $\int_a^x sgy^{\gamma-1}ds > \frac{1}{2}\int_a^x\left(-\frac{u'}{u^r}.u^{\gamma-1}\right)ds$

with $y = us$, $1 - w = -\frac{su'}{u}, = \frac{1}{2(\gamma-1)}u^{1-\gamma}(x)$.

On the other hand $\int^x sgy^{\gamma-1}ds = \int_a^x gs^\gamma u^{\gamma-1}ds = u^{\gamma-1}(a)\int_a^\xi gs^\gamma$, since u is decreasing.

$\le u^{\gamma-1}(a)\int_a^x gs^\gamma ds$.

Hence we have $\int_a^x gs^\gamma ds \ge C\ u^{1-\gamma}(x)$, or $u^{\gamma-1}(x) \ge \frac{C}{\int_a^x gs^\gamma}ds$ eventually.

Therefore, we have proved one side of the following theorem about the asymptotic behavior of unbounded positive solution of equation (1).

Theorem. Any unbounded positive solution y of (1), with $\int^\infty gs^\gamma ds = \infty$, $\int^\infty gs^{\frac{\gamma+1}{2}} < \infty$, is of the asymptotic order $\dfrac{x}{(\int_a^x gs^\gamma ds)^{\frac{1}{\gamma-1}}}$.

proof: we will prove the inequality $\leq$. Let y_0 be the bounded positive solution, then as before we have $G_y(x) \leq \left(\dfrac{y}{y_0}\right)^{\gamma+1} G_{y_0}(x), a \leq x < \infty$.

Let x_0 large fixed and consider $f(x) = G_y(x) - MG_{y_0}(x)$, $M = (\dfrac{y}{y_0})^{\gamma+1}(x_0)$.

Since y is an unbounded positive solution , $\left(\dfrac{y}{y_0}\right)' > 0$ and

$$f'(x) = Q\left[gy^{\gamma+1} - Mgy_0^{\gamma+1}\right] = Qgy_0^{\gamma+1}(x)\left[\left(\frac{y}{y_0}\right)^{\gamma+1}(x) - M\right] < 0 \text{ form } x_0 \text{ on.}$$

Hence f tends to a negative limit $-k, 0 < k \leq \infty$.

So, $G_y(x) - MG_{y_0}(x) \leq -\dfrac{2}{3}k$ on (x_1, ∞), some x_1.

Since $G_{y_0}(x) \to 0, x \to \infty$, we have $G_y(x) < -\dfrac{k}{2} < 0$ for x large.

This means $\dfrac{2}{\gamma+1}(s-a)gy^{\gamma+1} \leq y'y - (s-a)y'^2$ or $\dfrac{2}{\gamma+1}(s-a)gy^{\gamma-1} \leq \dfrac{1}{s}w(1-w) \leq \dfrac{1}{s}(1-w)$.

Then following the proof from equation (20) on (with reverse inequality) we get the inequality

$$u^{\gamma-1} \leq \frac{1}{\int^x gs^\gamma ds}.$$

The proof is completed .

Remark. We remark that for unbounded positive solution y , the corresponding $G_y(x)$ must tends to $-\infty$ as $x \to \infty$, because the M considered in the proof may be any large constant.

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