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REPRESENTING RIEMANN'S MINIMAL SURFACE BY WEIERSTRASS ζ FUNCTION

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It is more desirable to represent Weierstrass data of a minimal surface by Weierstrass ζ function rather than by complex integration. For example, Gray [2] figured out that the Costa minimal surface is given explicitly by

$$x_1 = \frac{1}{2}\Re\Big\{ -\zeta(u+iv) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}[\zeta(u+iv-\frac{1}{2}) - \zeta(u+iv-\frac{1}{2}i)]\Big\}$$

$$x_2 = \frac{1}{2}\Re\Big\{ -i\zeta(u+iv) + \pi v + \frac{\pi^2}{4e_1} - \frac{\pi}{2e_1}[i\zeta(u+iv-\frac{1}{2}) - i\zeta(u+iv-\frac{1}{2}i)]\Big\}$$

$$x_3 = \frac{1}{4}\sqrt{2\pi} ln\Big|\frac{\mathfrak{P}(u+iv)-e_1}{\mathfrak{P}(u+iv)+e_1}\Big|$$

To fix the notations for Riemann's minimal surface, let $w^2 = 4z(z^2 - e^2)$ be an elliptic curve conformal to the unit square in the u-plane, so $z = \mathfrak{P}(u)$ and $w = \mathfrak{P}'(u)$, where $\mathfrak{P}(u)$ is the Weierstrass $\mathfrak{P}$-function. The Weierstrass data in

$$x_1 = \Re\int \omega_1 = \Re\int \frac{1}{2}(1-g^2)\eta$$

$$x_2 = \Re\int \omega_2 = \Re\int \frac{i}{2}(1+g^2)\eta$$

$$x_3 = \Re\int \omega_3 = \Re\int g\eta$$

is given by

$$g = \lambda z = \lambda \mathfrak{P}(u)$$

and

$$\eta = \frac{du}{g} = \frac{1}{\lambda z}\frac{dz}{w}$$

where λ is a purely imaginary constant to be determined as follows.

Let γ_1 be the loop containing the two branch points 0 and −e and γ_2 containing 0 and +e. The real period $\Re\oint_{\gamma_1}\omega_2$ always vanishes. To eliminate $\Re\oint_{\gamma_2}\omega_2$, let's define

$$F = \int_0^e \frac{dz}{z\sqrt{z(e^2-z^2)}}$$

$$G = \int_0^e \frac{zdz}{\sqrt{z(e^2 - z^2)}}$$

so

$$\oint_{\gamma_2} \omega_2 = \int_0^e \frac{i}{\lambda} \frac{dz}{z\sqrt{z(z^2 - e^2)}} + \int_0^e \frac{i\lambda z dz}{\sqrt{z(z^2 - e^2)}} = \frac{1}{\lambda} F + \lambda G$$

therefore if we choose $\lambda = \sqrt{-F/G}$, then $\Re \oint_{\gamma_2} \omega_2 = 0$

In order to integrate the first term of

$$x_1 = \Re \int \frac{1}{2}(1 - g^2)\eta = \Re \int \frac{1}{2}\Big[\frac{1}{\lambda \wp(u)} - \lambda \wp(u)\Big] du$$

in terms of the Weierstrass ζ-function, we recall the addition formula for the Weierstrass $\wp$-function [1]:

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\Big(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\Big)^2$$

Let $v = \frac{1}{2} + \frac{i}{2}$

$$\wp(u + \frac{1}{2} + \frac{i}{2}) = -\wp(u) - 0 + \frac{1}{4}\Big(\frac{\wp'(u) - 0}{\wp(u) - 0}\Big)^2$$

Now $\wp'^2 = 4\wp(\wp^2 - e^2)$, so

$$\wp(u + \frac{1}{2} + \frac{i}{2}) = -\frac{e^2}{\wp(u)}$$

$$\int \frac{du}{\wp(u)} = \frac{1}{e^2}\zeta(u + \frac{1}{2} + \frac{i}{2})$$

It also follows that $\lambda = \frac{i}{e}$. Similarly we can get a parametric representation for Toubiana's generalization [3] of Riemann's minimal surface and easily generate computer graphics by Mathematica or Maple.

References

[1] Lars V. Ahlfors, *Complex Analysis*, McGraw-Hill (1979), 272-277.

[2] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, CRC Press (1997), 747-757.

[3] E. Toubiana, *On minimal surfaces of riemann.*, Comment. Math. Helvetici **67** (1992), 546-570.

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