行政院國家科學委員會專題研究計畫報告

Extensions of n-dimensional Euclidean
Vector Space E^n over R to
Pseudo Fuzzy Vector space over $F_p^1(1)$

計畫類別：個別型計畫

計畫名稱：n 維實數上的歐氏向量空間至$F_p^1(1)$上的擬似模糊向量空間的擴張

計畫編號：NSC91-2115-M-002-007

執行期間：91 年 8 月 1 日至 92 年 10 月 31 日

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中華民國 93 年 1 月 8 日

Extension of n-dimensional Euclidean Vector Space E^n over R to Pseudo Fuzzy Vector Space over $F_p^1(1)$

by

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Abstract: For any two points $P = (p^{(1)}, p^{(2)}, \ldots, p^{(n)}), Q = (q^{(1)}, q^{(2)}, \ldots, q^{(n)})$ of R^n, we define the crisp vector $\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) = Q(-)P$. Then we obtain an n-dimensional vector space $E^n = \{\overrightarrow{PQ} \mid \forall P, Q \in R^n\}$.Further, we extend the crisp vector into the fuzzy vector on fuzzy sets of R^n. Let $\widetilde{D}, \widetilde{E}$ be any two fuzzy sets on R^n, define the fuzzy vector $\overrightarrow{\widetilde{E}\widetilde{D}} = \widetilde{D} \ominus \widetilde{E}$, then we have a pseudo fuzzy vector space.

Keywords: Membership function, fuzzy vector space, pseudo fuzzy vector space, fuzzy vector.

1. Introduction

In [1,2, 4, 5, 6], fuzzy vector space is discussed theorectically. In Katsaras and Liu [2], E, denotes a vector space over K, where K is the space of real or complex numbers. A fuzzy set F in E is called a fuzzy subspace if (i) $F + F \subset F$. (ii) $\lambda F \subset F$ for every scalar λ. Katsaras and Liu introduced the concept of a fuzzy subspace of a vector space. In Das [1], E denotes a vector space over a field K. Let $I = [0, 1]$ and I^E be the collection of all mappings of E into I. We say $\mu \in I^E$ is a fuzzy space of E under a triangular norm T. (1, Definition 2.1), or simply a T-fuzzy subspace of E if $\forall x, y \in E$, $\forall a \in K$, $\mu(x + y) \geq T(\mu(x), \mu(y))$, and $\mu(ax) \geq \mu(x)$ respectively. In Lubczonok [5], a fuzzy vector space is a pair $\widetilde{E} = (E, \mu)$ where E is a vector space and $\mu : E \longrightarrow [0, 1]$ with the property that for all $a, b \in R$ and $x, y \in E$, we have $\mu(ax+by) \geq \mu(x) \wedge \mu(y)$. In Kumar [4], V is a vector space over F where F is the field of real numbers. A fuzzy subset μ of V is called a fuzzy subspace if it has the following properties. (i) $\mu(v_1 - v_2) \geq min(\mu(v_1), \mu(v_2))$ $\forall v_1, v_2 \in V$. (ii) $\mu(\alpha v) \geq \mu(v)$ $\forall \alpha \in F$, $v \in V$. There are various definitions of fuzzy vector spaces in these papers. All of them use the fuzzy set μ over a crisp vector space E, or $\mu : E \longrightarrow [0, 1]$ to define fuzzy vector. These are different from our work. Pick two points $P = (p_1, p_2, \cdots, p_n), Q = (q_1, q_2, \cdots, q_n)$ in R^n to form a vector $\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2, \cdots, q_n - p_n)$. Then extend this vector to the fuzzy vector $\overrightarrow{\widetilde{P}\widetilde{Q}} = \widetilde{Q} \ominus \widetilde{P}$ formed by fuzzy sets $\widetilde{P}, \widetilde{Q}$ on R^n. This is very useful compare to the abstract one defined in [1, 2, 4, 5].

Section 2 is a preparing work. Section 3 is the extension of the crisp n-dimentional Euclidean vector E^n to the pseudo fuzzy vector space. We talked in Section 4 about the length of the fuzzy vectors and fuzzy inner product. Section 5 is a more discussion.

2. Preparation

In order to consider the fuzzy vectors of fuzzy sets on R^n, we ought to know the following. First, from Kaufmann-and Gupta [3], and Zimmermann [8], we have the following definition.

Definition 2.1

(a). A fuzzy set $\widetilde{A}$ on $R = (-\infty, \infty)$ is convex iff every ordinary set $A(\alpha) = \{ x \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$ $\forall \alpha \in [0,1]$ is convex. Thus $A(\alpha)$ is a closed interval in R.

(b). A fuzzy set $\widetilde{A}$ on R is normal iff $\underset{x \in R}{\mathrm{V}} \mu_{\widetilde{A}}(x) = 1$.

We can extend this definition to R^n, say if $\widetilde{D}$ is a fuzzy set on R^n with membership function

$$\mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in [0,1] \, \forall \, (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in R^n \qquad (2.1)$$

then we have the following definition.

Definition 2.2. The α-cut of fuzzy set $\widetilde{D}$ on $R^n, 0 \leq \alpha \leq 1$ is defined by

$$D(\alpha) = \{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \mid \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq \alpha\} \qquad (2.2)$$

Insert Fig. 1 here.

Definition 2.3.

(a). A fuzzy set $\widetilde{D}$ on R^n is convex, iff for each $\alpha \in [0,1]$, every ordinary set

$$D(\alpha) = \{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \mid \mu_{\widetilde{D}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq \alpha\} \qquad (2.3)$$

is a convex closed subset of R^n.

(b). A fuzzy set $\widetilde{D}$ is normal iff

$$\bigvee_{(x^{(1)},x^{(2)},\ldots,x^{(n)})\in R^n} \mu_{\widetilde{D}}(x^{(1)},x^{(2)},\ldots,x^{(n)}) = 1 \tag{2.4}$$

Let F_c be the family of all fuzzy sets on R^n satisfying Definition 2.3 (a), (b).

Remark 2.4. When $\alpha = 0$, then the α-cut is

$$\{(x^{(1)},x^{(2)},\ldots,x^{(n)}) \,|\, \mu_{\widetilde{D}}(x^{(1)},x^{(2)},\ldots,x^{(n)}) \geq 0\}. \tag{2.5}$$

Let $D(0)$ be the smallest convex closed subset in R^n satisfying

$$\{(x^{(1)},x^{(2)},\ldots,x^{(n)}) \,|\, \mu_{\widetilde{D}}(x^{(1)},x^{(2)},\ldots,x^{(n)}) \geq 0\} \tag{2.6}$$

(see Example 4.11).

Definition 2.5. (Pu and Liu [7]) If the membership function of a fuzzy set $a_\alpha, 0 \leq \alpha \leq 1$; on R is

$$\mu_{a_\alpha} = \begin{cases} \alpha, & x = a \\ 0, & x \neq a \end{cases} \tag{2.7}$$

then we call a_α, a level α fuzzy point on R.

Let $F_p^1(\alpha) = \{a_\alpha \,|\, \forall a \in R\}$ be the family of all level α fuzzy points on R satisfying (2.7).

Definition 2.6. If the membership function of a fuzzy set $(a^{(1)},a^{(2)},\ldots,a^{(n)})_\alpha, 0 \leq \alpha \leq 1$; on R^n is

$$\begin{aligned} &\mu_{(a^{(1)},a^{(2)},\ldots,a^{(n)})_\alpha}(x^{(1)},x^{(2)},\ldots,x^{(n)}) \\ &= \begin{cases} \alpha, & \text{if } (x^{(1)},x^{(2)},\cdots,x^{(n)}) = (a^{(1)},a^{(2)},\ldots,a^{(n)}) \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \tag{2.8}$$

then we call $(a^{(1)},a^{(2)},\ldots,a^{(n)})_\alpha$, a level α fuzzy point on R^n.

Let $F_p^n(\alpha) = \{(a^{(1)},a^{(2)},\ldots,a^{(n)})_\alpha \,|\, \forall (a^{(1)},a^{(2)},\ldots,a^{(n)}) \in R^n\}$ be the family of all level α fuzzy points on R^n satisfying (2.8).

For every $a_\alpha \in F_p^1(\alpha)$, let $a_\alpha = (a, a, \ldots, a)_\alpha$, then a_α can be regarded as a special case of the level α fuzzy point $(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_\alpha$ degenerates to $a^{(1)} = a^{(2)} = \cdots = a^{(n)} = a$ Thus

$$\mu_{(a,a,\ldots,a)_\alpha}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = \begin{cases} \alpha, & (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (a, a, \ldots, a) \\ 0, & (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \neq (a, a, \ldots, a) \end{cases}$$
$$= \mu_{a_\alpha}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \tag{2.9}$$

Remark 2.7 We can regard a_α as a fuzzy set on R as the form in (2.7) or it can also be regarded as a fuzzy set on R^n, such as $a_\alpha = (a, a, \ldots, a)_\alpha$ in (2.9) according to how we want it to be. That is $0_1 = (0, 0, \ldots, 0)_1$, and $a_\alpha = (a, a, \ldots, a)_\alpha, \alpha \in [0, 1]$

From Kaufmann and Gupta [3], for $D, E \subset R^n, k \in R$, we have

$$D(+)E = \{(x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \ldots, x^{(n)} + y^{(n)}),$$
$$| \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D, (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in E\}$$

$$D(-)E = \{(x^{(1)} - y^{(1)}, x^{(2)} - y^{(2)}, \ldots, x^{(n)} - y^{(n)}),$$
$$| \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D, (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in E\}$$

$$k(\cdot)D = \{(kx^{(1)}, kx^{(2)}, \ldots, kx^{(n)}) | \forall (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in D\}$$

The $\alpha -$ cut of $\widetilde{D} \oplus \widetilde{E}$ is $D(\alpha)(+)E(\alpha)$.

The $\alpha -$ cut of $\widetilde{D} \ominus \widetilde{E}$ is $D(\alpha)(-)E(\alpha)$.

The $\alpha -$ cut of $k_1 \odot \widetilde{D}$ is $k(\cdot)D(\alpha)$. (2.10)

3. The extension of the crisp n-dimentional Euclidean vector space E^n to the pseudo fuzzy vector space SFR

In crisp case, for $P = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})$, $Q = (q^{(1)}, q^{(2)}, \ldots, q^{(n)})$, $A = (a^{(1)}, a^{(2)}, \ldots, a^{(n)})$, $B = (b^{(1)}, b^{(2)}, \ldots, b^{(n)}) \in R^n$, and $k \in R$, we can define the operations "$+, \cdot$" for the crisp vector $\overrightarrow{PQ}$, $\overrightarrow{AB}$ in E^n, the n-dimentional vector space over R^n, by :

$$\begin{aligned} \overrightarrow{AB} &= (b^{(1)} - a^{(1)}, b^{(2)} - a^{(2)}, \ldots, b^{(n)} - a^{(n)}) \\ \overrightarrow{PQ} &= (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) \end{aligned} \tag{3.1}$$

$$\begin{aligned} \overrightarrow{AB} + \overrightarrow{PQ} &= (b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)}, \ldots, \\ &\qquad b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)}) \\ k \cdot \overrightarrow{PQ} &= (kq^{(1)} - kp^{(1)}, kq^{(2)} - kp^{(2)}, \ldots, kq^{(n)} - kp^{(n)}) \end{aligned} \tag{3.2}$$

Let $O = (0, 0, \ldots, 0) \in R^n$, then $\overrightarrow{OP} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})$ and $\overrightarrow{OO} = (0, 0, \ldots, 0) \in E^n$.

Let E^n be an n-dimentional vector space over R. By Definition 2.6, $F_p^n(1) = \{(a^{(1)}, a^{(2)}, \cdots, a^{(n)})_1 \mid \forall\, (a^{(1)}, a^{(2)}, \cdots, a^{n)}) \in R^n\}$. This is a family of all level 1 fuzzy points on R^n.

We notice that there is an one-to-one onto mapping ρ between $(a^{(1)}, a^{(2)}, \ldots, a^{(n)}) \in R^n$ and $(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_1 \in F_p^n(1)$. i.e.

$$\rho : (a^{(1)}, a^{(2)}, \ldots, a^{(n)}) \in R^n \longleftrightarrow \rho((a^{(1)}, a^{(2)}, \ldots, a^{(n)})) = (a^{(1)}, a^{(2)}, \ldots, a^{(n)})_1 \in F_p^n(1) \tag{3.3}$$

$$\text{and } \mu_{(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_1}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = C_{(a^{(1)}, a^{(2)}, \ldots, a^{(n)})}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \tag{3.4}$$

where C_A is the characteristic function of A.

Let $\widetilde{P} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1$, $\widetilde{Q} = (q^{(1)}, q^{(2)}, \ldots, q^{(n)})_1 \in F_p^n(1)$. From (3.1), (3.3) we have the following definition:

$$\overrightarrow{\widetilde{P}\widetilde{Q}} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)})_1 = \widetilde{Q} \ominus \widetilde{P} \tag{3.5}$$

We call $\overrightarrow{\widetilde{P}\widetilde{Q}}$, a fuzzy vector.

Let $\widetilde{O} = (0, 0, \ldots, 0)_1 \in F_p^n(1)$, then $\overrightarrow{\widetilde{O}\widetilde{P}} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1$, $\overrightarrow{\widetilde{O}\widetilde{O}} = (0, 0, \ldots, 0)_1$.

Let $FE^n = \{\overrightarrow{\widetilde{P}\widetilde{Q}} \mid \forall \widetilde{P}, \widetilde{Q} \in F_p^n(1)\}$ be the family of all fuzzy vectors on $F_p^n(1)$. From (3.1), (3.5) we can have the one to one onto mapping ρ between E^n and FE^n by:

$$\begin{aligned} &\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) \quad (\in E^n) \\ &\longleftrightarrow \rho(\overrightarrow{PQ}) = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)})_1 = \overrightarrow{\widetilde{P}\widetilde{Q}} \quad \in FE^n \end{aligned} \tag{3.6}$$

Since $(p^{(1)}, p^{(2)}, \cdots, p^{(n)}) = \overrightarrow{OP}$, hence the point in R^n can be regarded as a vector in E^n. Also since $(p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1 = \overrightarrow{\widetilde{O}\widetilde{P}}$. hence the level 1 fuzzy points on R^n can be regarded as the fuzzy vectors in FE^n. Therefore the mapping in (3.3) is a special case of the mapping in (3.6).

The operations " $\oplus, \odot$ " of the fuzzy vectors in FE^n has the following property:

Property 3.1. For $\widetilde{P} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1$, $\widetilde{Q} = (q^{(1)}, q^{(2)}, \ldots, q^{(n)})_1$, $\widetilde{A} = (a^{(1)}, a^{(2)}, \ldots, a^{(n)})_1$, $\widetilde{B} = (b^{(1)}, b^{(2)}, \ldots, b^{(n)})_1 \in FE^n$ and $k \neq 0 \in \boldsymbol{R}$ we have

$$\begin{aligned} (1^\circ)\ &\overrightarrow{\widetilde{A}\widetilde{B}} \oplus \overrightarrow{\widetilde{P}\widetilde{Q}} \\ &= (b^{(1)} + q^{(1)} - a^{(1)} - p^{(1)}, b^{(2)} + q^{(2)} - a^{(2)} - p^{(2)}, \ldots, \\ &\quad b^{(n)} + q^{(n)} - a^{(n)} - p^{(n)})_1 \end{aligned} \tag{3.7}$$

$$(2^\circ)\ k_1 \odot \overrightarrow{\widetilde{P}\widetilde{Q}} = (kq^{(1)} - kp^{(1)}, kq^{(2)} - kp^{(2)}, \ldots, kq^{(n)} - kp^{(n)})_1 \tag{3.8}$$

Proof.

$$(1^\circ)\ \mu_{\underset{\sim}{\overrightarrow{AB}}\oplus\underset{\sim}{\overrightarrow{PQ}}}(z^{(1)},z^{(2)},\ldots,z^{(n)})$$

$$= \sup_{z^{(j)}=x^{(j)}+y^{(j)},\,j=1,2,\ldots,n} \{\mu_{(b^{(1)}-a^{(1)},b^{(2)}-a^{(2)},\ldots,b^{(n)}-a^{(n)})_1}(x^{(1)},x^{(2)},\ldots,x^{(n)})$$

$$\wedge\mu_{(q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\ldots,q^{(n)}-p^{(n)})_1}(y^{(1)},y^{(2)},\ldots,y^{(n)})\}$$

$$= \sup_{(y^{(1)},y^{(2)},\ldots,y^{(n)})} \{\mu_{(b^{(1)}-a^{(1)},b^{(2)}-a^{(2)},\ldots,b^{(n)}-a^{(n)})_1}(z^{(1)}-y^{(1)},z^{(2)}-y^{(2)},\ldots,$$

$$z^{(n)}-y^{(n)})$$

$$\wedge\mu_{(q^{(1)}-p^{(1)},q^{(2)}-p^{(2)},\ldots,q^{(n)}-p^{(n)})_1}(y^{(1)},y^{(2)},\ldots,y^{(n)})\}$$

$$= 1, \text{ if } z^{(j)}-y^{(j)}=b^{(j)}-a^{(j)}, y^{(j)}=q^{(j)}-p^{(j)}, j=1,2,\ldots,n$$

$$= 1, \text{ if } z^{(j)}-q^{(j)}+p^{(j)}=b^{(j)}-a^{(j)}, j=1,2,\cdots,n$$

$$= \mu_{(b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)},\ldots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)})}(z^{(1)},z^{(2)},\ldots,z^{(n)}),$$

$$\forall(z^{(1)},z^{(2)},\ldots,z^{(n)})\in R^n \tag{3.9}$$

i.e. $\overrightarrow{\widetilde{A}\widetilde{B}}\oplus\overrightarrow{\widetilde{P}\widetilde{Q}}$

$$= (b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)},\ldots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)})_1 \tag{3.10}$$

(2°) Similarly, we have (2°). In the case $k=0$, it follows by Property 3.7 (7°).

From Property 3.1, (3.2), (3.6), (3.7) - (3.8), we have

$$\rho(\overrightarrow{AB}+\overrightarrow{PQ})$$

$$= (b^{(1)}+q^{(1)}-a^{(1)}-p^{(1)},b^{(2)}+q^{(2)}-a^{(2)}-p^{(2)},\ldots,b^{(n)}+q^{(n)}-a^{(n)}-p^{(n)})_1$$

$$=\overrightarrow{\widetilde{A}\widetilde{B}}\oplus\overrightarrow{\widetilde{P}\widetilde{Q}}$$

$$= \rho(\overrightarrow{AB})\oplus\rho(\overrightarrow{PQ}) \tag{3.11}$$

$$\rho(k\cdot\overrightarrow{AB}) = k_1\odot\overrightarrow{\widetilde{A}\widetilde{B}} = \rho(k)\odot\rho(\overrightarrow{\widetilde{A}\widetilde{B}})$$

By Remark 2.7, $k=(k,k,\cdots,k)$. Hence by (3.3), $\rho(k)=\rho(k,k,\cdots,k)=$

$(k, k, \ldots, k)_1 = k_1$. From (3.6), (3.11), since E^n is a vector space over R, therefore FE^n satisfies the conditions to be a vector space too. We call FE^n, a fuzzy vector space over $F_p^1(1)$ and call $\overrightarrow{\widetilde{P}\widetilde{Q}}$ $(\in FE^n)$ a fuzzy vector.

Remark 3.2 The zero fuzzy vector $\overrightarrow{\widetilde{O}\widetilde{O}} = (0, 0, \cdots, 0)_1$ in FE^n will be obtained from the zero vector $\overrightarrow{OO} = (0, 0, \cdots, 0)$ in E^n by mapping $\overrightarrow{OO}$ to $\overrightarrow{\widetilde{O}\widetilde{O}}$

Obviously, there is an one-to-one mapping between R and $F_p^1(1)$ such that $a \in R \longleftrightarrow a_1 \in F_p^1(1)$. Thus, we have

Property 3.3 The fuzzy vector space FE^n over $F_p^1(1)$, is equivalent to the vector space E^n over R, denoted by $E^n \approx FE^n$.

Since the α-cut of the fuzzy point $\widetilde{P} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1$ in $F_p^n(1)$ is $(p^{(1)}, p^{(2)}, \cdots, p^{(n)})\, \forall \alpha \in [0, 1]$, hence it can be regarded as a special case in F_c, i.e. we can take $F_p^n(1)$ as a subfamily of F_c, i.e. $F_p^n(1) \subset F_c$. Therefore we can extend the fuzzy vector space FE^n to F_c. And have the following definition similarly as in (3.5).

Definition 3.4 For $\widetilde{X}, \widetilde{Y} \in F_c$, define

$$\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X} \tag{3.12}$$

We call $\overrightarrow{\widetilde{X}\widetilde{Y}}$, a fuzzy vector.

Let $SFR = \{\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X} \mid \forall \widetilde{X}, \widetilde{Y} \in F_c\}$.

Property 3.5 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$,

$$\overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{W}\widetilde{Z}} \quad \text{iff} \quad \widetilde{Y} \ominus \widetilde{X} = \widetilde{Z} \ominus \widetilde{W} \tag{3.13}$$

Proof. It follows from Definition 3.1 of fuzzy vector.

Property 3.6 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$, $k \in R$,

(1°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{A}\widetilde{B}}$, here $\widetilde{A} = \widetilde{X} \oplus \widetilde{W}, \widetilde{B} = \widetilde{Y} \oplus \widetilde{Z}$.

(2°) $k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{C}\widetilde{D}}$, here $\widetilde{C} = k_1 \odot \widetilde{X}, \widetilde{D} = k_1 \odot \widetilde{Y}$.

Proof.

(1°) For each $\alpha \in [0,1]$, from (i), (ii), (iv), (v) in (2.10), the α-cut of $\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X}$, $\overrightarrow{\widetilde{W}\widetilde{Z}} = \widetilde{Z} \ominus \widetilde{W}$ are $Y(\alpha)(-)X(\alpha), Z(\alpha)(-)W(\alpha)$ respectively. Let

$$\begin{aligned} D = &\{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha), \\ &(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in Z(\alpha), (w^{(1)}, w^{(2)}, \ldots, w^{(n)}) \in W(\alpha)\} \end{aligned} \tag{3.14}$$

Therefore the α-cut of $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}$ is

$$\begin{aligned} &(Y(\alpha)(-)X(\alpha))(+)(Z(\alpha)(-)W(\alpha)) \\ =&\{(y^{(1)} - x^{(1)} + z^{(1)} - w^{(1)}, y^{(2)} - x^{(2)} + z^{(2)} - w^{(2)}, \ldots \\ &y^{(n)} - x^{(n)} + z^{(n)} - w^{(n)}) \mid D\} \\ =&\{(y^{(1)} + z^{(1)}, y^{(2)} + z^{(2)}, \ldots, (y^{(n)} + z^{(n)}) \\ &\mid (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha), (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in Z(\alpha)\} \\ &(-)\{(x^{(1)} + w^{(1)}, x^{(2)} + w^{(2)}, \ldots, x^{(n)} + w^{(n)}) \\ &\mid (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (w^{(1)}, w^{(2)}, \ldots, w^{(n)}) \in W(\alpha)\} \end{aligned} \tag{3.15}$$

which is the α-cut of $(\widetilde{Y} \oplus \widetilde{Z}) \ominus (\widetilde{X} \oplus \widetilde{W}) = \overrightarrow{\widetilde{A}\widetilde{B}}$

(2°) Same way as in (1°).

Property 3.7 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}}, \overrightarrow{\widetilde{U}\widetilde{V}} \in SFR$, $k, t \in R$

(1°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{W}\widetilde{Z}} \oplus \overrightarrow{\widetilde{X}\widetilde{Y}}$

(2°) $(\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}) \oplus \overrightarrow{\widetilde{U}\widetilde{V}} = \overrightarrow{\widetilde{X}\widetilde{Y}} \oplus (\overrightarrow{\widetilde{W}\widetilde{Z}} \oplus \overrightarrow{\widetilde{U}\widetilde{V}})$

(3°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{O}\widetilde{O}} = \overrightarrow{\widetilde{X}\widetilde{Y}}$

(4°) $k_1 \odot (t_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) = (kt)_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}$

(5°) $k_1 \odot (\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}) = (k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \oplus (k_1 \odot \overrightarrow{\widetilde{W}\widetilde{X}})$

(6°) $1_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{X}\widetilde{Y}}$

(7°) $0_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = \overrightarrow{\widetilde{O}\widetilde{O}}$

Proof For each $\alpha \in [0,1]$, and from (iv), (v), and (vi) in (2.10),

(1°) The α-cut of $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = (\widetilde{Y} \ominus \widetilde{X}) \oplus (\widetilde{Z} - \widetilde{W})$ is

$$(Y(\alpha)(-)X(\alpha))(+)(Z(\alpha)(-)W(\alpha))$$

$$= (Z(\alpha)(-)W(\alpha)) + (Y(\alpha)(-)X(\alpha)) \tag{3.16}$$

which is the α-cut of $\overrightarrow{\widetilde{W}\widetilde{Z}} \oplus \overrightarrow{\widetilde{X}\widetilde{Y}}$. Therefore (1°) holds.

(2°) The proof is similar to (1°).

(3°) Since $\overrightarrow{\widetilde{O}\widetilde{O}} = (0,0,\cdots,0)_1$, the α-cut of $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{O}\widetilde{O}} = (\widetilde{Y} \ominus \widetilde{X}) \oplus (0,0,\ldots,0)_1$ is $(Y(\alpha)(-)X(\alpha))(+)(0,0,\ldots,0) = Y(\alpha)(-)X(\alpha)$ which is the α-cut of $\overrightarrow{\widetilde{X}\widetilde{Y}}$. Therefore $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{O}\widetilde{O}} = \overrightarrow{\widetilde{X}\widetilde{Y}}$

(4°) For each $\alpha \in [0,1]$, the α-cut of $k_1 \odot (t_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) = k_1 \odot (t_1 \odot (\widetilde{Y} \ominus \widetilde{X}))$ is $k(\cdot)(t(\cdot)(Y(\alpha)(-)X(\alpha))) = (kt)(\cdot)(Y(\alpha)(-)X(\alpha))$ which is the α-cut of $(kt)_1(\cdot)\ \overrightarrow{\widetilde{X}\widetilde{Y}}$

(5°) Similar as (4°)

(6°) Similar as (5°)

(7°) From (v), (vi), for each $\alpha \in [0,1]$, the α - cut of $0_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}$ is $(0,0,\ldots,0)$ which

is the α - cut of $\overrightarrow{\widetilde{O}\widetilde{O}}$

In order to be a fuzzy vector space, it needs the followings (8°), (9°) hold:

(8°) For any $\overrightarrow{\widetilde{X}\widetilde{Y}} \in SFR$, there exists $\overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$, such that $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{O}\widetilde{O}}$ (3.17)

(9°) $(m+n)_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = (m_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \oplus (n_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \quad \forall \overrightarrow{\widetilde{X}\widetilde{Y}} \in SFR$ (3.18)

Now since FE^n is a vector space, (3.17), (3.18) hold without any question. If $\widetilde{X}, \widetilde{Y} \in F_c$, but $\notin F_p^n(1)$ and $\overrightarrow{\widetilde{X}\widetilde{Y}} \neq \overrightarrow{\widetilde{O}\widetilde{O}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \neq \overrightarrow{\widetilde{O}\widetilde{O}}$. By (iv), (v), for each $\alpha \in [0,1]$ the α - cuts of $\overrightarrow{\widetilde{X}\widetilde{Y}}$ and $\overrightarrow{\widetilde{W}\widetilde{Z}}$ are $Y(\alpha)(-)X(\alpha) \neq (0,0,\ldots,0), Z(\alpha)(-)W(\alpha) \neq (0,0,\ldots,0)$ respectively. The α - cut of $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = (\widetilde{Y} \ominus \widetilde{X}) \oplus (\widetilde{Z} \ominus \widetilde{W})$ is

$$(Y(\alpha) - X(\alpha))(+)(Z(\alpha)(-)W(\alpha)) \neq (0,0,\ldots,0) \tag{3.19}$$

i.e. there exists no $\overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR$ such that $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{O}\widetilde{O}}$. i.e.(3.17) does not hold in this case.

For $\widetilde{X}, \widetilde{Y} \in F_c$ but $\notin F_p^n(1)$, from (i), (ii), (iii), (iv), (v) and (vi) in (2.10), the α - cut $(0 \leq \alpha \leq 1)$ of $(m+n)_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}$

$= (m+n)_1 \odot (\widetilde{Y} \ominus \widetilde{X})$ is

$$((m+n)Y(\alpha))(-)((m+n)X(\alpha))$$

$$= \{((m+n)y^{(1)} - (m+n)x^{(1)}, (m+n)y^{(2)} - (m+n)x^{(2)}, \ldots,$$

$$(m+n)y^{(n)} - (m+n)x^{(n)}) \mid (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha),$$

$$(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha)\}. \tag{3.20}$$

However, the α - cut of $(m_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \oplus (n_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) = (m_1 \odot (\widetilde{Y} \ominus \widetilde{X})) \oplus (n_1 \odot (\widetilde{Y} \ominus \widetilde{X}))$ is

$m(Y(\alpha)(-)X(\alpha))(+)(n(Y(\alpha)(-)X(\alpha))$

$$= \{ (m(y^{(1)'} - x^{(1)'}) + n(y^{(1)''} - x^{(1)''}), m(y^{(2)'} - x^{(2)'}) + n(y^{(2)''} - x^{(2)''}), \ldots$$

$$m(y^{(n)'} - x^{(n)'}) + n(y^{(n)''} - x^{(n)''}))$$

$$| (x^{(1)'}, x^{(2)'}, \ldots, x^{(n)'}), (x^{(1)''}, x^{(2)''}, \ldots, x^{(n)''})$$

$$\in X(\alpha), (y^{(1)'}, y^{(2)'}, \ldots, y^{(n)'}), (y^{(1)''}, y^{(2)''}, \ldots, y^{(n)''}) \in Y(\alpha) \} \qquad (3.21)$$

Therefore $(m+n)y^{(j)} - (m+n)x^{(j)} \neq m(y^{(j)'} - x^{(j)'}) + n(y^{(j)''} - x^{(j)''})$ if ($x^{(j)} \neq x^{(j)'}$ or $x^{(j)''}$) or ($y^{(j)} \neq y^{(j)'}$ or $y^{(j)''}$). Hence (3.18) does not hold.

Definition 3.8. The SFR which satisfies Property 3.7 (1°) – (7°) is called pseudo fuzzy vector space over $F_p^1(1)$, and call $\overrightarrow{\widetilde{X}\widetilde{Y}}$ ($\in SFR$), a fuzzy vector.

Then we have $E^n \approx FE^n \subset SFR$. i.e., we can regard SFR as an extension of E^n, but only obtain a pseudo fuzzy vector space in stead of a fuzzy vector space. Its addition $\oplus$ and multiplication $\odot$ are followed by Property 3.4.

Property 3.9 For $\overrightarrow{\widetilde{X}_j\widetilde{Y}_j} \in SFR$, $a_1^{(j)} \in F_p^1(1)$, $j = 1, 2, \cdots, m$

$$(a_1^{(1)} \odot \overrightarrow{\widetilde{X}_1\widetilde{Y}_1}) \oplus (a_1^{(2)} \odot \overrightarrow{\widetilde{X}_2\widetilde{Y}_2}) \oplus \cdots \oplus (a_1^{(m)} \odot \overrightarrow{\widetilde{X}_m\widetilde{Y}_m}) = \overrightarrow{\widetilde{A}\widetilde{B}} \qquad (3.22)$$

here $\widetilde{A} = \widetilde{C}_1 \oplus \widetilde{C}_2 \oplus \cdots \oplus \widetilde{C}_m$, $\widetilde{B} = \widetilde{D}_1 \oplus \widetilde{D}_2 \oplus \cdots \oplus \widetilde{D}_m$, and $\widetilde{C}_j = a_1^{(j)} \odot \widetilde{X}_j$, $\widetilde{D}_j = a_1^{(j)} \odot \widetilde{Y}_j$, $j = 1, 2, \ldots, m$

Proof. It follows from Property 3.6 (1°), (2°) and mathematical induction.

Property 3.10 For $\widetilde{Y} \in F_c$, but $\widetilde{Y} \notin F_p^n(1)$, $\widetilde{Y} \neq \widetilde{O}$, and $\widetilde{X} \in F_c$

(1°) $\widetilde{Y} \ominus \widetilde{Y} \neq \widetilde{O}$

(2°) $\overrightarrow{\widetilde{Y}\widetilde{Y}} \neq \overrightarrow{\widetilde{O}\widetilde{O}}$

(3°) $\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X} \neq \overrightarrow{\widetilde{O}\widetilde{O}}$

(4°) $\overrightarrow{\widetilde{Y}\widetilde{X}} = \widetilde{X} \ominus \widetilde{Y} \neq \overrightarrow{\widetilde{O}\widetilde{O}}$

Proof.

(1°) Since $\widetilde{Y} \neq \widetilde{O}$, the α-cut of $\widetilde{Y}$ is $Y(\alpha) \neq (0, 0, \cdots, 0)\, \forall \alpha \in [0, 1]$. By (ii), (v) in (2.10), the α-cut of $\widetilde{Y} \ominus \widetilde{Y}$ is

$$Y(\alpha)(-)Y(\alpha) = ((s^{(1)} - t^{(1)}, s^{(2)} - t^{(2)}, \ldots, s^{(n)} - t^{(n)}) \mid (s^{(1)}, s^{(2)}, \ldots, s^{(n)}), (t^{(1)}, t^{(2)}, \ldots, t^{(n)}) \in Y(\alpha)) \neq (0, 0, \ldots, 0) \tag{3.23}$$

Therefore $\widetilde{Y} \ominus \widetilde{Y} \neq \widetilde{O}$.

(2°) Follows by (1°).

(3°) Similar as (1°).

(4°) Similar as (1°).

Remark 11.

(a) If $\widetilde{Y} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1 \in F_p^n(1)$,
then $\widetilde{Y} \ominus \widetilde{Y} = (p^{(1)} - p^{(1)}, p^{(2)} - p^{(2)}, \ldots, p^{(n)} - p^{(n)})_1 = (0, 0, \ldots, 0)_1$.
Hence $\overrightarrow{\widetilde{Y}\widetilde{Y}} = \overrightarrow{\widetilde{O}\widetilde{O}}$.

(b) It is trivial that $\overrightarrow{\widetilde{O}\widetilde{O}} \oplus \overrightarrow{\widetilde{O}\widetilde{O}} = \overrightarrow{\widetilde{O}\widetilde{O}}$.

(c) Let $SFV = \{\overrightarrow{\widetilde{P}\widetilde{X}} \mid \forall \widetilde{P} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1, \in F_p^n(1), \widetilde{X} \in F_c\}$, then $E^n \approx FE^n \subset SFV \subset SFR$.

Property 3.12 For $\widetilde{X} \in F_c$

(1°) $0_1 \odot \widetilde{X} = \widetilde{O}$

(2°) $\widetilde{O} \oplus \widetilde{X} = \widetilde{X}$

Proof. It is obvious.

Example 3.13. $n = 2$. A car carrying a rocket departs from point $Q = (1, 2)$ passes through point $S = (5, 8)$, arrives at point $W = (10, 15)$, and launches the rocket from there. Suppose its target is located at $Z = (100, 200)$. Chances are the rocket will not hit at Z exactly. Instead it would probably drop in the vicinity of Z. Let

$$O((100, 200), 1) = \{(x, y) | (x - 100)^2 + (y - 200)^2 \leq 1\} \tag{3.24}$$

and the point hit is $\widetilde{Z}$, $(\widetilde{Z} \in F_c)$ with membership function

$$\mu_{\widetilde{Z}}(x, y) = \begin{cases} 1 - (x - 100)^2 - (y - 200)^2, & \text{if } (x, y) \in O((100, 200), 1) \\ 0, & \text{if } (x, y) \notin O((100, 200), 1) \end{cases} \tag{3.25}$$

The α - cut $(0 \leq \alpha \leq 1)$ of $\widetilde{Z}$ is

$$Z(\alpha) = \{(x, y) \,|\, \mu_{\widetilde{Z}}(x, y) \geq \alpha\} = \{(x, y) \,|\, (x - 100)^2 + (y - 200)^2 \leq 1 - \alpha\} \tag{3.26}$$

If we choose the route from base to target to be $Q \to S \to W \to Z$, then we have the crisp vectors $\overrightarrow{QS}$= (4, 6), $\overrightarrow{SW}$= (5, 7), $\overrightarrow{WZ}$= (90, 185). So the crisp vector from Q to Z is $\overrightarrow{QS} + \overrightarrow{SW} + \overrightarrow{WZ}$= (99, 198). And the route in the fuzzy sense is

$$\widetilde{Q} = (1, 2)_1 \to \widetilde{S} = (5, 8)_1 \to \widetilde{W} = (10, 15)_1 \to \widetilde{Z} \tag{3.27}$$

with (3.25) as the membership function of $\widetilde{Z}$. We then have fuzzy vectors $\overrightarrow{\widetilde{Q}\widetilde{S}}$= $(4, 6)_1$, $\overrightarrow{\widetilde{S}\widetilde{W}}$= $(5, 7)_1$ and $\overrightarrow{\widetilde{W}\widetilde{Z}}$= $\widetilde{Z} \ominus \widetilde{W}$. From (3.25), the membership function of $\overrightarrow{\widetilde{W}\widetilde{Z}}$ is

$$\begin{aligned} &\mu_{\overrightarrow{\widetilde{W}\widetilde{Z}}}(x, y) = \mu_{\widetilde{Z}}(x + 10, y + 15) \\ &= \begin{cases} 1 - (x - 90)^2 - (y - 185)^2, & \text{if } (x - 90)^2 + (y - 185)^2 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \tag{3.28}$$

The fuzzy vector from base $\widetilde{Q}$ to target $\widetilde{Z}$ by Property 3.4 (1°) is $\overrightarrow{\widetilde{Q}\widetilde{S}} \oplus \overrightarrow{\widetilde{S}\widetilde{W}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{Q}\widetilde{Z}}$, with membership function

$$\begin{aligned} &\mu_{\overrightarrow{\widetilde{Q}\widetilde{Z}}}(x, y) = \mu_{\widetilde{Z}}(x + 1, y + 2) \\ &= \begin{cases} 1 - (x - 99)^2 - (y - 198)^2, & \text{if } (x - 99)^2 + (y - 198)^2 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \tag{3.29}$$

Let $Z = (100, 200)$, $U = (99.5, 200.5)$, $V = (100.2, 199.7) \in O((100, 200), 1)$.

Insert Fig. 2 here

As shown in Fig. 2, the crisp vectors from Q to Z, U, V in $O((100, 200), 1)$ are $\overrightarrow{QZ}$= (99, 198), $\overrightarrow{QU}$= (98.5, 198.5), $\overrightarrow{QV}$= (99.2, 197.7) respectively. The grades of membership of these crisp vectors belong to fuzzy vector $\overrightarrow{\widetilde{Q}\widetilde{Z}}$ are

$\mu_{\underset{\widetilde{Q}\widetilde{Z}}{\rightarrow}}(\overrightarrow{QZ}) = \mu_{\underset{\widetilde{Q}\widetilde{Z}}{\rightarrow}}(99, 198) = 1$, $\mu_{\underset{\widetilde{Q}\widetilde{Z}}{\rightarrow}}(\overrightarrow{QU}) = \mu_{\underset{\widetilde{Q}\widetilde{Z}}{\rightarrow}}(98.5, 198.5) = 0.5$,

$\mu_{\underset{\widetilde{Q}\widetilde{Z}}{\rightarrow}}(\overrightarrow{QV}) = \mu_{\underset{\widetilde{Q}\widetilde{Z}}{\rightarrow}}(99.2, 197.7) = 0.87$

4.The length of fuzzy vectors in SFR and fuzzy inner product.

[I] The length of fuzzy vectors in SFR

Let $P = (p^{(1)}, p^{(2)}, \ldots, p^{(n)}), Q = (q^{(1)}, q^{(2)}, \ldots, q^{(n)}) \in R^n$. The vector $\overrightarrow{PQ}$ in E^n, $\overrightarrow{PQ}$= $(q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) = Q(-)P$ has length $\mid \overrightarrow{PQ} \mid = \sqrt{\sum_{j=1}^{n}(q^{(j)} - p^{(j)})^2}$. Called this, the length of the vector $\overrightarrow{PQ}$. Now let $\widetilde{P} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)})_1$ and $\widetilde{Q} = (q^{(1)}, q^{(2)}, \ldots, q^{(n)})_1 \in F_p^n(1)$. Since $FE^n \approx E^n$, we can define the length of fuzzy vector $\overrightarrow{\widetilde{P}\widetilde{Q}}$= $(q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)})_1$ by $\mid \overrightarrow{\widetilde{P}\widetilde{Q}} \mid = \mid \overrightarrow{PQ} \mid = \sqrt{\sum_{j=1}^{n}(q^{(j)} - p^{(j)})^2}$.

Since $FE^n \subset SFR$, we may extend this thought to SFR. Similar to the fuzzy vectors in FE^n, for the vector $\overrightarrow{\widetilde{X}\widetilde{Y}} = \widetilde{Y} \ominus \widetilde{X} \in SFR$, its α-cut $(0 \leq \alpha \leq 1)$ is

$Y(\alpha)(-)X(\alpha)$

$= \{(y^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \ldots, y^{(n)} - x^{(n)})$

$\mid (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha)\}$ (4.1)

where $X(\alpha), Y(\alpha)$ are the α -cuts of $\widetilde{X}, \widetilde{Y}$, respectively. For each point $P_X(\alpha) = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha)$, and $P_Y(\alpha) = (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha)$, the crisp

vector $\overrightarrow{P_X(\alpha)P_Y(\alpha)}= (y^{(1)} - x^{(1)}, y^{(2)} - x^{(2)}, \ldots, y^{(n)} - x^{(n)})$ has length

$$| \overrightarrow{P_X(\alpha)P_Y(\alpha)} | = \sqrt{\sum_{j=1}^{n}(y^{(j)} - x^{(j)})^2} \tag{4.2}$$

which is the distance between two points $P_X(\alpha)$ and $P_Y(\alpha)$. For any $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha)$, denoted simply by $(x^{(j)}) \in X(\alpha)$, let

$$d^*(Y(\alpha)(-)X(\alpha)) = \sup_{(x^{(j)})\in X(\alpha),(y^{(j)})\in Y(\alpha)} | \overrightarrow{P_X(\alpha)P_Y(\alpha)} |$$

$$= \sup_{(x^{(j)})\in X(\alpha),(y^{(j)})\in Y(\alpha)} \sqrt{\sum_{j=1}^{n}(y^{(j)} - x^{(j)})^2} \tag{4.3}$$

Since $\widetilde{X}, \widetilde{Y} \in F_c$, and by Definition 2.3(a), $X(\alpha), Y(\alpha)$ are convex closed subsets of R^n, so $d^*(Y(\alpha)(-)X(\alpha))$ exists. And $d^*(Y(\alpha)(-)X(\alpha))$ is the longest one among all the distances between points $P_X(\alpha)$ in $X(\alpha)$ and $P_Y(\alpha)$ in $Y(\alpha)$, which makes sense for using this as the distance between $X(\alpha)$ and $Y_(\alpha)$. Therefore, we have the following definition.

Definition 4.1 For $\overrightarrow{\widetilde{X}\widetilde{Y}} \in SFR$, define the length of $\overrightarrow{\widetilde{X}\widetilde{Y}}$ to be

$$| \overrightarrow{\widetilde{X}\widetilde{Y}} |^* = \sup_{0\le\alpha\le 1} d^*(Y(\alpha)(-)X(\alpha)) \tag{4.4}$$

Property 4.2 For $\overrightarrow{\widetilde{X}\widetilde{Y}} \in SFR$, let the 0-cuts ($\alpha$-cuts, $\alpha = 0$) of $\widetilde{X}\widetilde{Y}$ be $X(0), Y(0)$ by Remark 2.4. Then there exist $(x_m^{(1)}(0), x_m^{(2)}(0), \ldots, x_m^{(n)}(0)) \in X(0), (y_m^{(1)}(0), y_m^{(2)}(0), \ldots, y_m^{(n)}(0)) \in Y(0)$ such that

$$| \overrightarrow{\widetilde{X}\widetilde{Y}} |^* = \sup_{0\le\alpha\le 1} d^*(Y(\alpha)(-)X(\alpha)) = \sqrt{\sum_{j=1}^{n}(y_m^{(j)}(0) - x_m^{(j)}(0))^2} \tag{4.5}$$

Proof. Let the α-cuts $(0 \leq \alpha \leq 1)$ of $\widetilde{X}, \widetilde{Y}$ be

$X(\alpha) = \{(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) | \mu_{\widetilde{X}}(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \geq \alpha\}$ and

$Y(\alpha) = \{(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) | \mu_{\widetilde{Y}}(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \geq \alpha\}$. (4.6) It is obvious that $X(\alpha) \subset X(\beta), Y(\alpha) \subset Y(\beta)$ if $0 \leq \beta \leq \alpha \leq 1$.

$$\text{Since} \qquad d^*(Y(\alpha)(-)X(\alpha)) = \sup_{(x^{(j)}) \in X(\alpha), (y^{(j)}) \in Y(\alpha)} \sqrt{\sum_{j=1}^{n} (y^{(j)} - x^{(j)})^2} \tag{4.7}$$

$$\text{we have} \qquad d^*(Y(\alpha)(-)X(\alpha)) \leq d^*(Y(\beta)(-)X(\beta)) \qquad \forall\, 0 \leq \beta \leq \alpha \leq 1 \tag{4.8}$$

So

$$\sup_{0 \leq \alpha \leq 1} d^*(Y(\alpha)(-)X(\alpha)) = \sup_{(x^{(j)}(0)) \in X(0), (y^{(j)}(0)) \in Y(0)} \sqrt{\sum_{j=1}^{n} (y^{(j)}(0) - x^{(j)}(0))^2} \tag{4.9}$$

Since $\widetilde{X}, \widetilde{Y} \in F_c$, by the definition of F_c, we know that $X(0), Y(0)$ are convex closed subsets of R^n. Hence there exist $(x_m^{(1)}(0), x_m^{(2)}(0), \ldots, x_m^{(n)}(0)) \in X(0), (y_m^{(1)}(0), y_m^{(2)}(0), \ldots, y_m^{(n)}(0)) \in Y(0)$ such that

$$| \overrightarrow{\widetilde{X}\widetilde{Y}} |^* = \sup_{0 \leq \alpha \leq 1} d^*(Y(\alpha)(-)X(\alpha)) = \sqrt{\sum_{j=1}^{n} (y_m^{(j)}(0) - x_m^{(j)}(0))^2} \tag{4.10}$$

Example 4.3. In Example 1, the rocket eject at W takes the route $Q = (1, 2) \longrightarrow S = (5, 8) \longrightarrow W = (10, 15)$, aimes at $Z = (100, 200)$. The membership function of fuzzy target $\widetilde{Z}$ is

$$\mu_{\widetilde{Z}}(x, y) = \begin{cases} 1 - (x - 100)^2 - (y - 200)^2, & \text{if } (x - 100)^2 + (y - 200)^2 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \tag{4.11}$$

We obtain fuzzy vectors $\overrightarrow{\widetilde{Q}\widetilde{S}}= (4,6)_1, \overrightarrow{\widetilde{S}\widetilde{W}}= (5,7)_1, \overrightarrow{\widetilde{W}\widetilde{Z}}$ and $\overrightarrow{\widetilde{Q}\widetilde{Z}}$. The former two have lengths $|\overrightarrow{\widetilde{Q}\widetilde{S}}|^* = \sqrt{4^2+6^2} = 7.21$, and $|\overrightarrow{\widetilde{S}\widetilde{W}}|^* = \sqrt{5^2+7^2} = 8.6$ respectively. As to the length of $\overrightarrow{\widetilde{W}\widetilde{Z}}$, since for each $\alpha \in [0,1]$, the α - cuts of $\widetilde{W}, \widetilde{Z}$ are $W(\alpha) = (10,15)$, $Z(\alpha) = \{(x,y)|\,(x-100)^2+(y-200)^2 \leq 1-\alpha\}$. respectively.The longest distance between points $P_{W(\alpha)} = (10,15) \in W(\alpha)$ and $P_{Z(\alpha)} = (x,y) \in Z(\alpha)$ is

$$\begin{aligned} d^*(Z(\alpha)(-)W(\alpha)) &= \sqrt{(100-10)^2+(200-15)^2} + \sqrt{1-\alpha} \\ &= \sqrt{42325} + \sqrt{1-\alpha} = 205.73 + \sqrt{1-\alpha} \end{aligned} \tag{4.12}$$

Hence by Definition 4.1 $|\overrightarrow{\widetilde{W}\widetilde{Z}}|^* = \sup\limits_{0\leq\alpha\leq 1} d^*(Z(\alpha)(-)W(\alpha)) = 206.73$

Similarly, we can calculate the length of $\overrightarrow{\widetilde{Q}\widetilde{Z}}$: For each $\alpha \in [0,1]$, the α - cut of $\widetilde{Q}$ is $Q(\alpha) = (1,2)$, so

$$\begin{aligned} d^*(Z(\alpha)(-)Q(\alpha)) &= \sqrt{(100-1)^2+(200-2)^2} + \sqrt{1-\alpha} \\ &= \sqrt{49005} + \sqrt{1-\alpha} = 221.37 + \sqrt{1-\alpha} \end{aligned} \tag{4.13}$$

Therefore $|\overrightarrow{\widetilde{Q}\widetilde{Z}}|^* = \sup\limits_{0\leq\alpha\leq 1} d^*(Z(\alpha)(-)Q(\alpha)) = 222.37$

Remark 4.4. By Cauchy-Schwartz inequality

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \tag{4.14}$$

we have

$$\sum_{j=1}^{n} a_jb_j \leq |\sum_{j=1}^{n} a_jb_j| \leq \sqrt{\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2}. \tag{4.15}$$

Therefore

$$\sum_{j=1}^{n} a_j^2 + \sum_{j=1}^{n} b_j^2 + 2\sum_{j=1}^{n} a_jb_j \leq \sum_{j=1}^{n} a_j^2 + \sum_{j=1}^{n} b_j^2 + 2\sqrt{\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2} \tag{4.16}$$

i.e.

$$\sqrt{\sum_{j=1}^{n}(a_j+b_j)^2} \le \sqrt{\sum_{j=1}^{n}a_j^2} + \sqrt{\sum_{j=1}^{n}b_j^2} \tag{4.17}$$

Property 4.5 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{U}\widetilde{V}} \in SFR, k_1 \in F_p^1(1), k \neq 0$

(1°) $|k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}|^* = |k| \cdot |\overrightarrow{\widetilde{X}\widetilde{Y}}|^*$

(2°) $|\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}|^* \le |\overrightarrow{\widetilde{X}\widetilde{Y}}|^* + |\overrightarrow{\widetilde{W}\widetilde{Z}}|^*$

Proof

(1°) For each $\alpha \in [0,1]$, the α-cut of $k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} = k_1 \odot (\widetilde{Y}(-)\widetilde{X})$ is

$$\begin{aligned} & k(\cdot)(Y(\alpha)(-)X(\alpha)) \\ =&\{(ky^{(1)} - kx^{(1)}, ky^{(2)} - kx^{(2)}, \ldots, ky^{(n)} - kx^{(n)}) \\ & |(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha),\ (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha)\} \end{aligned} \tag{4.18}$$

For each $\alpha \in [0,1]$, $d^*(k(\cdot)(Y(\alpha)(-)X(\alpha)))$

$$= \sup_{(x^{(j)})\in X(\alpha),(y^{(j)})\in Y(\alpha)} \sqrt{\textstyle\sum_{j=1}^{n}(ky^{(j)} - kx^{(j)})^2} = |k| d^*(Y(\alpha) - X(\alpha)) \tag{4.19}$$

Therefore

$$\begin{aligned} |k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}|^* &= \sup_{0\le\alpha\le 1} d^*(k(\cdot)(Y(\alpha)(-)X(\alpha))) \\ &= |k| \sup_{0\le\alpha\le 1} d^*(Y(\alpha)(-)X(\alpha)) = |k| \cdot |\overrightarrow{\widetilde{X}\widetilde{Y}}|^* \end{aligned} \tag{4.20}$$

(2°) From Property 3.6 (1°), $\overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{A}\widetilde{B}} = \widetilde{B} \ominus \widetilde{A}$, where $\widetilde{A} = \widetilde{X} \oplus \widetilde{W}$, $\widetilde{B} = \widetilde{Y} \oplus \widetilde{Z}$. For each $\alpha \in [0,1]$, the α-cuts of $\widetilde{X}, \widetilde{Y}, \widetilde{W}, \widetilde{Z}$ are $X(\alpha), Y(\alpha), W(\alpha), Z(\alpha)$, respectively, and the α-cut of $\widetilde{B} \ominus \widetilde{A}$ is $(Y(\alpha)(+)Z(\alpha))(-)(X(\alpha)(+)W(\alpha))$

$= \{(y^{(1)} + z^{(1)} - x^{(1)} - w^{(1)},\ y^{(2)} + z^{(2)} - x^{(2)} - w^{(2)}, \ldots,$

$y^{(n)} + z^{(n)} - x^{(n)} - w^{(n)})$

$|\ (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(2)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha),$

$$(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in Z(\alpha), (w^{(1)}, w^{(2)}, \ldots, w^{(n)}) \in W(\alpha)\}. \tag{4.21}$$

For each $\alpha \in [0,1]$, let

$$\begin{aligned} D = \{&(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha), \\ &(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in Z(\alpha), (w^{(1)}, w^{(2)}, \ldots, w^{(n)}) \in W(\alpha)\} \end{aligned} \tag{4.22}$$

From Remark 4.4,

$$\sqrt{\sum_{j=1}^{n}(a_j + b_j)^2} \leq \sqrt{\sum_{j=1}^{n} a_j^2} + \sqrt{\sum_{j=1}^{n} b_j^2} \tag{4.23}$$

and inequality $\sup(A+B) \leq \sup A + \sup B$, we have $d^*((Y(\alpha)(+)Z(\alpha))(-)(X(\alpha)(+)W(\alpha)))$

$= \sup_D \{\sum_{j=1}^{n}(y^{(j)} + z^{(j)} - x^{(j)} - w^{(j)})^2\}^{\frac{1}{2}}$

$\leq \sup_{(x^{(j)}) \in X(\alpha), (y^{(j)}) \in Y(\alpha)} \sqrt{\sum_{j=1}^{n}(y^{(j)} - x^{(j)})^2}$

$+ \sup_{(w^{(j)}) \in W(\alpha), (z^{(j)}) \in Z(\alpha)} \sqrt{\sum_{j=1}^{n}(z^{(j)} - w^{(j)})^2}$

$= d^*(Y(\alpha)(-)X(\alpha)) + d^*(Z(\alpha)(-)W(\alpha)).$ (4.24)

By Definition 4.1, we have $| \overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} |^* \leq | \overrightarrow{\widetilde{X}\widetilde{Y}} |^* + | \overrightarrow{\widetilde{W}\widetilde{Z}} |^*$

4.2 The fuzzy inner product and the angle between fuzzy vectors for the fuzzy vectors in SFR.

Corresponding to the equation

$$\begin{aligned} &d'(Y(\alpha)(-)X(\alpha), V(\alpha)(-)U(\alpha)) \\ &= \sup_{(x^{(j)}) \in X(\alpha), (y^{(j)}) \in Y(\alpha), (u^{(j)}) \in U(\alpha), (v^{(j)}) \in V(\alpha)} \sum_{j=1}^{n}(y^{(j)} - x^{(j)})(v^{(j)} - u^{(j)}) \end{aligned}$$

We define the fuzzy inner product as follows:

Definition 4.6 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{U}\widetilde{V}} \in SFR$, define the fuzzy inner product of them to be

$$\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}} = \sup_{0\le\alpha\le 1} d'(Y(\alpha)(-)X(\alpha), V(\alpha)(-)U(\alpha)) \tag{4.25}$$

Property 4.7 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{U}\widetilde{V}} \in SFR$, let the 0-cuts ($\alpha$-cuts, $\alpha = 0$) of $\widetilde{X}, \widetilde{Y}, \widetilde{U}, \widetilde{V}$ be $X(0), Y(0), U(0), V(0)$, respectively. Then there exist $(x_m^{(1)}(0), x_m^{(2)}(0), \ldots, x_m^{(n)}(0)) \in X(0)$,

$(y_m^{(1)}(0), y_m^{(2)}(0), \ldots, y_m^{(n)}(0)) \in Y(0)$, $(u_m^{(1)}(0), u_m^{(2)}(0), \ldots, u_m^{(n)}(0)) \in U(0)$,

$(v_m^{(1)}(0), v_m^{(2)}(0), \ldots, v_m^{(n)}(0)) \in V(0)$ (4.26)

such that

$$\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}} = \sum_{j=1}^{n} (y_m^{(j)}(0) - x_m^{(j)}(0))(v_m^{(j)}(0) - u_m^{(j)}(0)) \tag{4.27}$$

Proof Use the same way of the proof of Property 4.1 to prove it.

Property 4.8 For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{U}\widetilde{V}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR, k_1 \in F_p^1(1), k > 0$,

(1°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}} = \overrightarrow{\widetilde{U}\widetilde{V}} \odot^* \overrightarrow{\widetilde{X}\widetilde{Y}}$.

(2°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* (\overrightarrow{\widetilde{U}\widetilde{V}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}) \le (\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}}) \oplus (\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{W}\widetilde{Z}})$.

(3°) $k(\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}}) = (k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \odot^* \overrightarrow{\widetilde{U}\widetilde{V}} = \overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* (k_1 \odot \overrightarrow{\widetilde{U}\widetilde{V}})$.

(4°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{X}\widetilde{Y}} = |\overrightarrow{\widetilde{X}\widetilde{Y}}|^{*2}$.

(5°) $|\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}}| \le |\overrightarrow{\widetilde{X}\widetilde{Y}}|^* \cdot |\overrightarrow{\widetilde{U}\widetilde{V}}|^*$.

Proof

(1°) By Property 4.7,

$$\begin{aligned} \overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}} &= \sum_{j=1}^{n} (y_m^{(j)}(0) - x_m^{(j)}(0))(v_m^{(j)}(0) - u_m^{(j)}(0)) \\ &= \sum_{j=1}^{n} (v_m^{(j)}(0) - u_m^{(j)}(0))(y_m^{(j)}(0) - x_m^{(j)}(0)) = \overrightarrow{\widetilde{U}\widetilde{V}} \odot^* \overrightarrow{\widetilde{X}\widetilde{Y}} \end{aligned} \tag{4.28}$$

(2°) By Definition 4.6 and Property 3.6(1°), $\overrightarrow{\widetilde{U}\widetilde{V}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} = \overrightarrow{\widetilde{A}\widetilde{B}} = \widetilde{B} \ominus \widetilde{A}$, where $\widetilde{A} = \widetilde{U} \oplus \widetilde{W}$, $\widetilde{B} = \widetilde{V} \oplus \widetilde{Z}$. The α-cut of $\widetilde{B} \ominus \widetilde{A}$ is $(V(\alpha)(+)Z(\alpha))(-)(U(\alpha)(+)W(\alpha))$. Hence for each $\alpha \in [0,1]$, set

$$
\begin{aligned}
H = \{&(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha), \\
&(u^{(1)}, u^{(2)}, \ldots, u^{(n)}) \in U(\alpha), (v^{(1)}, v^{(2)}, \ldots, v^{(n)}) \in V(\alpha), \\
&(w^{(1)}, w^{(2)}, \ldots, w^{(n)}) \in W(\alpha), (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in Z(\alpha)\} \\
E = \{&(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha), \\
&(u^{(1)}, u^{(2)}, \ldots, u^{(n)}) \in U(\alpha), (v^{(1)}, v^{(2)}, \ldots, v^{(n)}) \in V(\alpha)\} \\
D = \{&(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha), \\
&(w^{(1)}, w^{(2)}, \ldots, w^{(n)}) \in W(\alpha), (z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in Z(\alpha)\}
\end{aligned}
\tag{4.29}
$$

Then for each $\alpha \in [0,1]$,

$$
\begin{aligned}
& d'(Y(\alpha)(-)X(\alpha)), (V(\alpha)(+)Z(\alpha))(-)(U(\alpha)(+)W(\alpha))) \\
&= \sup_{H} \sum_{j=1} (y^{(j)} - x^{(j)})(v^{(j)} + z^{(j)} - u^{(j)} - w^{(j)}) \\
&\leq \sup_{E} \sum_{j=1}^{n} (y^{(j)} - x^{(j)})(v^{(j)} - u^{(j)}) \\
&\quad + \sup_{D} \sum_{j=1}^{n} (y^{(j)} - x^{(j)})(z^{(j)} - w^{(j)}) \\
&= d'(Y(\alpha)(-)X(\alpha), V(\alpha)(-)U(\alpha)) + d'(Y(\alpha)(-)X(\alpha), Z(\alpha)(-)W(\alpha))
\end{aligned}
\tag{4.30}
$$

By Definition 4.6, (2°) is proved.

(3°) Follows from Property 3.6 (2°), and (v), (vi).

(4°) Follows from Property 4.2 and 4.7.

(5°) For each α, where $(0 \leq \alpha \leq 1)$,

$(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in X(\alpha), (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in Y(\alpha)$,

$$(u^{(1)}, u^{(2)}, \ldots, u^{(n)}) \in U(\alpha), (v^{(1)}, v^{(2)}, \ldots, v^{(n)}) \in V(\alpha), \tag{4.31}$$

By Cauchy - Schwartz inequality,

$$\text{i.e.} \quad -\sqrt{\sum_{j=1}^{n}(y^{(j)} - x^{(j)})^2}\sqrt{\sum_{j=1}^{n}(v^{(j)} - u^{(j)})^2}$$

$$\leq \sum_{j=1}^{n}(y^{(j)} - x^{(j)})(v^{(j)} - u^{(j)}) \leq \sqrt{\sum_{j=1}^{n}(y^{(j)} - x^{(j)})^2}\sqrt{\sum_{j=1}^{n}(v^{(j)} - u^{(j)})^2} \tag{4.32}$$

Since $\sup AB \leq \sup A \sup B$, if $A > 0, B > 0$. Then, we have

$$-d^*(Y(\alpha)(-)X(\alpha))d^*(V(\alpha)(-)U(\alpha)) \leq d'((Y(\alpha)(-)X(\alpha)), (V(\alpha)(-)U(\alpha)))$$

$$\leq d^*(Y(\alpha)(-)X(\alpha))d^*(V(\alpha)(-)U(\alpha)), \forall \alpha \in [0,1] \tag{4.33}$$

Therefore $-|\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^* \leq |\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}| \leq |\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^*$ Hence $|\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}| \leq |\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^*$.

Remark 4.9. If $|\overrightarrow{\tilde{X}\tilde{Y}}|^* > 0$, and $|\overrightarrow{\tilde{U}\tilde{V}}|^* > 0$, by Property 4.8 (5°),

$$-1 \leq \frac{\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}}{|\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^*} \leq 1 \tag{4.34}$$

. So we have the following definition.

Definition 4.10 For $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}} \in SFR$, if $\overrightarrow{\tilde{X}\tilde{Y}} \neq \overrightarrow{\tilde{O}\tilde{O}}, \overrightarrow{\tilde{U}\tilde{V}} \neq \overrightarrow{\tilde{O}\tilde{O}}$, define the angle θ between $\overrightarrow{\tilde{X}\tilde{Y}}$ and $\overrightarrow{\tilde{U}\tilde{V}}$ by

$$cos\theta = \frac{\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}}{|\overrightarrow{\tilde{X}\tilde{Y}}|^* \cdot |\overrightarrow{\tilde{U}\tilde{V}}|^*} \tag{4.35}$$

.

Example 4.11 $n = 2$. Let us eject the rocket from $(2,3)$ aiming at $(6,8)$. The rocket falls in the circle centered at $(6,8)$ with radius 2. Also eject another rocket aiming at $(10,4)$, the rocket falls in the circle centered at $(10,4)$ with radius 1.

Insert Fig. 3 here

Then we have the membership functions of the fuzzy sets $\widetilde{X}, \widetilde{Y}, \widetilde{U}, \widetilde{V}$:

$$\mu_{\widetilde{X}}(2,3) = \begin{cases} 1 & \text{, if } x^{(1)} = 2,\ x^{(2)} = 3 \\ 0 & \text{, elsewhere} \end{cases}$$

$$\mu_{\widetilde{Y}}(y^{(1)}, y^{(2)}) = \begin{cases} \frac{1}{4}[4 - (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2] & \text{, if } (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2 \leq 4 \\ 0 & \text{, elsewhere} \end{cases}$$

$$\mu_{\widetilde{U}}(4,1) = \begin{cases} 1 & \text{, if } u^{(1)} = 4,\ u^{(2)} = 1 \\ 0 & \text{, elsewhere} \end{cases}$$

$$\mu_{\widetilde{V}}(v^{(1)}, v^{(2)}) = \begin{cases} 1 - (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2 & \text{, if } (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \tag{4.36}$$

The α-cuts, $0 \leq \alpha \leq 1$ of $\widetilde{X}, \widetilde{Y}, \widetilde{U}, \widetilde{V}$ are

$$\begin{aligned} X(\alpha) &= (2,3) \\ Y(\alpha) &= \{(y^{(1)}, y^{(2)}) \mid (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2 \leq 4(1-\alpha)\} \\ U(\alpha) &= (4,1) \\ V(\alpha) &= \{(v^{(1)}, v^{(2)}) \mid (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2 \leq (1-\alpha)\} \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} X(0) &= (2,3) \\ Y(0) &= \{(y^{(1)}, y^{(2)}) \mid (y^{(1)} - 6)^2 - (y^{(2)} - 8)^2 \leq 4\} \\ U(0) &= (4,1) \\ V(0) &= \{(v^{(1)}, v^{(2)}) \mid (v^{(1)} - 10)^2 - (v^{(2)} - 4)^2 \leq 1\} \end{aligned}$$

By Fig. 3, we have

$$\begin{aligned} x_m^{(1)}(0) &= 2, & x_m^{(2)}(0) &= 3 \\ \tfrac{y_m^{(1)}(0)-2}{4} &= \tfrac{\sqrt{41}+2}{\sqrt{41}}, & y_m^{(1)}(0) &= 6 + \tfrac{8}{\sqrt{41}} \\ \tfrac{y_m^{(2)}(0)-3}{5} &= \tfrac{\sqrt{41}+2}{\sqrt{41}}, & y_m^{(2)}(0) &= 8 + \tfrac{10}{\sqrt{41}} \end{aligned} \tag{4.38}$$

By Property 4.2, the length is

$$| \overrightarrow{\tilde{X}\tilde{Y}} |^* = \sqrt{(y_m^{(1)}(0) - x_m^{(1)}(0))^2 + (y_m^{(2)}(0) - x_m^{(2)}(0))^2} = 8.403 \tag{4.39}$$

Similarly, we have

$$\begin{aligned} u_m^{(1)}(0) &= 4, & u_m^{(2)}(0) &= 1 \\ v_m^{(1)}(0) &= 10 + \tfrac{6}{\sqrt{45}}, & v_m^{(2)}(0) &= 4 + \tfrac{3}{\sqrt{45}} \end{aligned} \tag{4.40}$$

and the length

$$| \overrightarrow{\tilde{U}\tilde{V}} |^* = \sqrt{(v_m^{(1)}(0) - u_m^{(1)}(0))^2 + (v_m^{(2)}(0) - u_m^{(2)}(0))^2} = 7.708 \tag{4.41}$$

By Property 4.7,

$$\begin{aligned} \overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}} &= (y_m^{(1)}(0) - x_m^{(1)}(0))(v_m^{(1)}(0) - u_m^{(1)}(0)) \\ &\quad + (y_m^{(2)}(0) - x_m^{(2)}(0))(v_m^{(2)}(0) - u_m^{(2)}(0)) \\ &= 58.81125 \end{aligned} \tag{4.42}$$

By Definition 4.10, the angle between $\overrightarrow{\tilde{X}\tilde{Y}}$ and $\overrightarrow{\tilde{U}\tilde{V}}$ has

$$cos\theta = \frac{\overrightarrow{\tilde{X}\tilde{Y}} \odot^* \overrightarrow{\tilde{U}\tilde{V}}}{| \overrightarrow{\tilde{X}\tilde{Y}} |^* \cdot | \overrightarrow{\tilde{U}\tilde{V}} |^*} = 0.907996 \tag{4.43}$$

In crisp case, the vector $\overrightarrow{\tilde{X}\tilde{Y}}$ from $X = (2, 3)$ to $Y = (6, 8)$ is $(4, 5)$, and the vector $\overrightarrow{\tilde{U}\tilde{V}}$ from $U = (4, 1)$ to $V = (10, 4)$ is $(6, 3)$. Their lengths are $| \overrightarrow{\tilde{X}\tilde{Y}} | = 6.403$ and $| \overrightarrow{\tilde{U}\tilde{V}} | = 6.708$; $\overrightarrow{\tilde{X}\tilde{Y}} \cdot \overrightarrow{\tilde{U}\tilde{V}} = 4 \cdot 6 + 5 \cdot 3 = 39$ and

$$cos\theta = \frac{\overrightarrow{\tilde{X}\tilde{Y}} \cdot \overrightarrow{\tilde{U}\tilde{V}}}{| \overrightarrow{\tilde{X}\tilde{Y}} | \cdot | \overrightarrow{\tilde{U}\tilde{V}} |} = 0.908005 \tag{4.44}$$

Example 4.12 Let

$$\mu_{\underset{X}{\sim}}(x^{(1)}, x^{(2)})$$
$$= \begin{cases} 1-(x^{(1)}-5)^2-(x^{(2)}-10)^2, & \text{if } (x^{(1)}-5)^2+(x^{(2)}-10)^2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$
$$\mu_{\underset{Y}{\sim}}(y^{(1)}, y^{(2)})$$
$$= \begin{cases} \frac{1}{4}\{4-(y^{(1)}-14)^2-(y^{(2)}-15)^2\}, & \text{if } (y^{(1)}-14)^2+(y^{(2)}-15)^2 \le 4 \\ 0, & \text{elsewhere} \end{cases}$$
$$\mu_{\underset{U}{\sim}}(u^{(1)}, u^{(2)})$$
$$= \begin{cases} \frac{1}{4}\{4-(u^{(1)}-8)^2-(u^{(2)}-2)^2\}, & \text{if } (u^{(1)}-8)^2+(u^{(2)}-2)^2 \le 4 \\ 0, & \text{elsewhere} \end{cases}$$
$$\mu_{\underset{V}{\sim}}(v^{(1)}, v^{(2)})$$
$$= \begin{cases} 1-(v^{(1)}-17)^2-(v^{(2)}-7)^2, & \text{if } (v^{(1)}-17)^2+(v^{(2)}-7)^2 \le 1 \\ 0, & \text{elsewhere} \end{cases} \quad (4.45)$$

Insert Figure 4 here

We have the $\alpha = 0$-cuts of $\widetilde{X}, \widetilde{Y}, \widetilde{U}, \widetilde{V}$:

$$\begin{aligned} X(0) &= \{(x^{(1)}, x^{(2)}) | (x^{(1)}-5)^2+(x^{(2)}-10)^2 \le 1\} \\ Y(0) &= \{(y^{(1)}, y^{(2)}) | (y^{(1)}-14)^2+(y^{(2)}-15)^2 \le 4\} \\ U(0) &= \{(u^{(1)}, u^{(2)}) | (u^{(1)}-8)^2+u^{(2)}-2)^2 \le 4\} \\ V(0) &= \{(v^{(1)}, v^{(2)}) | (v^{(1)}-17)^2+(v^{(2)}-7)^2 \le 1\} \end{aligned} \quad (4.46)$$

As in Example 4.11, from Fig.4, we have

$$\begin{aligned} x_m^{(1)}(0) &= 5-\tfrac{9}{\sqrt{106}}, & x_m^{(2)}(0) &= 10-\tfrac{5}{\sqrt{106}} \\ y_m^{(1)}(0) &= 14+\tfrac{18}{\sqrt{106}}, & y_m^{(2)}(0) &= 15+\tfrac{10}{\sqrt{106}} \\ u_m^{(1)}(0) &= 8-\tfrac{18}{\sqrt{106}}, & u_m^{(2)}(0) &= 2-\tfrac{10}{\sqrt{106}} \\ v_m^{(1)}(0) &= 17+\tfrac{9}{\sqrt{106}}, & v_m^{(2)}(0) &= 7+\tfrac{5}{\sqrt{106}} \end{aligned} \quad (4.47)$$

By Property 4.2, the lengths

$$
\begin{aligned}
|\overrightarrow{\widetilde{X}\widetilde{Y}}|^* &= \sqrt{(y_m^{(1)}(0)-x_m^{(1)}(0))^2+(y_m^{(2)}(0)-x_m^{(2)}(0))^2} \\
&= \sqrt{(9+\frac{27}{\sqrt{106}})^2+(5+\frac{15}{\sqrt{106}})^2} = 13.29563 \\
|\overrightarrow{\widetilde{U}\widetilde{V}}|^* &= \sqrt{(v_m^{(1)}(0)-u_m^{(1)}(0))^2+(v_m^{(2)}(0)-u_m^{(2)}(0))^2} \\
&= \sqrt{(9+\frac{27}{\sqrt{106}})^2+(5+\frac{15}{\sqrt{106}})^2} = 13.29563
\end{aligned}
\tag{4.48}
$$

and by Property 4.7,

$$
\begin{aligned}
&\overrightarrow{\widetilde{X}\widetilde{Y}} \odot^* \overrightarrow{\widetilde{U}\widetilde{V}} \\
&= (y_m^{(1)}(0)-x_m^{(1)}(0))\cdot(v_m^{(1)}(0)-u_m^{(1)}(0))+(y_m^{(2)}(0)-x_m^{(2)}(0))\cdot(v_m^{(2)}(0)-u_m^{(2)}(0)) \\
&= (9+\frac{27}{\sqrt{106}})^2+(5+\frac{15}{\sqrt{106}})^2 = 13.29563^2
\end{aligned}
\tag{4.49}
$$

By Definition 4.3, the angle between $\overrightarrow{\widetilde{X}\widetilde{Y}}$ and $\overrightarrow{\widetilde{U}\widetilde{V}}$ has

$$
cos\theta = \frac{\overrightarrow{\widetilde{X}\widetilde{Y}}\cdot\overrightarrow{\widetilde{U}\widetilde{V}}}{|\overrightarrow{\widetilde{X}\widetilde{Y}}|\cdot|\overrightarrow{\widetilde{U}\widetilde{V}}|} = 1 \tag{4.50}
$$

Hence $\theta = 0$.*i.e.* $\overrightarrow{\widetilde{X}\widetilde{Y}} \,//\, \overrightarrow{\widetilde{U}\widetilde{V}}$.

In the crisp case, the vector $\overrightarrow{\widetilde{X}\widetilde{Y}}$ from $X=(5,10)$ to $Y=(14,15)$ is $(9,5)$, the vector $\overrightarrow{\widetilde{U}\widetilde{V}}$ from $U=(8,2)$ to $V=(17,7)$ is $(9,5)$. The lengths $|\overrightarrow{\widetilde{X}\widetilde{Y}}| = |\overrightarrow{\widetilde{U}\widetilde{V}}| = 10.2956$ and the angle between them has $cos\phi = 1$. i.e. $\overrightarrow{\widetilde{X}\widetilde{Y}} \,//\, \overrightarrow{\widetilde{U}\widetilde{V}}$.

5. Discussion

(A) The comparison of the second definition 5.1 of the length of the fuzzy vector $\overrightarrow{\widetilde{X}\widetilde{Y}} \in SFR$ and the length $|\overrightarrow{\widetilde{X}\widetilde{Y}}|^*$ of Definition 4.1.

Method 2: Definition 5.1

(a) The length of the fuzzy vector $\overrightarrow{\widetilde{X}\widetilde{Y}} \in SFR$ defined as

$$| \overrightarrow{\widetilde{X}\widetilde{Y}} | = \int_0^1 d^*(Y(\alpha)(-)X(\alpha))d\alpha \tag{5.1}$$

(b) The inner product of $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{U}\widetilde{V}} \in SFR$ is defined as

$$\overrightarrow{\widetilde{X}\widetilde{Y}} \odot' \overrightarrow{\widetilde{U}\widetilde{V}} = \int_0^1 d'(Y(\alpha)(-)X(\alpha), V(\alpha)(-)U(\alpha))d\alpha \tag{5.2}$$

(c-1) By Definition 4.1, since $| \overrightarrow{\widetilde{X}\widetilde{Y}} |^* = \sup_{0\leq\beta\leq 1} d^*(Y(\beta) - X(\beta)) \geq d^*(Y(\alpha) - X(\alpha)) \forall \alpha \in [0,1]$, so we have

$$| \overrightarrow{\widetilde{X}\widetilde{Y}} |^* \geq \int_0^1 d^*(Y(\alpha)(-)X(\alpha))d\alpha = | \overrightarrow{\widetilde{X}\widetilde{Y}} | \tag{5.3}$$

Use the same way as in section 4, we can prove the following:

(c-2) For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR, k \in F_p^1(1), k \neq 0$, we have

(1°) $|k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}} | = |k|| \overrightarrow{\widetilde{X}\widetilde{Y}} |$

(2°) $| \overrightarrow{\widetilde{X}\widetilde{Y}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}} | \leq | \overrightarrow{\widetilde{X}\widetilde{Y}} | + | \overrightarrow{\widetilde{W}\widetilde{Z}} |$

This leads to the same results as Property 4.5.

(c-3) For $\overrightarrow{\widetilde{X}\widetilde{Y}}, \overrightarrow{\widetilde{U}\widetilde{V}}, \overrightarrow{\widetilde{W}\widetilde{Z}} \in SFR, k_1 \in F_p^1(1), k \geq 0$

(1°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \odot' \overrightarrow{\widetilde{U}\widetilde{V}} = \overrightarrow{\widetilde{U}\widetilde{V}} \odot' \overrightarrow{\widetilde{X}\widetilde{Y}}$

(2°) $\overrightarrow{\widetilde{X}\widetilde{Y}} \odot' (\overrightarrow{\widetilde{U}\widetilde{V}} \oplus \overrightarrow{\widetilde{W}\widetilde{Z}}) \leq (\overrightarrow{\widetilde{X}\widetilde{Y}} \odot' \overrightarrow{\widetilde{U}\widetilde{V}}) \oplus (\overrightarrow{\widetilde{X}\widetilde{Y}} \odot' \overrightarrow{\widetilde{W}\widetilde{Z}})$

(3°) $k(\overrightarrow{\widetilde{X}\widetilde{Y}} \odot' \overrightarrow{\widetilde{U}\widetilde{V}}) = (k_1 \odot \overrightarrow{\widetilde{X}\widetilde{Y}}) \odot' \overrightarrow{\widetilde{U}\widetilde{V}} = \overrightarrow{\widetilde{X}\widetilde{Y}} \odot' (k_1 \odot \overrightarrow{\widetilde{U}\widetilde{V}})$

These are the same results as Property 4.8 (1°), (2°), (3°). However, (4°), (5°) of

Property 4.8 do not hold in Method 2. Therefore we can not define the angle between two fuzzy vectors in SFR as in Definition 4.10.

(B) In P.Lubczonok [5], he defined the fuzzy vector space as the following:

Definition 5.2 (Definition 2.1 in [5]) Fuzzy vector space is a pair $\widetilde{E} = (E, \mu)$ where E is a vector space and $\mu : E \to [0, 1]$ with property that for all $a, b \in R$, and $x, y \in E, \mu(ax + by) \geq \mu(x) \wedge \mu(y)$ holds. Then he obtained some results from this in [5].

In this paper, we get fuzzy vector space FE^n over $F_p^1(1)$ through n-dimensional vector space E^n over R, then extend this to the pseudo fuzzy vector space SFR over $F_p^1(1)$. It is strongly linked with E^n throughout this process, so it has more practical usage.

Since $E^n \approx FE^n$, we may consider the fuzzy vector space under the sense of [5]. The mapping

$$\overrightarrow{PQ} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) \in E^n$$

$$\longleftrightarrow \overrightarrow{\widetilde{P}\widetilde{Q}} = (q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)} \ldots, q^{(n)} - p^{(n)})_1 \in FE^n \tag{5.4}$$

is one-to -one onto. In [5], let $E = E^n$ and $\mu = \nu$. For $\overrightarrow{PQ} \in E^n$, define $\nu(\overrightarrow{PQ}) = \mu_{\underset{\widetilde{PQ}}{\longrightarrow}}(q^{(1)} - p^{(1)}, q^{(2)} - p^{(2)}, \ldots, q^{(n)} - p^{(n)}) = 1$. Let

$\overrightarrow{ST}= (t^{(1)} - s^{(1)}, t^{(2)} - s^{(2)}, \ldots, t^{(n)} - s^{(n)}) \in E^n$ and $a, b \in R$. Then

$$\begin{aligned}
&a\ \overrightarrow{PQ} + b\ \overrightarrow{ST} \\
&= (a(q^{(1)} - p^{(1)}) + b(t^{(1)} - s^{(1)}), a(q^{(2)} - p^{(2)}) + b(t^{(2)} - s^{(2)}), \ldots, \\
&\qquad a(q^{(n)} - p^{(n)}) + b(t^{(n)} - s^{(n)})) \in E^n \ \forall a, b \in R \\
&\longleftrightarrow (a_1 \odot \overrightarrow{\widetilde{P}\widetilde{Q}}) \oplus (b_1 \odot \overrightarrow{\widetilde{S}\widetilde{T}}) \\
&= (a(q^{(1)} - p^{(1)}) + b(t^{(1)} - s^{(1)}), a(q^{(2)} - p^{(2)}) + b(t^{(2)} - s^{(2)}), \ldots, \\
&\qquad a(q^{(n)} - p^{(n)}) + b(t^{(n)} - s^{(n)}))_1 \in FE^n
\end{aligned} \tag{5.5}$$

Hence by the definition of ν, we have

$$\begin{aligned}
&\nu(a\ \overrightarrow{PQ} + b\ \overrightarrow{ST}) \\
&= \mu_{(a_1 \odot \overrightarrow{\widetilde{PQ}} \oplus b_1 \odot \overrightarrow{\widetilde{ST}})} (a(q^{(1)} - p^{(1)}) + b(t^{(1)} - s^{(1)}), a(q^{(2)} - p^{(2)}) + b(t^{(2)} - s^{(2)}), \ldots, \\
&a(q^{(n)} - p^{(n)}) + b(t^{(n)} - s^{(n)})) = 1.
\end{aligned} \tag{5.6}$$

Then $\nu(a\ \overrightarrow{PQ} + b\ \overrightarrow{ST}) = 1 = \nu(\overrightarrow{PQ}) \wedge \nu(\overrightarrow{ST})$. Thus we get the fuzzy vector space $[E^n, \nu]$ by Definition 5.2 under the sense of [5].

(C) In this paper, we emphasize on solving the practical problem in stead of just working it out theoretically.

Acknowledgement

I deeply thank to the Emeritus Professor Jing-Shing Yao of National Taiwan University for his many valuable suggestions.

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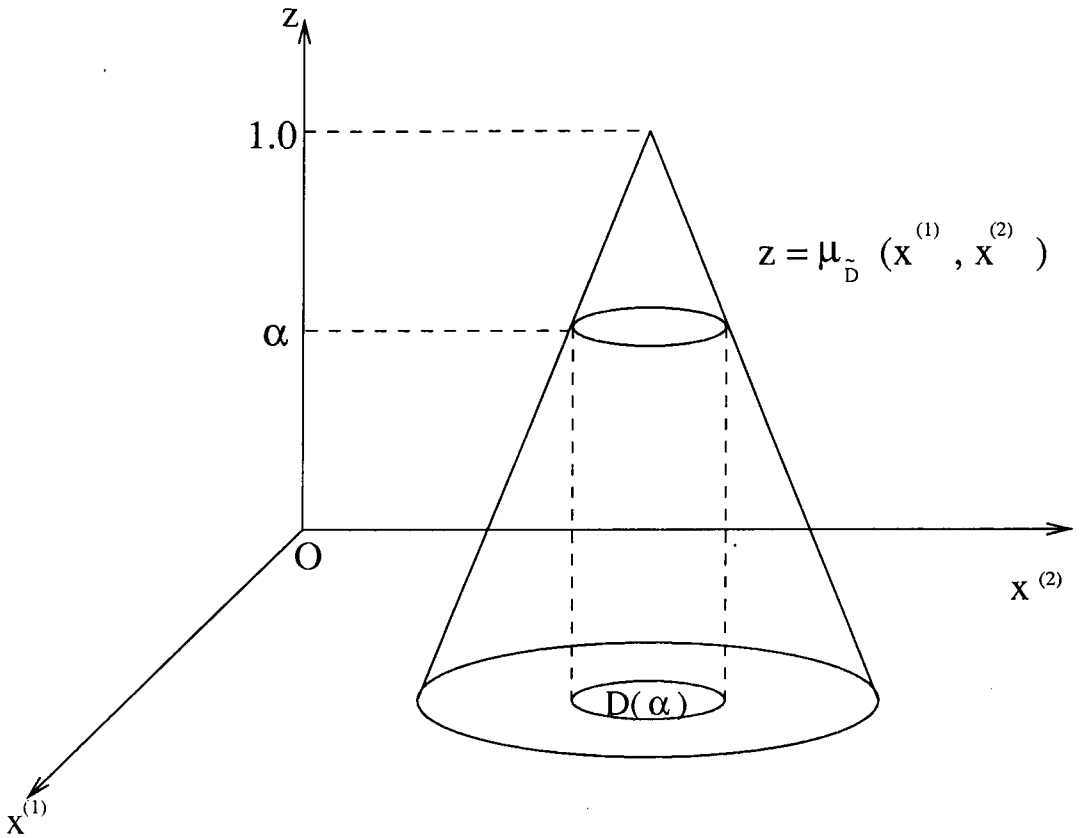

Fig.1 α–cut of fuzzy set $\tilde{D}$ on R

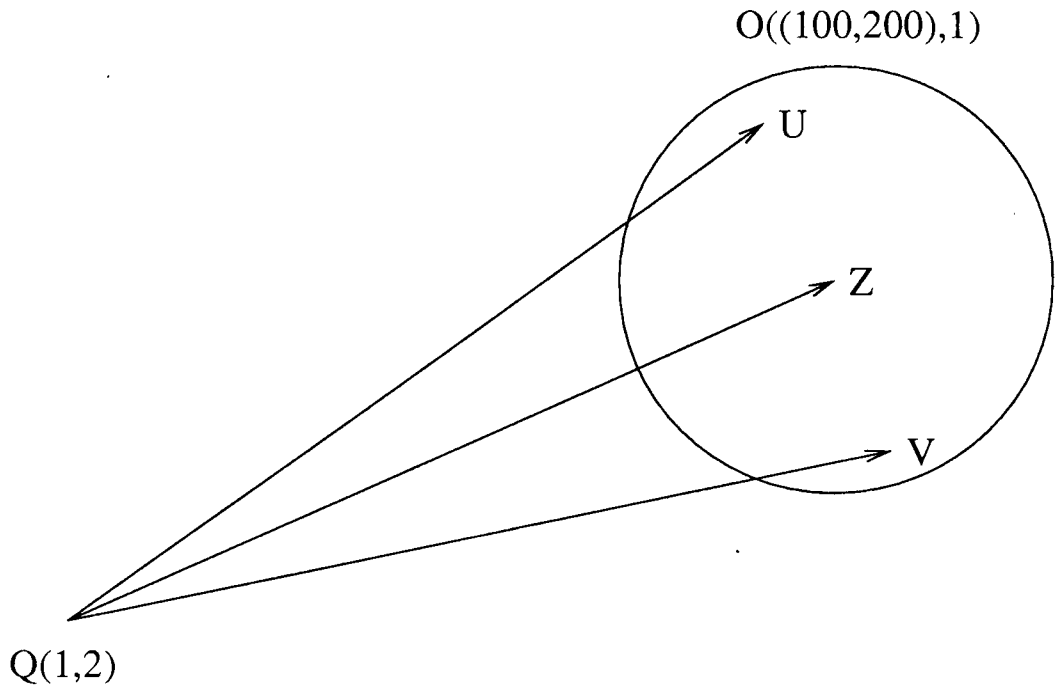

Fig. 2 Fuzzy vector $\overrightarrow{\tilde{Q}\tilde{Z}}$

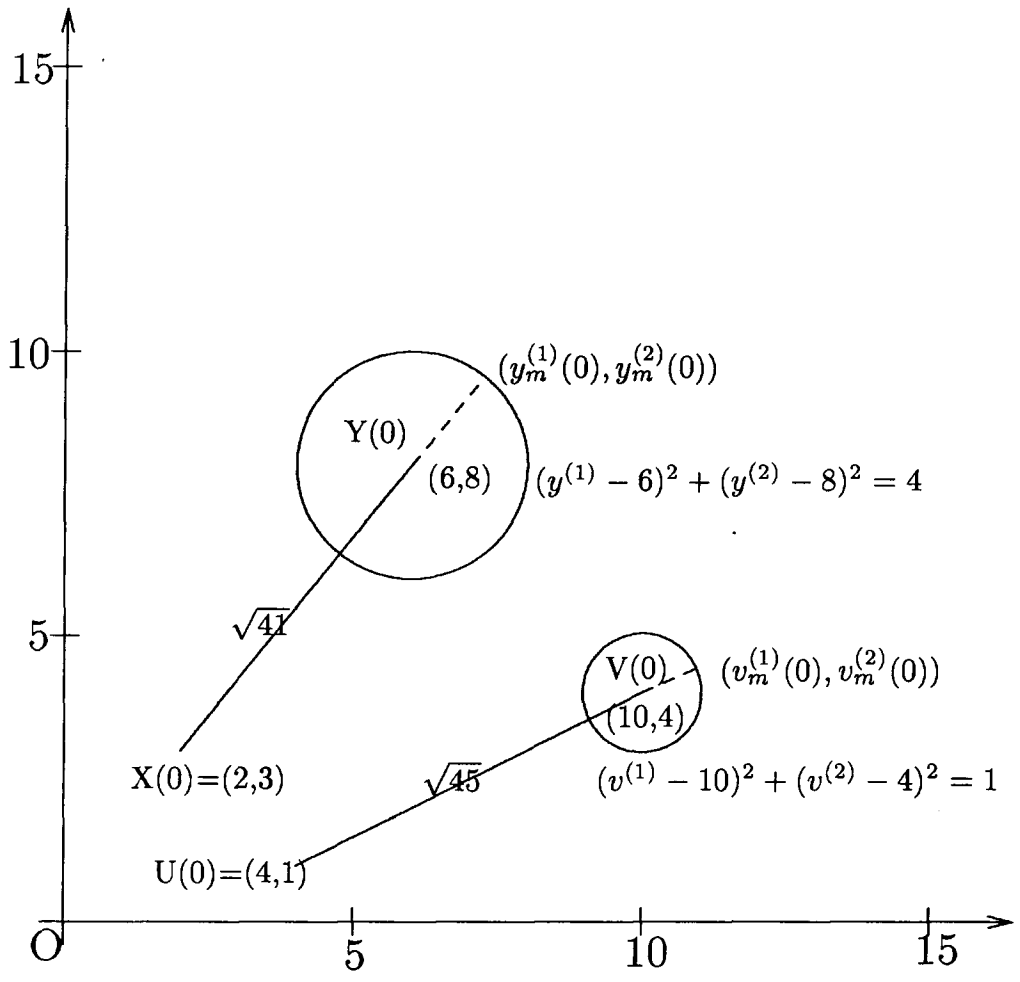

Fig. 3 Fuzzy vector $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}}$

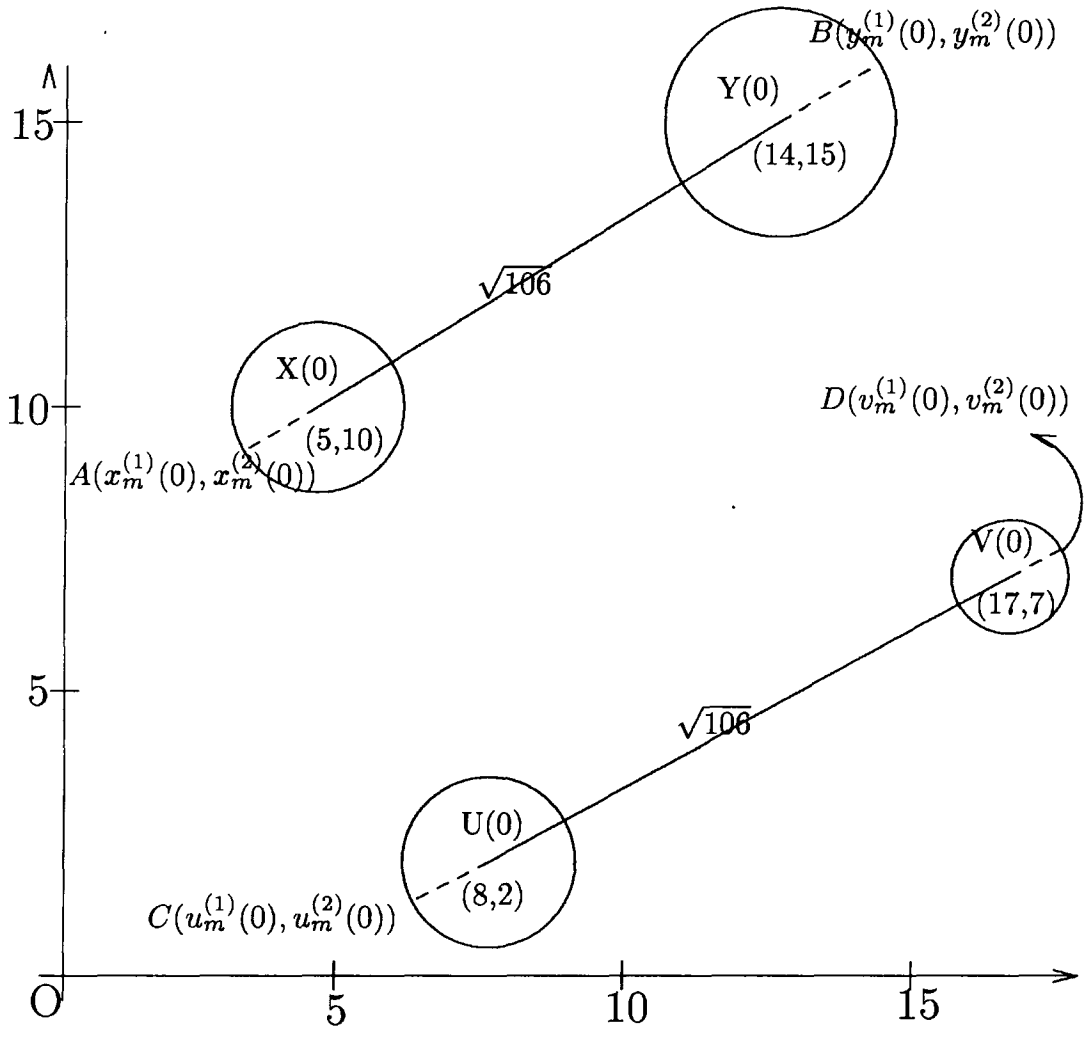

Fig. 4 Fuzzy inner product of $\overrightarrow{\tilde{X}\tilde{Y}}, \overrightarrow{\tilde{U}\tilde{V}}$