# 行政院國家科學委員會專題研究計畫　成果報告

## 交互作用粒子系統的流力極限 (6)

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## Hydrodynamic Limit of Interacting Particle Systems (6)

計畫編號：NSC 91 – 2115 – M002 – 010

執行期限：91 年 8 月 1 日至 92 年 9 月 30 日

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### 一、中文摘要

本計劃中我們探討了 1 維與 2 維格子點空間$Z^1$、$Z^2$上對稱簡單互斥過程中數個位置被粒子佔據的時間差（總平均為零）的大離差估計。

**關鍵詞**：對稱簡單互斥過程、粒子佔據的時間差、大離差估計

### Abstract

In this project we study the large deviations estimate of occupation time difference of symmetric simple exclusion processes (SEP) on one and two dimensional lattice spaces $Z^1$ and $Z^2$.

**Keywords**: symmetric simple exclusion process, occupation time difference, large deviations estimate

### 二、緣由與目的

此處先說明期限延長與經費使用的情形。原核定「出席國際會議」之部分，因 91 年 5 月時不確定能否獲得補助，故未能在 91 年 5 月限期截止前報名，以致未能如預定計畫參加 ICM 2002 於 91 年 8 月 29 日至 9 月 3 日在北京舉行的 satellite conference: stochastic analysis （主要行程）及在京都於 9 月 4 日至 7 日舉行的 RIMS 2002 international project research: stochastic analysis and related topics （次要行程）。92 年 6 月時因受國內 SARS 疫情影響恐申請外國簽證不易、或出國後在外國入境時遭隔離檢疫而影響行程，故申請延期 2 個月並變更「出席國際會議」之補助為「國外差旅費」。此申請案於 6 月 13 日獲准（臺會綜二字第 0920028200 號），故計畫執行期限延至 92 年 9 月 30 日。但原計畫訪問的姚鴻澤教授於暑假中忙於自紐約 Courant Institute 搬家至加州的 Stanford University，而 University of Maryland 的李宗祐教授則為胃癌侵襲所苦，故國外訪問終未能成行。以下敘述計畫之緣由與目的。

Consider the symmetric simple exclusion process (SEP) on d-dimensional lattice $Z^d$, d=1 or 2. The configurations of this process are denoted by $\eta$ so that $\eta(x)$ is equal to 1 or 0 if site x in $Z^d$ is occupied or not for $\eta$. For each $\alpha$ in [0,1], denote by $\nu(\alpha)$ the Bernoulli product measure on the configuration space $\Omega$ with marginals given by $\nu(\alpha)\{\eta, \eta(x)=1\}=\alpha$, x in $Z^d$. It is well-known that $\{\nu(\alpha), 0\leq\alpha\leq 1\}$ is a one-parameter family of reversible invariant measures. In this project we study SEP accelerated by T starting from the reversible measure $\nu(\alpha)$ for a fixed $\alpha$ in (0,1). Given a local function $b(\eta)$ on $\Omega$ satisfying $\nu(\alpha)\ [b(\eta)\ |\ \eta^{--}]=0$, denote the occupation time difference B(T) associated with b by ( c(T)=square root of ln T )

$$B(T) = c(T) \int_0^1 b(\eta_s)\, ds.$$

We are interested in the large deviations of the occupation time difference B(T).

### 三、結果與討論

Given $T > 0$, on the configuration space $\Omega = \{0,1\}^{\mathbb{Z}^d}$, d=1,2, consider the *accelerated* symmetric simple exclusion process (SEP) generated by $L_T$ given by

$$(L_T f)(\eta) \;=\; \frac{T}{2} \sum_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|=1}} [f(\sigma^{x,y}\eta) - f(\eta)] \;,$$

where the summation is carried over all nearest neighbor sites $x, y, |x-y| = 1$, of $\mathbb{Z}^d$. In this formula, $f$ is a local function and $\sigma^{x,y}\eta$ is the configuration obtained from $\eta$ by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x,\, y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x \;. \end{cases}$$

For each $0 \le \alpha \le 1$, denote by $\nu_\alpha$ the Bernoulli product measure on $\Omega$ with marginals given by

$$\nu_\alpha\{\eta,\, \eta(x) = 1\} \;=\; \alpha$$

for $x \in \mathbb{Z}^d$. Clearly, $\{\nu_\alpha, 0 \le \alpha \le 1\}$ is a one-parameter family of reversible invariant measures. For $0 \le \alpha \le 1$, denote by $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ the probability on the path space $D(\mathbb{R}_+, \Omega)$ corresponding to SEP starting from $\nu_\alpha$. From now on we fix an $\alpha \in (0,1)$.

Define the occupation time of the origin:

$$A_T \;=\; \int_0^1 \eta_s(0)\, ds \;.$$

The large deviations principle of $A_T$ under $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ as $T \to \infty$, which is established in [1], states that the order is $T/\log T$ and the rate function $\Upsilon_\alpha : [0,1] \to \mathbb{R}_+$ is given by

$$\Upsilon_\alpha(\beta) \;=\; \frac{\pi}{2}\Big\{ \sin^{-1}(2\beta-1) - \sin^{-1}(2\alpha-1) \Big\}^2 .$$

Let $b$ be a local function on $\Omega$ satisfying $\nu_\alpha[b(\eta)\,|\,\bar\eta\,] = 0$, where $\nu_\alpha[\cdot\,|\,\bar\eta\,]$ represents the $\nu_\alpha$–expectation conditioned on the average number of particles $\bar\eta$. Typical examples are $\eta(0) - \eta(e_1)$ and $\eta(e_1) + \eta(-e_1) + \eta(e_2) + \eta(-e_2) - 4\eta(0)$.

Denote the occupation time difference $B_T$ associated with $b$ by

$$B_T = \sqrt{\log T}\Big( \int_0^1 b(\eta_s)\, ds \Big) \in \mathbb{R} \;.$$

There are at least two methods to study the large deviations principle of the joint distribution $(A_T, B_T)$. The first is a probabilistic approach which basically is similar to the one used in [1] (see [3] also) with some natural modifications. As the object now is more complicated, one can expect that more detailed analysis is needed and the arguments would be much harder than those in [1]. The second is a PDE approach developed by T.Y. Lee and has been applied successfully in several examples, see [2]. Here we outline the basic idea of PDE approach without proof. It is remarked that to verify each step described below also requires lengthy and sophisticated arguments.

For simplicity consider $d = 2$ case only. To investigate the LDP of the joint distribution of $A_T$ and $B_T$, by Laplace-Varadhan theorem, it suffices to study

$$\begin{aligned} &\lim_{T\to\infty} \frac{\log T}{T} \log \mathbb{E}_\alpha \Big[ \exp\Big\{ \frac{T}{\log T}\Big(\sigma A_T + \lambda B_T\Big)\Big\}\Big] \\ &= \lim_{T\to\infty} \frac{\log T}{T} \log \mathbb{E}_\alpha [V_T(1, \eta; \sigma, \lambda)] \,, \end{aligned}$$

where $V_T$ is defined for $t \in [0,1]$, $(\sigma, \lambda) \in \mathbb{R}^2$ as

$$\begin{aligned} &V_T(t,\eta) = V_T(t,\eta;\sigma,\lambda) \\ &= \mathbb{E}_\eta \Big[ \exp\Big\{ \int_0^t \Big( \frac{\sigma T}{\log T}\eta_s(0) + \frac{\lambda T}{\sqrt{\log T}} b(\eta_s)\Big)\, ds \Big\}\Big] . \end{aligned}$$

By Feynman-Kac formula, $V_T$ solves the differential equation

$$\begin{cases} \partial_t V_T(t,\eta) = L_T V_T + \Big( \dfrac{\sigma T}{\log T}\eta(0) \\ \qquad\qquad + \dfrac{\lambda T}{\sqrt{\log T}} b(\eta)\Big) V_T,\; t \in [0,1]\,, \\ V_T(0,\eta) = 1 \;. \end{cases}$$

For convenience let

$$\begin{aligned} \log V_T &= v_T \Leftrightarrow V_T = \exp(v_T)\,, \\ \phi_T(\eta;\sigma,\lambda) &= \frac{\sigma T}{\log T}\eta(0) + \frac{\lambda T}{\sqrt{\log T}} b(\eta)\,. \end{aligned}$$

It follows that $v_T(t,\eta)=v_T(t,\eta;\sigma,\lambda)$ satisfies, $t\in[0,1]$,

$$\begin{cases} \partial_t v_T(t,\eta) = e^{-v_T}(L_T e^{v_T}) + \phi_T(\eta) \\ \qquad\qquad = L_T v_T + \phi_T(\eta) + R_T(v_T), \\ v_T(0,\eta) = 0\,. \end{cases} \tag{1}$$

Here

$$\begin{aligned} R_T(f) &= e^{-f}(L_T e^f) - L_T f \\ &= \frac{T}{2}\sum_{\substack{x,y\in\mathbb{Z}^2\\|x-y|=1}}\Big[\exp\Big\{f(\sigma^{x,y}\eta)-f(\eta)\Big\} \\ &\quad -1-\Big(f(\sigma^{x,y}\eta)-f(\eta)\Big)\Big] \\ &= Q_T(f)+e_T(f)\,, \end{aligned}$$

and

$$\begin{aligned} Q_T(f) &= \frac{T}{4}\sum_{\substack{x,y\in\mathbb{Z}^2\\|x-y|=1}}\Big[f(\sigma^{x,y}\eta)-f(\eta)\Big]^2, \\ e_T(f) &= R_T(f)-Q_T(f)\,. \end{aligned}$$

**The first claim is that $e_T$ is negligible in the large deviations limit $T\to\infty$.** Denote by $v^{(1)}(t,\eta)=v_T^{(1)}(t,\eta;\sigma,\lambda), t\in[0,1]$, the solution of differential equation (1) in which $R_T$ is replaced solely by $Q_T$. The claim implies that it suffices to study $v^{(1)}(t,\eta)$.

To study $v_T^{(1)}(t,\eta)$ we need to introduce an auxiliary function. Note that by assumption $b$ has $\nu_\alpha$–mean 0 on each hyperplane $\bar\eta$ =constant. Therefore there exists a $g_T(\eta)=g_T(\eta;\lambda)$ such that

$$L_T g_T(\eta)+\frac{\lambda T}{\sqrt{\log T}}b(\eta)=0\,.$$

For simplicity we take a typical example:

$$g_T(\eta)=\frac{2\lambda}{\sqrt{\log T}}\eta(0),\, b=4\eta(0)-\sum_{\substack{j=1,2\\\chi=1,-1}}\eta(\chi e_j)\,.$$

Let $v_T^{(2)}(t,\eta;\sigma,\lambda) = v_T^{(1)}(t,\eta;\sigma,\lambda) - g_T(\eta;\lambda)$. When we write down the differential equation for $v_T^{(2)}$, and replace the term $Q_T(v_T^{(2)}+g_T)$ by $Q_T(v_T^{(2)})+Q_T(g_T)$, we actually write down a new equation. Denote by $v_T^{(3)}$ the solution of this new differential equation. **The second claim is that the contribution of $g_T$ within $v_T^{(j)}, j=1,2$, is negligible and one can substitute $Q_T(v_T^{(2)}+g_T)$ with $Q_T(v_T^{(2)})+Q_T(g_T)$.** In summary we have

$$\begin{aligned} &\lim_{T\to\infty}\frac{\log T}{T}\log\mathbb{E}_\alpha\Big[\exp\{v_T(1,\eta)\}\Big] \\ =\;&\lim_{T\to\infty}\frac{\log T}{T}\log\mathbb{E}_\alpha\Big[\exp\{v_T^{(3)}(1,\eta)\}\Big]\,. \end{aligned}$$

Observe that

$$\begin{aligned} &Q_T(f+h)=Q_T(f)+Q_T(h) \\ &+\frac{T}{2}\sum_{\substack{x,y\in\mathbb{Z}^2\\|x-y|=1}}\Big[f(\sigma^{x,y}\eta)-f(\eta)\Big]\Big[h(\sigma^{x,y}\eta)-h(\eta)\Big]\,, \\ &Q_T(g_T)=\frac{\lambda^2 T}{\log T}\sum_{\substack{l=1,2\\\chi=-1,1}}\Big[\eta(\chi e_j)-\eta(0)\Big]^2 \\ =\;&\frac{\lambda^2 T}{\log T}\Big\{4\eta(0)+\Big[1-2\eta(0)\Big]\Big(\sum_{\substack{l=1,2\\\chi=-1,1}}\eta(\chi e_j)\Big)\Big\} \end{aligned}$$

which satisfies $\nu_\alpha[\,Q_T(g_T)\,|\,\bar\eta\,]\neq 0$. Let

$$U_T(t,\eta)=\mathbb{E}_\eta\Big[\exp\Big\{\int_0^t\frac{T}{\log T}\mathcal{U}(\eta_s)\,ds\Big\}\Big],$$

$u_T=\log U_T$, and $u_T^{(1)}(t,\eta)$ the solution of the following differential equation:

$$\begin{cases} \partial_t u_T^{(1)}(t,\eta) = L_T u_T^{(1)} + Q_T(u_T^{(1)}) \\ \qquad\qquad + \dfrac{T}{\log T}\mathcal{U}(\eta),\ t\in[0,1]\,, \\ u_T^{(1)}(0,\eta) = 0\,. \end{cases}$$

Now it is clear that by choosing

$$\begin{aligned} &\mathcal{U}(\eta)=\sigma\eta(0)+4\lambda^2\eta(0) \\ &+\lambda^2\Big[1-2\eta(0)\Big]\Big(\sum_{\substack{l=1,2\\\chi=-1,1}}\eta(\chi e_j)\Big)\,, \end{aligned}$$

and applying superexponential estimate and large deviations estimate of the occupation time established in [1] one can derive the rate function for the large deviations of the joint distribution $(A_T,B_T), T\to\infty$, as

$$\mathbf{I}_\alpha(c,d)=\frac{d^2}{32c(1-c)}+\Upsilon_\alpha(c)\,.$$

## 四、計畫成果自評

The large deviations results obtained in this project are contained in [4].

## 五、參考文獻


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