

行政院國家科學委員會專題研究計畫 成果報告

指數型 Mean Field 方程的 Leray-Schauder 度 II

計畫類別：個別型計畫

計畫編號：NSC91-2115-M-002-014-

執行期間：91年08月01日至92年10月31日

執行單位：國立臺灣大學數學系暨研究所

計畫主持人：陳俊全

報告類型：精簡報告

報告附件：出席國際會議研究心得報告及發表論文

處理方式：本計畫可公開查詢

中 華 民 國 93 年 6 月 6 日

æ

Let M be a compact Riemann surface without boundary, h be a C^2 function on M , $h > 0$, and $\rho \in \mathbf{R}$. We consider the following equation

$$(0.1) \quad \Delta u + \rho \left(\frac{he^u}{\int_M he^u} - 1 \right) = 0 \quad \text{on } M,$$

where Δ is the Betrami-Laplace operator on M . This is the second part of two papers concerning this equation. In the first paper, we obtained sharp estimates for blowup solutions. Based on these estimates, we will calculate the Leray-Schauder degree defined in a standard way for (0.1) in this paper.

When $\rho = 8\pi$ and M is a sphere, (0.1) is related to the problem of prescribed Gaussian curvature, called the Kazdan-Warner problem or the Nirenberg problem, see Kazdan-Warner [28]. In some Chern-Simons-Higgs model as discussed in Taubes [48]-[49], Hong, Kim and Pac [26], Jackiw and Weinberg [27], Spruck and Yang [44], Caffarelli and Yang [4], Tarantello [47], Struwe and Tarantello [45], Nolasco and Tarantello [39]-[41], Riciardi and Tarantello [42]-[43], Ding, Jost, Li and Wang [20]-[23] and the references therein, (0.1) play a role when M is a torus and $\rho = 4\pi N$, where u denotes a condensate solution and N is the vortex number. (0.1) on torus or on a bounded domain of \mathbf{R}^2 with Dirichlet boundary condition also arises from the mean field limit of point vortices of Euler flow as studied in Caglioti, Lions, Marchioro and Pulvirenti [5]-[6] and Kiessling [29].

The existence of a solution (0.1) for $\rho < 8\pi$ can be obtained by variational methods using the Moser-Trudinger inequality. For $\rho \geq 8\pi$, the existence problem is more difficult. When $\rho = 8\pi$ and M is the standard S^2 , there have been a lot of related works. When $\rho = 8\pi$ and M is a general Riemann surface, Ding, Jost, Li and Wang in [20] gave sufficient conditions on h which implies the existence of (0.1). Using another approach, Nolasco and Tarantello in [39] obtained similar result for torus. When M is a flat torus with fundamental domain $[0, 1] \times [0, 1]$ and $h \equiv 1$, Struwe and Tarantello in [45] proved that (0.1) has a nontrivial solution for $8\pi < \rho < 4\pi^2$ by the mountain pass method. More recently, Ding, Jost, Li and Wang in [22] proved for $8\pi < \rho < 16\pi$ and $h > 0$, there exists a non-minimum solution of (0.1).

Another approach to study the existence problem of (0.1) is using the Leray-Schauder degree. Since $u + \text{constant}$ is still a solution for each solution u of (0.1), we can normalize u to satisfy $\int_M u = 0$. Then it can be shown by the Moser-Trudinger inequality known that when ρ lies in a compact subset of $(-\infty, 8\pi)$, all solutions of (0.1) stay bounded in $C^{2,\alpha}(M)$. When ρ is in a compact subset of $(8\pi m, 8\pi(m+1))$ with m a positive integer, the same conclusion were obtained by Brezis and Merle [2] and Li and Shafrir [32]. Let

$$X_\alpha = \left\{ v \in C^{2,\alpha}(M) : \int_M v = 0 \right\}$$

for $0 < \alpha < 1$,

$$B_R = \{v \in X_\alpha : \|v\|_{C^{2,\alpha}(M)} < R\},$$

and $T(\rho) : X_\alpha \rightarrow X_\alpha$ be defined by

$$(0.2) \quad T(\rho) = \rho \Delta^{-1} \left(\frac{he^u}{\int_M he^u} - 1 \right).$$

For large R , due to the results mentioned above, the Leray-Schauder degree

$$d_\rho \equiv \text{deg}(I + T(\rho), B_R, 0)$$

is well defined when $\rho \neq 8\pi m$. Due to the homotopy invariance, d_ρ for $\rho \neq 8\pi m$

The purpose of this paper is to find a complete formula of d_ρ for $\rho \neq 8\pi m$. Our main result is as follows. Let g denote the genus of M and $\chi(M)$ be the Euler characteristic of M , that is, $\chi(M) = 2 - 2g$. For two integers m_1 and m_2 with $m_2 \geq m_1 \geq 0$, let

$$\binom{m_2}{m_1} = \begin{cases} \frac{m_2(m_2-1)\cdots(m_2-m_1+1)}{m_1!} & \text{for } m_1 > 0 \\ 1 & \text{for } m_1 = 0 \end{cases}$$

Theorem 1.1. *Assume $8\pi m < \rho < 8\pi(m+1)$ with m a nonnegative integer. Then $d_\rho = \binom{m-\chi(M)}{m}$, that is,*

$$d_\rho = \begin{cases} \frac{1}{m!}(-\chi(M)+1)(-\chi(M)+2)\cdots(-\chi(M)+m) & \text{for } m > 0 \\ 1 & \text{for } m = 0. \end{cases}$$

As an application of Theorem 1.1, we have

Theorem 1.2. *For $g \geq 1$, there is ρ_h depending on h such that if $\rho > \rho_h$, then (0.1) has a solution. Moreover, we can choose*

$$\rho_h = \max_M(2K - \Delta \log h),$$

where K is the Gaussian curvature of M .

The main ideas of the proof of Theorem 1.1 is as follows. To find d_ρ for $\rho \neq 8\pi m$, it suffices to calculate the jump values at $8\pi m$. Instead of choosing $h = 1$ as in [33] to calculate the degree, we take a Morse like function h . When ρ approaches $8\pi m$, we will show the solutions of (0.1) separate into two groups: bounded solutions and blowup solutions. For solutions in the first group, there is a uniform constant \bar{C}_m depending on m only such that the sup-norms of these solutions are bounded by \bar{C}_m . For solution in the second group, the sup-norms tend to infinity as $\rho \rightarrow 8\pi m$. Since the Leray-Schauder degree of bounded solutions remains constant when ρ across $8\pi m$, the jump value of d_ρ at $8\pi m$ depends on the difference of the degree of blowup solutions for $\rho = 8\pi m - \epsilon$ and the degree of blowup solutions for $\rho = 8\pi m + \epsilon$, where $\epsilon > 0$ is a small number. Therefore, to determine the jump value, it is essential to study the blowup behavior of (0.1).

Let u_i be a solution of (0.1) with $\rho = \rho_i$ satisfying $\int_M u_i = 0$. Assume $\rho_i \rightarrow 8\pi m$ and $\sup u_i \rightarrow \infty$ as $i \rightarrow \infty$. By the results of Brezis and Merle in [2] and Li and Shafrir in [32], there are m local maximum points $p_{i,j}$ for $j = 1, \dots, m$ of u_i such that $u_i(p_{i,j}) \rightarrow \infty$ and after passing to a subsequence, $p_{i,j} \rightarrow p_j$ for some p_j and

$$(0.4) \quad u_i(x) \rightarrow 8\pi \sum_j G(x, p_j)$$

in $C_{loc}^2(M \setminus \{p_1, \dots, p_m\})$, where $G(x, p)$ is the Green function of $-\Delta$ on M satisfying

$$(0.5) \quad \begin{aligned} -\Delta G(x, p) &= \delta_p - 1, \\ \int_M G &= 0. \end{aligned}$$

On the other hand, let $\tilde{u}_i = u_i - \log \int_M h e^{u_i}$ and

$$V_{i,j}(y) = \log \frac{e^{\tilde{u}_i(p_{i,j})}}{\left(1 + \frac{\rho_i h(p_{i,j})}{8} e^{\tilde{u}_i(p_{i,j})} |y|^2\right)^2}.$$

Then $\int_M h e^{\tilde{u}_i} = 1$ and it is known for suitable $p_{i,j}$, there is $C > 0$ such that

$$(0.6) \quad |\tilde{u}_i - V_{i,j}(y)| \leq C$$

on a small neighborhood of $p_{i,j}$, where y is an isothermal coordinates centered at $p_{i,j}$. For many applications, inequality (1.6) is not good enough since the neighborhood which it holds on may shrink to a point as $i \rightarrow \infty$. Using the method of moving planes, Li proved in [30] that (1.6) holds on a fixed domain containing $p_{i,j}$ and moreover, $|\tilde{u}_i(p_{i,j}) - \tilde{u}_i(p_{i,l})| < C$ for $j \neq l$. This turns out to be very essential for further studying phenomena related to blowing up.

For the purpose to calculate the Leray-Schauder degree, more sharp estimates for blowup behavior of equation (1.1) are needed. This has been done by the authors in the previous paper [15]. Let

$$\lambda_{i,j} = \tilde{u}_i(p_{i,j})$$

and

$$(0.7) \quad \tilde{G}(x, p) = \frac{1}{2\pi} \log \text{dist}(x, p) + G(x, p),$$

where $\text{dist}(x, p)$ is the distance between x and p . Let

$$\eta_{i,j} = \tilde{u}_i - V_{i,j}(y) - 8\pi[\tilde{G}(x, p) - \tilde{G}(p, p)]$$

in a neighborhood of $p_{i,j}$. We improved (1.6) in Theorem 1.4 of [15] and obtained

$$(0.8) \quad |\eta_{i,j}| \leq C e^{-\lambda_{i,j}} \lambda_{i,j}^2.$$

Also a more sharp estimate for (1.4) are obtained as follows

$$(0.9) \quad |u_i(x) - 8\pi \sum_j G(x, p_j)| \leq C e^{-\lambda_{i,1}} \lambda_{i,1}$$

for x outside a fixed neighborhood of $\{p_1, \dots, p_m\}$, see Lemma 5.3 in [15]. Based on these two estimates, further results concerning the location of $p_{i,j}$, the difference between $\lambda_{i,j}$ and $\lambda_{i,l}$, and the relation between $\rho_i - 8\pi m$, $\{\lambda_{i,j}\}$ and $\{p_{i,j}\}$ can be obtained as in Theorem 1.4 and Theorem 1.1 in [15]. That is, we have

$$(0.10) \quad \left. \nabla \log h(x) + 8\pi [\nabla \tilde{G}(x, p_{i,j}) + \sum_{l \neq j} G(x, p_{i,l})] \right|_{x=p_{i,j}} = O(e^{-\lambda_{i,j}} \lambda_{i,j}),$$

$$(0.11) \quad \begin{aligned} & \lambda_{i,j} + 8\pi \tilde{G}(p_{i,j}, p_{i,j}) + 8\pi \sum_{l \neq j} G(p_{i,l}, p_{i,l}) + 2 \log \frac{\rho_i h(p_{i,j})}{8} \\ &= \lambda_{i,k} + 8\pi \tilde{G}(p_{i,k}, p_{i,k}) + 8\pi \sum_{l \neq k} G(p_{i,k}, p_{i,l}) + 2 \log \frac{\rho_i h(p_{i,k})}{8} + O(e^{-\lambda_{i,j}} \lambda_{i,j}^2) \end{aligned}$$

$$(0.12) \quad \begin{aligned} \rho_i - 8\pi m &= \frac{2}{m} \sum_{j=1}^m (\Delta \log h(p_{i,j}) + 8\pi m - 2K(p_{i,j})) h(p_{i,j})^{-1} \frac{\lambda_{i,j}}{e^{\lambda_{i,j}}} \\ &+ O(e^{-\lambda_{i,j}}), \end{aligned}$$

where K is the Gaussian curvature of M . These relations contain a lot of information of blowup behavior. For example, if the signs on both sides of (1.12) are not the same, then blowup can not happen. When $m = 1$, this gives a further explanation of the result by Ding, Jost, Li and Wang in [20] which states that the minimizers of the variational problem corresponding to (1.1) can not blow up when ρ tends to 8π from below if the sign in the right hand side of (1.12) is positive.

As mentioned above, to find the jump values of d_ρ , it suffices to calculate the degree of blowup solutions. To do this, we will first construct approximate blowup solutions which satisfy (1.10), (1.11) and (1.12). Then the neighborhoods of these approximate solutions contain all blowup solutions. The problem now is reduced

to find all possible approximate blowup solutions and to calculate the action of the operator $T(\rho)$ defined in (1.2) on these neighborhoods. When $m > 1$, since there are more than one Green function in (1.10), it is hard to characterize all possible blowup points by using these relations. To overcome this difficulty, the key point is to explain the main term of (1.10) as the gradient of a function f_h defined on M^m , the m times product space of M . The function can be written as

$$f_h(x_1, x_2, \dots, x_m) = \sum_j \left[\log h(x_j) + 4\pi\varphi(x_j) + \sum_{l \neq j} 8\pi G(x_j, x_l) \right],$$

where $(x_1, x_2, \dots, x_m) \in M^m$ and $\varphi(x_j) = \tilde{G}(x_j, x_j)$ is the regular part of the Green function. Then it can be shown after ignoring the permutation of (x_1, x_2, \dots, x_m) , the approximate solutions are one to one corresponding to critical points of f_h on M^m which also satisfy the compatibility conditions (1.11) and (1.12). When considering an equation (see (0.16) below) similar to equation (0.1) on a bounded domain in \mathbf{R}^2 with a constant h , the function

$$f = \sum_j \left[4\pi\varphi(x_j) + \sum_{l \neq j} 8\pi G(x_j, x_l) \right]$$

has been used by Suzuki [46] and Nagasaki and Suzuki [38] to describe the locations of blowup points. When f has nondegenerate critical points, Baraket and Pacard [] constructed solutions with m bubbles.

Using a similar idea of Bahri and Coron in [1] (see also Li [31] for further application), the functions in a small neighborhood of an approximate solution can be represented in suitable terms orthogonal to each other. Then the dominant terms of $T(\rho)$ on the neighborhoods of the approximate solutions are found according to this representation. This is the main estimates of the paper and it will be stated in Lemma 3.2. When f_h is a Morse function and satisfies some additional nondegenerate condition at critical points, based on the calculation of the action of $T(\rho)$ near each approximate solution, the jump value of d_ρ can be found to equal plus or minus

$$(0.13) \quad \frac{1}{m!} \sum_{\nabla f_h(Q)=0} (-1)^{ind Q},$$

where $Q \in M^m$ and $ind Q$ is the Morse index of f_h at Q considered as a function on M^m . If we let

$$\Gamma = \{(x_1, x_2, \dots, x_m) \in M^m : x_j = x_k \text{ for some } j \neq k\},$$

the value in (1.13) can be further shown to be a topological invariant and to equal the Euler characteristic of $M \setminus \Gamma$. To get a final formula of d_ρ for $\rho \neq 8\pi m$, we need some more effort to calculate this Euler characteristic. It turns out this characteristic can be express in a very simple formula and d_ρ can be written down in the explicit form as in Theorem 1.1. When f_h is a Morse function on $M \setminus \Gamma$, our argument also obtains a formula for d_ρ at $8\pi m$. In this case, d_ρ depends on the function h .

Some phenomena similar to the ones of (1.1) can be found for the conformal scalar curvature equation

$$(0.14) \quad \Delta u - \frac{n(n-2)}{4}u + \frac{n-2}{4(n-1)}R(x)u^{\frac{n+2}{n-2}} = 0, u > 0$$

on S^n with $n \geq 3$. The Leray-Schauder degree for (1.14) is also close related to the critical points of $R(x)$. However, (1.1) is different from (1.14) in some aspects. The first is one needs to consider f_h on a product space M^m for (1.1), while one considers R , which is corresponding to the role play by f_h , on the space S^n itself for (1.14). This difference maybe is not essential since one can consider the function $\sum_{j=1}^m R(x_j)$ on the product space $(S^n)^m$ for equation (1.14) also. The second difference comes from the stronger interaction between the Green function $G(x, p)$ and the local data of a function near x for two dimension than for higher dimensions. For equation (1.1), we need to use some Green functions and h together to decide the locations of blowup points (that is, to decide the critical points of f_h and check the compatibility conditions (1.11) and (1.12)), while for equation (1.14), we only use R itself first to determine whether a point is a possible blowup point or not (that is, to find the critical points of R and the sign of ΔR on these points) and then only in a later step, the Green function is used to help determine the allowable combination of multiple blowing up. The effect of Green functions for (1.14) only comes in a more subtle or secondary way. For statements of theorems on (1.1), this fact may make the assumptions on h more complicated. To state the third different feature, we use (1.11) to write $\lambda_{i,j}$ in terms of $\lambda_{i,1}$ first and put them in to (1.12). Then it follows that

$$(0.15) \quad \rho - 8\pi m = l(Q)e^{-\lambda_{i,1}}\lambda_{i,1} + O(e^{-\lambda_{i,1}}),$$

for some $l(Q)$ which is given in (2.26) in Section 2 below, where Q is some critical point of f_h . For (1.14), both ∇R and ΔR are important for deciding the blowup points, but for (1.1), instead of using Δf_h , we use ∇f_h and $l(Q)$. It is $l(Q)$, not Δf_h , plays the role for (1.1) as ΔR for the higher dimensional problem. Usually, $l(Q)$ is not equal to $\Delta f_h(Q)$ at a critical point Q .

There is one more difference between (1.1) and (1.14). To construct an approximate solution, we need to glue bubbles near blowup points and Green functions away from blowup points together. As suggested by (1.10) and (1.12), the difference between a solution and the corresponding approximate solution should be within the order $e^{-\lambda_{i,j}} \lambda_{i,j}$. Unfortunately, in general, the error term defines in (0.8) is of order $e^{-\lambda_{i,j}} \lambda_{i,j}^2$ when h is not a constant. Therefore it is hard to obtain good approximate solutions. However, by Theorem 1.4 in [15],

$$\eta_{i,j}(x) = c_o e^{-\lambda_{i,j}} \lambda_{i,j}^2 + O(e^{-\lambda_{i,j}} \lambda_{i,j})$$

when x is away from blowup points, where c_o is a constant. Hence, we can add the constants $c_o e^{-\lambda_{i,j}} \lambda_{i,j}^2$ to the approximate solution around neighborhoods of blowup points and overcome the difficulty. For higher dimension problem (1.14), no such error term $\eta_{i,j}$ appears.

In [15], we showed similar sharp estimates for bubble solutions also hold for a similar equation

$$(0.16) \quad \begin{cases} \Delta u + \rho \frac{he^u}{\int_{\Omega} he^u} = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a $C^{2,\beta}$ bounded domain in \mathbf{R}^2 , $\rho \in \mathbf{R}$, and h is a positive $C^{2,\beta}$ function on the closure $\bar{\Omega}$ of Ω . When $\rho \neq 8\pi m$, since the sup-norm estimate holds, we can define the Leray-Schauder degree for (0.16)

$$d_{\Omega,\rho} \equiv \deg(Iu + \rho \Delta^{-1} \left(\frac{he^u}{\int_{\Omega} he^u} \right), B_R, 0).$$

Since the solutions of (0.16) can not blowup at the boundary, the method to calculate the degree for (0.1) can also apply to (0.16). Let $\chi(\Omega)$ denote the Euler characteristic of Ω , that is, $\chi(\Omega) = 1 - g_{\Omega}$ with g_{Ω} the genus of Ω . We have a similar formula for $d_{\Omega,\rho}$ as the one in Theorem 1.1.

Theorem 1.3. *Assume $8\pi m < \rho < 8\pi(m+1)$ with m a nonnegative integer. Then $d_{\Omega,\rho} = \binom{m-\chi(\Omega)}{m}$, that is,*

$$d_{\Omega,\rho} = \begin{cases} \frac{1}{m!} (-\chi(\Omega) + 1)(-\chi(\Omega) + 2) \cdots (-\chi(\Omega) + m) & \text{for } m > 0 \\ 1 & \text{for } m = 0. \end{cases}$$

References

- [1] A. Bahri and J.M. Coron, The scalar-curvature problem on the standard three-dimensional sphere. *J. Funct. Anal.* 95 (1991), no. 1, 106–172.
- [2] Brezis, H. & Merle, F., Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equation* 16(1991), 1223-1254.
- [3] H. Brezis, Y.Y. Li and I. Shafrir, A sup + inf inequality for some nonlinear elliptic equations involving exponential nonlinearities, *J. Functional Analysis* 115(1993), 344-358.
- [4] L. Caffarelli and Y. Yang, Vortex condensation in the Chern-Simons Higgs model: an existence theorem, *Comm. Math. Phys.* 168 (1995), 321-336.
- [5] E. Caglioti, P.L. Lions, C. Marchioro, A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description, *Comm. Math. Phys.* 143 (1992), 501-525.
- [6] E. Caglioti, P.L. Lions, C. Marchioro, and M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description, part II, *Comm. Math. Phys.* 174 (1995), 229-260.
- [7] L. Carleson and S.Y. Chang, On the existence of an extremal function for an inequality of Moser, *Bull. Sci. Math.* 110 (1986), 113-127.
- [8] S. Chanillo and M. Kiessling, Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry, *Comm. Math. Phys.*, 160 (1994), 217-238.
- [9] S.Y. Chang, M. Gursky and P. Yang, The scalar curvature equation on 2- and 3-spheres. *Calc. Var. Partial Differential Equations* 1 (1993), no. 2, 205–229.
- [10] S.Y. Chang and P. Yang, A perturbation result in prescribing scalar curvature on S^n , *Duke Math. J.* 64 (1991), 27-69.
- [11] C.C. Chen and C.S. Lin, A sharp sup+inf inequality for a nonlinear equation in \mathbf{R}^2 , *Comm. Anal. Geom.* 6 (1998), 1-19.

- [12] C.C. Chen and C.S. Lin, On the symmetry of blowup solutions to a mean field equation, preprint.
- [13] C.C. Chen and C.S. Lin, Estimate of the conformal scalar curvature equation via the method of moving planes. II. *J. Differential Geom.* 49 (1998), no. 1, 115–178.
- [14] C.C. Chen and C.S. Lin, preprint.
- [15] C.C. Chen and C.S. Lin, preprint.
- [16] W. Chen, Remarks on the existence of branch bubbles on the blowup analysis of equation $-\Delta u = e^{2u}$ in dimension two, *Comm. Anal. Geom.*, to appear.
- [17] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.*, 63 (1991), 615-623.
- [18] K.S. Cheng and C.S. Lin, conformal metrics with prescribed nonpositive Gaussian curvature on R^2 . *Calc. Var. Partial Differential Equations* 11 (2000), no. 2, 203–231.
- [19] K.S. Cheng and C.S. Lin, Multiple solutions of conformal metrics with negative total curvature. *Adv. Differential Equations* 5 (2000), no. 10-12, 1253–1288.
- [20] W. Ding, J. Jost, J. Li and G. Wang, The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface, *Asian J. Math.* 1 (1997), 230-248.
- [21] W. Ding, J. Jost, J. Li and G. Wang, An analysis of two-vortex case in Chern-Simons Higgs model, *Calc. Var.* 7 (1998), 87-97.
- [22] W. Ding, J. Jost, J. Li and G. Wang, Existence results for mean field equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 16 (1999), no. 5, 653–666.
- [23] W. Ding, J. Jost, J. Li and G. Wang, Multiplicity results for the two-vortex Chern-Simons Higgs model on the two-sphere. *Comment. Math. Helv.* 74 (1999), no. 1, 118–142.
- [24] B.Gidas, W.M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in \mathbf{R}^n , *Math. Anal. and Applications, Part A, Advances in Math. Suppl. Studies* 7A, (ed. L. Nachbin), Academic Pr. (1981), 369-402.
- [25] Gilbarg and Trudinger

- [26] J. Hong, Y. Kim and P.Y. Pac, Multivortex solutions of the Abelian Chern Simons theory, Phys. Rev. Letter 64 (1990), 2230-2233.
- [27] R. Jackiw and E.J. Weinberg, Selfdual Chern Simons vortices, Phys. Rev. Lett. 64 (1990), 2234-2237.
- [28] J. Kazdan and F. Warner, Curvature functions for compact 1-manifolds, Ann. of Math. 99 (1974), 14-47
- [29] M.H.K. Kiessling, Statistical mechanics of classical particles with logarithmic interaction, Comm. Pure Appl. Math. 46 (1993), 27-56.
- [30] Y.Y. Li, Harnack type inequality: the method of moving planes, Comm. Math. Phy., 200(1999), 421-444.
- [31] Y.Y. Li Prescribing scalar curvature on S^n and related problems, Part II: Existence and compactness, Comm. Pure Appl. Math. 49 (1996), 541-597.
- [32] Y.Y. Li and I. Shafrir, Blowup analysis for solutions $-\Delta u = Ve^u$ in dimension two, Indiana Univ. Math. J. 43 (1994), 1255-1270.
- [33] C.S. Lin, Topological degree for mean field equations on S^2 . Duke Math. J. 104 (2000), no. 3, 501–536.
- [34] C.S. Lin, Uniqueness of solutions to the mean field equations for the spherical Onsager vortex. Arch. Ration. Mech. Anal. 153 (2000), no. 2, 153–176.
- [35] C.S. Lin, Uniqueness of conformal metrics with prescribed total curvature in R^2 . Calc. Var. Partial Differential Equations 10 (2000), no. 4, 291–319.
- [36] J.L. Moseley, Asymptotic solutions for a Dirichlet problem with an exponential nonlinearity, SIAM, J. Math. Anal. 14 (1983), 719-735.
- [37] J.L. Moseley, A two-dimensional Dirichlet problem with an exponential nonlinearity, SIAM J. Math. Anal. 14 (1983), 934-946.
- [38] K. Nagasaki and T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearity, Asymptotic Analysis 3(1990), 173-188.

- [39] M. Nolasco and G. Tarantello, On a sharp type inequality on two dimensional compact manifolds, *Arch. Rational Mech. Anal.* 145 (1998) 161-195.
- [40] M. Nolasco and G. Tarantello, Double vortex condensates in the Chern-Simons-Higgs theory. *Calc. Var. Partial Differential Equations* 9 (1999), no. 1, 31–94.
- [41] M. Nolasco and G. Tarantello, Vortex condensates for the $SU(3)$ Chern-Simons theory. *Comm.Math. Phys.* 213 (2000), no. 3, 599–639.
- [42] T. Riciardi and G. Tarantello, On a periodic boundary value problem with exponential nonlinearities. *Differential Integral Equations* 11 (1998), no. 5, 745–753.
- [43] T. Riciardi and G. Tarantello, Vortices in the Maxwell-Chern-Simons theory. *Comm. Pure Appl. Math.* 53 (2000), no. 7, 811–851.
- [44] J. Spruck and Y. Yang, Topological solutions in the self-dual Chern-Simons theory: existence and approximation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 12(1995), 75-97.
- [45] M. Struwe and G. Tarantello, On multivortex solutions in Chern-Simons Gauge theory, *Boll. Unione Math. Ital. Sez. B Artic. Ric. Mat.* (8) 1 (1998), 109-121.
- [46] T. Suzuki, Global analysis for a two-dimensional elliptic eigenvalues problem with the exponential nonlinearity, *Ann. Inst. Henri Poincaré, Analyse nonlinéaire* 9(1992), 367-398.
- [47] G. Tarantello, Multiple condensate solutions for the Chern-Simons-Higgs theory, *J. Math. Phys.* 37(1996), 3769-3796.
- [48] C.H. Taubes, Arbitrary N-vortex solutions to the first order Ginzburg-Landau equation, *Comm. Math. Phys.* 72 (1980), 277-292.
- [49] C.H. Taubes, On the equivalence of the first and second order equations for gauge theories, *Comm. Math. Phys.* 75 (1980), 207-227.
- [50] V.H. Weston, On the asymptotic solution of a partial differential equation with an exponential nonlinearity, *SIAM J. Math. Anal.* 9(1978), 1030-1053.
- [51] S.W. Yang