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摘　　要

在一圖(graph)上面，我們考慮其中的邊縮成一點時，產生新的圖。利用這種過程，在一個特定範圍內的圖將形成一鏈複體(chain complex)。由此得到的homology叫做"Graph homology"。我們希望Graph homology有一些消逝定理，由此證明Perturbative Chern-Simons理論中的重要結果。

關鍵詞：

The Homology Theory of Graphs

by

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Abstract

For a graph, the contraction of its edges produces some other graphs. The contractions are considered as the boundary operator, and we have a chain complex of graphs. Motivated from the theory of knot invariant, we consider the graphs with a special fixed set of points as part of the vertices and the associated chain complex. If we can get some particular results for such chain complexes of graphs, we can show the important results for the knot invariants from the perturbative Chern-Simons theory.

A graph is an abstract 1-dimensional simplicial complex, that is, a set of vertices and a set of edges, each edge consists of two distinct vertices.

We shall define chain complexes of modules (or vector spaces), each module is freely generated by a set of graphs. Similar to the simplexes in simplicial homology, the graphs need orientations. There are different methods to define the orientation of graph, one of the simplest way is to choose a linear order for the edges, two orders represent the same orientation if they are different by an even permutation. For a graph, a simplicial isomorphism from the graph to itself is said to be an automorphism, if the simplicial isomorphism preserves all structures which are assigned to the graph; the set of all automorphisms forms a group which is called the automorphism group of the graph. If there is an automorphism of the graph, which reverses the orientation of the graph, then this graph is said to be non-orientable. The non-orientable graphs are considered as zero in the chain complex, when we use the real number as the coefficients.

Next important thing for chain complexes is the boundary operator. For a graph and an edge of the graph, we can contract this edge to a vertex and get a quotient graph;

the summation of all such quotient graphs multiplied by a proper sign is defined as the boundary of the graph. Suppose we use the linear orders of edges as the orientation and Γ is a graph with orientation, then the boundary $\partial(\Gamma) = \sum(-1)^j \partial^{(j)}(\Gamma)$, where $\partial^{(j)}(\Gamma)$ is the quotient graph of Γ by contracting the j-th edge, with orientation the restriction order. Similar to the boundary operator in simplicial homology, $\partial(\partial(\Gamma)) = 0$, it is the only geometric property of a chain complex.

Remark: In Perturbative Chern-Simons theory, the graphs represent differential forms and the boundary operator is exactly part of exterior differentiation of the associative differential forms. Thus the above homology theory of graphs is usually called the graph cohomology as in Bott and Cattaneo []. But, in spirit, it is a homology theory and is dual to the cohomology theory of differential forms by the Stoke's Theorem.

1 Graphs and chain complex

1.1 Based graphs

In this subsection, we will introduce the notion of graphs based on a set.

We always assume these graphs with some particular points as part of the vertices, such vertices are called the base points of the graphs and the graphs are called the based graphs.

Definition 1.1.1: Suppose $x_1, x_2, \cdots, x_m$ are m distinct points. A graph Γ is said to be a graph based on the ordered set $(x_1, x_2, \cdots, x_m)$, if the points $x_1, x_2, \cdots, x_m$ are part of vertices of Γ. The points $x_1, x_2, \cdots, x_m$ are called the base points of Γ.

Notations 1.1.2:

(i) We use Γ, Γ', Γ_1, Γ_2 to denote the graphs.

(ii) For a graph Γ, $V(\Gamma)$ denotes the set of all vertices of Γ and $\mathcal{E}(\Gamma)$ denotes the set of all edges in Γ. Thus, for any $E \in \mathcal{E}(\Gamma)$, $E = \{v, w\}$, $v, w \in V(\Gamma)$, and $v \neq w$.

(iii) The vertices other than the base points are called the inner vertices of Γ.

Equivalence of based graphs

Definition 1.1.3: Suppose Γ_1 and Γ_2 are two graphs based on $(x_1, x_2, \cdots, x_m)$. A bijection $f : V(\Gamma_1) \longrightarrow V(\Gamma_2)$ is an equivalence of based graphs, if $f(x_i) = x_i, i = 1, 2, \cdots, m$, $f(E) \in \mathcal{E}(\Gamma_2)$, for $E \in \mathcal{E}(\Gamma_1)$, and $f^{-1}(E') \in \mathcal{E}(\Gamma_1)$, for $E' \in \mathcal{E}(\Gamma_2)$. (The last two conditions on the edges are the conditions for the bijection f to be a simplicial isomorphism.)

If two graphs are equivalent, it is hard to distinguish one from the other. Thus we need only to choose one graph from each equivalence class of graphs, or just consider the whole equivalence class instead of the particular graph. There are a few assumptions which are crucial to our result. Under these restrictions on the graphs, there are only finite number of equivalence classes of based graphs.(See the **Proposition 1.1.5**).

Assumption 1.1.4:

(i) **Valency Assumption:** The valency of a vertex in a graph is the number of edges which contain the vertex. Now, we assume that every inner vertices of a based graph are of valency at least 3. For the base points x_i, there is no restriction on the valency, it could be 0 or any positive integer.

(ii) **Order Restriction:** The order of a graph is defined as the number of its edges minus the number of its inner vertices. The definition of order makes sense only under the above Valency Assumption.

Because the order is invariant under edge contraction, we usually fix the order of graphs in a chain complex.

Proposition 1.1.5: Under the valency assumption and the order restriction, we have only finite number of equivalence classes of graphs based on $(x_1, x_2, \cdots, x_m)$.

Proof: First, we show that there are only finite graphs in which the inner vertices are all of valency exact 3.

Assume the order of graph is n.

Let s_i denote the valency of the base point x_i, $i = 1, 2, \cdots, m$, $s = s_1 + s_2 + \cdots + s_m$, r denote the number of inner vertices and k be the number of edges.

Then $s + 3r = 2k$.

Thus $r \leq s + r = 2k - 2r = 2n$,

and hence, $k = n + r \leq 3n$.

Up to equivalence, we may assume that all the graphs have the vertices in the set $\{x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_{2n}\}$. Thus the possible graphs is finite.

From such "trivalent" graphs, we can have all the order n graphs by contracting the edge a finite number of times, and each "trivalent" graph can only produce finite number of graphs. Thus all possible order n graphs are finite.

1.2 Chain complex of graphs

All graphs satisfy the valency assumption: the valency of inner vertices are at least three.

We also need a number, the degree of graph. Suppose the vertices of graphs are points of α-dimensional Euclidean space. (We may say that the graphs are on $\mathbf{R}^\alpha$.) Each edge gives a $(\alpha-1)$-form and each inner vertex can move in $\mathbf{R}^\alpha$. If Γ has k edges and r inner vertices, then the integration of the wedge of the k $(\alpha-1)$-form on the $r \times \alpha$-dimensional configuration space produces a differential form of degree $k \times (\alpha-1) - r \times \alpha$. But the degree is only good for cohomology theory. To get a homology theory, we define the degree as $-k \times (\alpha - 1) + r \times \alpha$. (For the details of the relations of graphs and the differential forms, please see the historical backgrounds in Appendix.) For the purpose of our main applications, we may consider only the graphs in $\mathbf{R}^2$ temporarily. We shall also be interested in the graphs in $\mathbf{R}^3$ in the following related works. Thus we give the following definition.

Definition 1.2.1: (Degree of plane graph) The degree of a based graph is the double of the number of its inner vertices minus the number of its edges.

Definition 1.2.2: For a positive integer m and an integer i, let C_i^m be the vector space over the real number generated by all oriented degree i graphs based on $(x_1, x_2, \cdots, x_m)$, modulo the following **Orientation-equivalence relation**:

> Two oriented graphs having an orientation preserving equivalence between them are considered as the same element in C_i^m; if the equivalence is orientation reversing, one is equal to the other multiplied by a negative sign.

Orientation systems

There are two different orientation systems, including the linear order of edges mentioned previously.

Orientation system (I): (Linear order of edges)

It is the orientation system used in the proof of the theorems in the paper.

Suppose a based graph Γ has k distinct edges $E_1, E_2, \cdots, E_k$. Consider a linear order $(E_1, E_2, \cdots, E_k)$, all the informations of the graph are contained in $(E_1, E_2, \cdots, E_k)$. Thus we still use $(E_1, E_2, \cdots, E_k)$ to denote the oriented graph, and also use the same notation to denote the corresponding element in C_i^m. Interchanging the positions of two edges in a linear order set, we get the negative element. Thus, if $E_j = E_{j'}$, for some $1 \leq j < j' \leq k$, then $(E_1, E_2, \cdots, E_k) = 0$; ordinarily, we do not meet such an oriented graph, but it does happen on the "degenerate boundary" of a graph discussed in the following section.

If $f : V(\Gamma) \longrightarrow V(\Gamma')$ is an equivalence of based graphs from Γ to Γ', then $(f(E_1), f(E_2), \cdots, f(E_k))$

is an oriented graph and

$$(f(E_1), f(E_2), \cdots, f(E_k)) = (E_1, E_2, \cdots, E_k) .$$

When $\Gamma = \Gamma'$, f is an automorphism and $(f(E_1), f(E_2), \cdots, f(E_k))$ is a permutation of $(E_1, E_2, \cdots, E_k)$; if the permutation is odd, then the above equality implies that $(E_1, E_2, \cdots, E_k) = 0$ in C_i^m. In this situation, Γ is said to be non-orientable.

Orientation system (II): (Linear order of vertices together with directions on every edges)

If the graphs are on $\mathbb{R}^3$, we should get this orientation system from the differential forms. (For the graphs on $\mathbb{R}^2$, we get the Orientation system (I).)

For an orientation of Γ, we need to choose a linear order of the vertices of Γ and directions on every edges of Γ. Two such orientations for Γ are the same , if the total number of changes in the order of vertices and the directions of the edges is an even number. Because the direction of an edge is also an order of the two endpoints, we may get the order by restricting the linear order of vertex set. Thus the linear order $(v_1, v_2, \cdots, v_s)$ of vertices can determine an orientation, denoted by $[v_1, v_2, \cdots, v_s]$, in the Orientation system (II).

What happen when interchanging the positions of two vertices? If $\{v_1, v_2\}$ is an edge of Γ, then

$$[v_2, v_1, \cdots, v_s] = [v_1, v_2, \cdots, v_s] ;$$

if $\{v_1, v_2\}$ is not an edge of Γ, then

$$[v_2, v_1, \cdots, v_s] = -[v_2, v_1, \cdots, v_s] .$$

We may also have the non-orientable based graph in the Orientation system (II).

Remark 1.2.3: Because the two orientation systems have different non-orientable based graphs for the same numbers m and i, C_i^m can not be the same vector space in the two systems. But the main results of the paper hold in both systems. (The proofs are also completely similar in both systems.)

The boundary operator

To obtain a chain complex, we need a boundary operator from C_i^m to C_{i-1}^m. For both orientation systems, the boundary operator can be defined. Here we define it only in Orientation system (I). For Orientation system (II), please see Bott and Cattaneo [].

The boundary operator is the sum of edge-contractions, but the edges consisting of two base points can not be contracted, such edges shall be shown to have essential contributions to the graph homology. Thus we need the following definitions.

Definition 1.2.4: (Basic edges) Suppose Γ is a graph based on $(x_1, x_2, \cdots, x_m)$. An edge E is said to be a basic edge, if E consists of two base points, say, x_{j_1}, x_{j_2}, $1 \le j_1 < j_2 \le m$; otherwise, it is called the non-basic edge.

Thus a non-basic edge may consists of two inner vertices, or, a base point together with an inner vertex.

Suppose Γ is a graph based on $(x_1, x_2, \cdots, x_m)$ and $(E_1, E_2, \cdots, E_k)$ is a linear order of the edges of Γ.

For each non-basic edge $E_j, 1 \le j \le k$, let $\pi_j : V(\Gamma) \longrightarrow V(\Gamma)/E_j$ denote the quotient map. Then $(\pi_j(E_1), \pi_j(E_2), \cdots, \pi_j(E_{j-1}), \pi_j(E_{j+1}), \cdots, \pi_j(E_k))$ is a linear order of edges of the quotient graph Γ/E_j.

Convention 1.2.5: If the edge E_j consists of a base point x_l and an inner vertex v, then we identify the quotient point $\pi_j(x_l)(= \pi_j(v))$ with the base point x_l.

Thus the quotient graph Γ/E_j still be a graph based on $(x_1, x_2, \cdots, x_m)$ and $(\pi_j(E_1), \pi_j(E_2), \cdots, \pi_j($ is the associated oriented graph; it is the j-th boundary of $(E_1, E_2, \cdots, E_k)$ as in the following notation.

Notation 1.2.6: If E_j is a non-basic edge of Γ,

$$\partial^{(j)}(E_1, E_2, \cdots, E_k) = (\pi_j(E_1), \pi_j(E_2), \cdots, \pi_j(E_{j-1}), \pi_j(E_{j+1}), \cdots, \pi_j(E_k)) \ .$$

If E_j is a basic edge of Γ, $\partial^{(j)}(E_1, E_2, \cdots, E_k) = 0$.

Now, we can define the boundary operator

$$\partial : C_i^m \longrightarrow C_{i-1}^m$$

by the formula

$$\partial(E_1, E_2, \cdots, E_k) = \sum_{j=1}^{k} (-1)^j \partial^{(j)}(E_1, E_2, \cdots, E_k)$$

For a non-basic edge E_j, $\partial^{(j)}(E_1, E_2, \cdots, E_k)$ usually is not equal to zero except that the graph Γ/E_j is non-orientable or the edge E_j is part of a triangle, as described below.

Degenerate boundary (Triangular edge)

$E_j = \{v, w\}$ is said to be a triangular edge, if there is a vertex u of Γ such that both $\{u, v\}$ and $\{u, w\}$ are edges of Γ. For the edge E_j, we consider the boundary $\partial^{(j)}(E_1, E_2, \cdots, E_k)$. $\pi_j(\{u, v\}) = \pi_j(\{u, w\})$. When interchanging the two edges, we have the equality

$$\partial^{(j)}(E_1, E_2, \cdots, E_k) = -\partial^{(j)}(E_1, E_2, \cdots, E_k) \ .$$

Thus $\partial^{(j)}(E_1, E_2, \cdots, E_k) = 0$.

Therefore, for any positive integer m, we have the chain complex

$$\mathcal{C}^m = \{C_i^m, \quad \partial : C_i^m \longrightarrow C_{i-1}^m, i = \cdots, -2, -1, 0, 1, \cdots\}$$

Our main result is the following theorem.

Theorem 1 Let $C(m, \mathbf{R}^2) = \{(z_1, z_2, \cdots, z_m) \in \prod_m \mathbf{R}^2 : z_p \neq z_q, \text{ for } p \neq q\}$, the configuration space of m distinct points in the plane. Then

$$H_i(\mathcal{C}^m) \cong H^{-i}(C(m, \mathbf{R}^2), \mathbf{R})$$

where $H^{-i}(C(m, \mathbf{R}^2), \mathbf{R})$ is the cohomology group of the space $C(m, \mathbf{R}^2)$ with the coefficient $\mathbf{R}$ at the dimension $-i$. ▮

Thus $H_i(\mathcal{C}^m)$ is non-zero only at the degree ≤ 0.

Remark 1.2.7: The homotopy structure of $C(m, \mathbf{R}^2)$ is easy to describe, this space is homotopy equivalent to the product space $X_{m-1} \times X_{m-2} \times \cdots \times X_1$, where $X_p = S^1 \vee S^1 \vee \cdots \vee S^1$, the wedge of p copies of S^1, S^1 is the unit circle. Thus we can compute the graph homology easily by this theorem. We give some applications in the following.

Splitting $\mathcal{C}^m$ by the order restriction

As mention before, the order of a graph is the number of edges minus the number of inner vertices and it is invariant under the boundary operator.

Let $\mathcal{C}^{m,n} = \{C_i^{m,n}\}$ be the subchain complex of $\mathcal{C}^m$ generated by the order n oriented graphs based on $(x_1, x_2, \cdots, x_m)$. Then

$$\mathcal{C}^m = \bigoplus_n \mathcal{C}^{m,n}$$

We have shown that there are only finite equivalence classes for a fixed number of base points m and a fixed order n. Thus $\mathcal{C}^{m,n}$ is a vector space of finite rank (finite dimension), $\mathcal{C}^m$ can not be a finite rank vector space.

There are some trivial examples:

(i) For any positive integer m, the unique graph of order 0 is the graph Γ_0 without any edge ($\mathcal{E}(\Gamma_0)$ is empty), it is in degree 0. Thus $C_0^{m,0} = \mathbf{R}$, $C_i^{m,0} = 0$, for $i \neq 0$, and the homology $H_i(\mathcal{C}^{m,0}) = C_i^{m,0}$, for all degree i.

(ii) The graph of order 1 is a graph with one edge and no inner vertex. Thus all the order 1 graph are of degree -1. $H_i(\mathcal{C}^{m,1}) = C_i^{m,1} = 0$, except $i = -1$.

(iii) Consider the case $m = 2$. The unique graph of order 1 is the graph with the unique basic edge $\{x_1, x_2\}$. Thus $H_i(\mathcal{C}^{2,1}) = \mathcal{C}_i^{2,1} = 0$, for $i \neq -1$, and $H_{-1}(\mathcal{C}^{2,1}) = \mathcal{C}_{-1}^{2,1} = \mathbf{R}$.

Consider the "simplest" configuration space of 2 points, $C(2, \mathbf{R}^2)$, it is homotopy equivalent to S^1. By the theorem, $H_i(\mathcal{C}^2) = 0$, for $i \neq -1$, and $H_{-1}(\mathcal{C}^2) = \mathbf{R}$. The results $H_{-1}(\mathcal{C}^{2,1}) = \mathbf{R}$ and $H_0(\mathcal{C}^{2,0}) = \mathbf{R}$ imply that $H_i(\mathcal{C}^{2,n}) = 0$, for all order $n \geq 2$ and all degree i. This simple fact is related to the problem of zero-anomaly in the perturbative Chern-Simons theory for knot invariant.

Using the same way, we can compute all homology of $\mathcal{C}^{m,n}$ easily. For example, $H_{1-m}(\mathcal{C}^{m,m-1})$ has rank $(m-1)!$, it is the lowest degree in which the homology of $\mathcal{C}^m$ is non-zero, and $H_i(\mathcal{C}^{m,m-1}) = 0$, for $i \neq 1 - m$. These are important to the theory of knot invariant.

2 The subchain complexes of $\mathcal{C}^{m,n}$

In Section 1, we find that when the number of base points, m, and the order of graphs, n, are fixed, the homology of the chain complex $\mathcal{C}^{m,n}$ are completely determined. And for many cases, the homology of $\mathcal{C}^{m,n}$ are trivial group, such chain complexes $\mathcal{C}^{m,n}$ are called acyclic. To apply the acyclic results of $\mathcal{C}^{m,n}$ from Section 1, we should find a subchain complex of $\mathcal{C}^{m,n}$, which is generated by the based graphs with trivalent inner vertices, and is chain homotopy equivalent to $\mathcal{C}^{m,n}$.

2.1 A filtration of $\mathcal{C}^{m,n}$

We need some notations to formulate the theory.

Definition 2.1.1: Suppose Γ is a graph based on $(x_1, x_2, \cdots, x_m)$. An edge E of Γ is said to be an **inner edge**, if E consists of two inner vertices. An edge E of Γ is said to be an **outer edge**, if E consists of a base point and an inner vertex.

Thus these two types of edges are the edges we can make the edge contraction, and the edges other than these two types are the **basic edges** which contain two base points.

It is easy to see that the inner vertices and inner edges form a subgraph, which is called the **inner subgraph** of the based graph.

For a based graph Γ, let s denote the number of outer edges of Γ, t denote the number of inner edges of Γ, and u denote the number of basic edges of Γ. Thus Γ has $s + t + u$ edges. In the theory of knot invariants, the uni-trivalent graphs are considered and the number $s + 2u$ is related to the number of univalent vertices, i.e., the number of legs. We shall use the number $s+u$ to give a filtration of $\mathcal{C}^{m,n}$. Precisely, for a based graph Γ, let $p = n - s - u$ denote the base degree of Γ. (n, the order, is fixed, thus p is just a shift of $s+u$.)

Let $\mathcal{F}(p)$ denote the subchain complex of $\mathcal{C}^{m,n}$ generated by the based graphs with base degree not larger than p. (or, equivalently, generated by the based graphs for which the number of outer edges and basic edges is at least $n-p$.)

$$\mathcal{C}^{m,n} = \mathcal{F}(n) \supset \mathcal{F}(n-1) \supset \cdots \supset \mathcal{F}(0) \supset \mathcal{F}(-1) \supset \cdots \supset \mathcal{F}(-n)$$

2.2 The double sequence $\mathcal{D}_{p,q}$

Let $\mathcal{D}_p^{m,n}$ be the quotient chain complex $\mathcal{F}(p)/\mathcal{F}(p-1)$, its degree i vector space $(\mathcal{D}_p^{m,n})_i$ shall be denoted by $\mathcal{D}_{p,i-p}^{m,n}$, or, simply, $\mathcal{D}_{p,i-p}$. Then we have a double sequence of vector spaces $\{\mathcal{D}_{p,q}, p = \cdots, -1, 0, 1, 2, \cdots, q = \cdots, -2, -1, 0, \cdots\}$. The vector space $\mathcal{D}_{p,q}$ can be interpreted as the real vector space generated by the degree $p+q$ based graphs for which the number of outer edges and basic edges are equal to $n-p$. The number p and $q = i - p$ is an important number of the graphs and there are some observations as follows:

(i) For a based graph Γ, let r denote the number of inner vertices. Then the order $n = s + t + u - r$, thus $p = t - r$. p is the negative Euler Characteristic of the inner subgraph.

(ii) The degree $i = 2r - (s+t+u)$. $q = (2r - s - t - u) - (t - r) = 3r - s - 2t$. $s + 2t$ is equal to the sum of valencies of all inner vertices. By assumption, the valency of inner vertex is at least 3, thus $-q$ is the summation of (Valency $-$ 3) over all inner vertices. q is a non-positive integer.

(iii) If $q = 0$ for a based graph, then this graph is a trivalent graph (the valency of inner vertices are all equal to three).

The study of the chain complex $\mathcal{D}_p^{m,n}$ is an important step for the further works. As Yu's work, in many cases (order $n \le 5$), the homologies of $\mathcal{D}_p^{m,n}$ are all equal to 0 except $q = 0$. Thus, originally, we conjecture that the homologies of $\mathcal{D}_p^{m,n}$ are all equal to 0 except $q = 0$, for any m, n, and p. (In Section 4, we prove the conjecture for the case that $m = 2$ and $n = 5$.) If the conjecture is true, then the acyclic property $\mathcal{C}^{m,n}$ implies the acyclic property of $\mathcal{K}^{m,n} = \{K_p^{m,n}, \overline{\partial}\}$, where $K_p^{m,n}$ is the kernel of $\partial : \mathcal{D}_{p,0} \longrightarrow \mathcal{D}_{p,-1}$ and $\overline{\partial} : K_p^{m,n} \longrightarrow K_{p-1}^{m,n}$ is a natural boundary operator induced from $\mathcal{C}^{m,n}$. Thus an element of $K_p^{m,n}$ is a linear combination of oriented trivalent based graphs. By Bott and Taubes' result, the formal boundary relationship of graphs produces the same formula for the corresponding differential forms. But, after a long time trying, we can not find a reasonable method to attack the conjecture. Therefore, we weaken the conjecture such that it is still good enough for our purpose. Finally, we find that the new conjecture looks much more reasonable. For our special purpose, we may restricts ourselves to the based graphs in which the inner subgraphs are connected.

Definition 2.2.1: Suppose Γ is a graph based on $(x_1, x_2, \cdots, x_m)$. Let $\Gamma^\diamond$ denote the subgraph of Γ consisting of all the inner vertices of Γ and the edges connecting two inner vertices in Γ. The graph $\Gamma^\diamond$ shall be called the **inner subgraph** of Γ. Γ is said to be **quasi-connected**, if the inner subgraph $\Gamma^\diamond$ of Γ is connected.

Definition 2.2.2: The **loop number** of Γ shall denote the loop number of its inner subgraph $\Gamma^\diamond$, that is, the first betti-number of $\Gamma^\diamond$. If Γ is quasi-connected, then its loop number is equal to the number of inner edges minus the number of inner vertices and plus 1.

Obviously, for a quasi-connected based graph, the loop number l is exactly equal to $p+1$ (note: $p = t - r$). Because the loop number can be read from the graphs directly, we shall use the number l instead of the number p when we consider only the quasi-connected graphs.

Let $\mathcal{Q}_q^{m,n,l}$ be the sub linear space of $\mathcal{D}_{l-1,q}^{m,n}$ generated by all quasi-connected based graphs with order n and with loop number l. Furthermore, let $\mathcal{Q}^{m,n,l}$ denote the following chain complex:

$$\cdots \xrightarrow{\partial_I} \mathcal{Q}_0^{m,n,l} \xrightarrow{\partial_I} \mathcal{Q}_{-1}^{m,n,l} \xrightarrow{\partial_I} \mathcal{Q}_{-2}^{m,n,l} \xrightarrow{\partial_I} \cdots,$$

where $\partial_I : \mathcal{Q}_q^{m,n,l} \longrightarrow \mathcal{Q}_{q-1}^{m,n,l}$ is the boundary operator induced from the quotient chain complex of $\mathcal{C}^{m,n}$ and can also be defined explicitly on the based graphs in $\mathcal{D}_{l-1}^{m,n}$ as follows:

For an oriented based graph $(E_1, E_2, \cdots, E_k)$ in $\mathcal{Q}_q^{m,n,l}$,
$\partial_I(E_1, E_2, \cdots, E_k) = \sum(-1)^j \partial^{(j)}(E_1, E_2, \cdots, E_k)$,
over all inner edges E_j.

The operator ∂_I is called the inner boundary.

In our consideration, the base points $(x_1, x_2, \cdots, x_m)$ and the order number n are always fixed. Thus we may simplify the notations $\mathcal{Q}^{m,n,l}$ and $\mathcal{Q}_q^{m,n,l}$ to $\mathcal{Q}^l$, and $\mathcal{Q}_q^l$, respectively, without any ambiguity.

Actually, as long as we have the number u of basic edges and the number s of outer edges, the loop number l also determines the order $n = s + u + l - 1$.

Our future main work shall be to prove the following

Conjecture 2.2.3: Assume the loop number l is larger than 1, then $H_q(\mathcal{Q}^l) = 0$, as

$q \leq -l$. ∎

For the case that the loop number $l = 0$, the corresponding statement of **Conjecture 2.2.3** is not adequate. But we already know that the homologies of $\mathcal{Q}^0$ are all trivial except the top dimension $q = 0$; it is the correct statement we need. When the loop number is small, say, $l \leq 4$, it is not hard to check the conjecture in some way. We shall mention these results in further works.

2.3 Restricted based graph and standard graph

In this subsection, we define the notion of restricted based graph Γ and the associated base-map φ_Γ.

For a quasi-connected based graph, the inner loop number l and the order n can determine the number of edges which are not in its inner subgraph, that is, the number $s + u$. In applications, there is no basic edges usually. Thus, in $\mathcal{Q}^l$, all the graphs have the same number of outer edges, no matter how many base points $x_1, x_2, \cdots, x_m$ are under consideration. Therefore, it looks that the conjecture is nothing to do with the base points, and we may assume that different outer edges have different base points in the edges, that is, forgetting the existence of basic edges temporarily, the base points are all univalent.

Precisely, let $\mathcal{R}^{m,n,l}$ be the subchain complex of $\mathcal{Q}^{m,n,l}$ generated by all the based graphs Γ for which each base point of Γ has exactly one outer edge connecting to it, such based graphs are called the **restricted based graphs**. Then we have the following straightforward result.

Proposition 2.3.1: Now we fix the loop number l and the order number n and consider the chain complexes $\mathcal{R}^{m,n,l}$, for all possible m. If the Conjecture 2.2.3 holds for all $\mathcal{R}^{m,n,l}$, $m = 1, 2, \cdots$, then the Conjecture 2.2.3 holds for all $\mathcal{Q}^{m,n,l}$, $m = 1, 2, \cdots$. ∎

(Thus in the proposition we consider the graphs on different based set $(x_1, x_2, \cdots, x_m)$.)

Therefore, to prove Conjecture 2.2.3, it is enough to consider the restricted based graphs and the chain complexes $\mathcal{R}^{m,n,l}$. For convenience, we also simplify the notations $\mathcal{R}^{m,n,l}$ and $\mathcal{R}^{m,n,l}_q$ to $\mathcal{R}^l$, and $\mathcal{R}^l_q$ and consider the following conjecture.

Conjecture 2.3.2: Assume the loop number l is larger than 1, then $H_q(\mathcal{R}^l) = 0$, as $q \leq -l$. ∎

Definition 2.3.3: (Base-map of restricted graph) For a restricted based graph Γ, there is a map φ_Γ from the based set $\{x_1, x_2, \cdots, x_m\}$ to the set of inner vertices, which assigns each base point to the inner vertex connecting to it by a unique outer edge of Γ. For simplicity, we write φ_Γ as $\{1, 2, \cdots, m\} \longrightarrow V(\Gamma^\circ)$.

Thus, for each i, $1 \leq i \leq m$, $\{x_i, \varphi_\Gamma(i)\}$ is an outer edge of Γ.

Definition 2.3.4: A restricted based graph Γ is said to be a standard graph, if its base-map φ_Γ is a one-to-one map; or else, it is a non-standard graph.

It is easy to see that all the non-standard restricted based graphs form a subchain complex $\mathcal{N}^l$ of $\mathcal{R}^l$, and the associated quotient chain complex $\mathcal{R}^l/\mathcal{N}^l$, denoted by $\mathcal{S}^l$, can be considered as the chain complex of standard graphs with the following "partial" inner boundary:

For an oriented standard based graph $\Gamma = (E_1, E_2, \cdots, E_k)$ $\partial_s(E_1, E_2, \cdots, E_k) = \sum(-1)^j \partial^{(j)}(E_1, E_2, \cdots, E_k)$, the summation is over the edges E_j which are not contained in the image set of φ_Γ. (Thus the graph $\partial^{(j)}(E_1, E_2, \cdots, E_k)$ is also a standard graph.)

In the following, for a standard graph, the "partial" inner boundary defined above shall be called the **standard inner boundary**, or simply, the **standard boundary**.

Proposition 2.3.5 The homology of $\mathcal{N}^l$ are all trivial, and hence, the two chain complexes $\mathcal{R}^l$ and $\mathcal{S}^l$ have isomorphic homologies. (The isomorphism is naturally induced from the quotient map.) ∎

The proof of the proposition is similar to that of Theorem 1, and it is given after the proof of Theorem 1.

Remark 2.3.6: In the study of $\mathcal{R}^l$ (or, $\mathcal{Q}^l$), we may assume the number of basic edge, u, is a fixed number. Then the number of the base points is already determined by the order number n and the loop number l, that is, $m = s = n - u - l + 1$. If we forget the basic edges, or substitute the order number n by the number $n - u$, we shall finally find the loop number is the only essential thing we really need and the $n - u - l + 1$ base points appear as the coefficients in some homology theory.

Structure of standard graph:

(i) The inner subgraph of a standard graph Γ does not admit any univalent vertex. Each bi-valent vertex of the inner subgraph must be equal to $\varphi_\Gamma(j)$, for some base point X_j.

(ii) There is a "graph" with every valencies at least 3 and homeomorphic to the inner subgraph of Γ, such a "graph" is called the **central graph** of the standard graph. Thus, for a standard graph, its central graph is the inner subgraph forgetting the bivalent vertices. The central graph admits double edges and is not a simplicial complex.

Utilizing the concept of central graph, we may consider the standard graph as a central graph together with a one-to-one map from $\{1, 2, \cdots, m\}$ into the central graph. Thus, the central graph is actually the "central part" of a standard graph.

3 The proofs

3.1 Proof of Theorem 1

We shall define a chain homotopy type linear map $\tau : C_i^m \longrightarrow C_{i+1}^m$ and consider the associated chain map $\lambda = \tau \circ \partial + \partial \circ \tau$ of $\mathcal{C}^m$. We can show that (1) the subchain complex $\mathrm{Ker}(\lambda) = \{x \in \mathcal{C}^m : \lambda(x) = 0\}$, the kernel space of λ, is chain homotopy equivalent to $\mathcal{C}^m$, and (2) $\mathrm{Ker}(\lambda)$ is equal to the tensor product of $\mathcal{C}^{m-1}$ and $\mathcal{E}^{m-1}$, $\mathcal{E}^{m-1}$ is the dual of $H^*(X_{m-1})$, as in Theorem 1.

Definition of $\tau : C_i^m \longrightarrow C_{i+1}^m$:

For any degree i graph Γ with orientation $(E_1, E_2, \cdots, E_k)$, let $\eta(\Gamma)$ be the graph Γ with an additional vertex a and an additional edge $E'_{k+1} = \{a, x_1\}$. We assign $\eta(\Gamma)$ the orientation $(E_1, E_2, \cdots, E_k, E'_{k+1})$. Let ϕ be the permutation of the vertex set $V(\Gamma) \cup \{a\}$ interchanging a and x_1, that is, $\phi(a) = x_1, \phi(x_1) = a$, and $\phi(v) = v$, for any other vertices v.

Define $\tau(E_1, E_2, \cdots, E_k) = (-1)^{k+1}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E'_{k+1}))$, the corresponding graph is denoted by $\tau(\Gamma)$. Then $\tau(\Gamma)$ is also a graph based on $(x_1, x_2, \cdots, x_k)$, with an additional inner vertex a and is simplicially isomorphic to $\eta(\Gamma)$. If the valency of x_1 in Γ is at least 2, then the valency of a in $\tau(\Gamma)$ is at least 3, and hence, the associated oriented graph $\tau(E_1, E_2, \cdots, E_k)$ is a qualified element in C_{i+1}^m, i is the degree of Γ; if the valency of x_1 in Γ is equal to 0 or 1, then $\tau(\Gamma)$ can not satisfy the valency assumption and we just define $\tau(E_1, E_2, \cdots, E_k)$ as 0 in C_{i+1}^m. If Γ' is a graph equivalent to Γ, then $\tau(\Gamma')$ is also equivalent to $\tau(\Gamma)$ and it is straightforward to prove the remaining well-defined property.

Because in the graph $\tau(\Gamma)$, the valency of x_1 is equal to 1, $\tau(\tau(E_1, E_2, \cdots, E_k))$ is always 0 in C_{i+2}^m. This proves the following lemma.

Lemma 3.1.1: The linear homomorphism $\tau \circ \tau : C_i^m \longrightarrow C_{i+2}^m$ is a zero-map. ∎

Lemma 3.1.2: Suppose Γ is graph based on $(x_1, x_2, \cdots, x_m)$.
If Γ satisfies the following $\star$-condition:

($\star$): every edge containing x_1 is basic and the valency of $x_1 \leq 1$, then

$$(\tau \circ \partial + \partial \circ \tau)([\Gamma]) = 0 \ ;$$

if Γ does not satisfy the $\star$-condition, then

$$(\tau \circ \partial + \partial \circ \tau)([\Gamma]) = [\Gamma]$$

where $[\Gamma]$ denotes the graph Γ with orientation. ∎

We prove **Lemma 3.1.2** later and use it to prove Theorem 1.

At first, we check that the linear homomorphism $\lambda = \tau \circ \partial + \partial \circ \tau$ is a chain map, that is, to show the equality $\lambda \circ \partial = \partial \circ \lambda$ as follows:

$$\lambda \circ \partial = (\tau \circ \partial + \partial \circ \tau) \circ \partial = \tau \circ \partial \circ \partial + \partial \circ \tau \circ \partial = \partial \circ \tau \circ \partial \ ,$$

$$\partial \circ \lambda = \partial \circ (\tau \circ \partial + \partial \circ \tau) = \partial \circ \tau \circ \partial + \partial \circ \partial \circ \tau = \partial \circ \tau \circ \partial \ .$$

Let $\mathrm{Ker}(\lambda) = \{x \in \mathcal{C}^m : \lambda(x) = 0\}$, the kernel space of λ, and $\mathrm{Im}(\lambda)$ be the image space of λ. Then both $\mathrm{Ker}(\lambda)$ and $\mathrm{Im}(\lambda)$ are subchain complexes of $\mathcal{C}^m$.

By **Lemma 3.1.2**, $\mathrm{Ker}(\lambda)$ contains the linear subspace $\mathcal{D}_1$ of $\mathcal{C}^m$, generated by the set $\{[\Gamma] : \Gamma$ satisfies the $\star$-condition $\}$. We may also consider the linear subspace $\mathcal{D}_2$ of $\mathcal{C}^m$ generated by the set $\{[\Gamma] : \Gamma$ does not satisfy the $\star$-condition $\}$, then $\mathcal{C}^m = \mathcal{D}_1 \oplus \mathcal{D}_2$. Because λ is equal to 0 on $\mathcal{D}_1$ and is equal to the identity map on $\mathcal{D}_2$ (also by **Lemma 1.4.11**), λ is a projection map of $\mathcal{C}^m$, that is, satisfying the equality $\lambda \circ \lambda = \lambda$.

Of course, this leads to the result that $\mathcal{D}_1 = \mathrm{Ker}(\lambda)$ and $\mathcal{D}_2 = \mathrm{Im}(\lambda)$.

By **Lemma 3.1.1** and a similar computation as above, $\lambda \circ \tau = \tau \circ \lambda$. Thus τ provides a chain homotopy between the identity map and the 0-map in the chain complex $\mathrm{Im}(\lambda)$, and hence, $H_*(\mathrm{Im}(\lambda)) = 0$.

This implies that $H_*(\mathcal{C}^m) \cong H_*(\mathrm{Ker}(\lambda))$.

We summarize the arguments above to the following proposition.

Proposition 3.1.3: Suppose $\mathcal{C} = \{C_i^m, \partial_i : C_i^m \longrightarrow C_{i-1}^m, i = \cdots, -2, -1, 0, \cdots\}$ is a chain complex ($\partial_{i-1} \circ \partial_i = 0,$), and $\tau_i : C_i^m \longrightarrow C_{i+1}^m$, $i = \cdots, -2, -1, 0, \cdots$, are linear maps increasing the grade by 1 which also satisfy the condition of coboundary, $\tau_{i+1} \circ \tau_i = 0$. Furthermore, assume that the associative chain map of $\{\tau_i\}$, $\{\lambda_i = \tau_{i-1} \circ \partial_i + \partial_{i+1} \circ \tau_i : C_i^m \longrightarrow C_i^m, i = \cdots, -2, -1, 0, \cdots\}$ satisfies the condition of projection map, that is, $\lambda_i \circ \lambda_i = \lambda_i$, for all i.

Then the kernel subchain complex $\mathrm{Ker}(\lambda) = \{$kernel of λ_i, for all $i\}$ has the homologies isomorphic to that of $\mathcal{C}$. ∎

For the different possible basic edge containing x_1, we split $\mathrm{Ker}(\lambda)$ into the subchain complexes which are isomorphic to $\mathcal{C}^{m-1}$.

Let $\mathcal{K}(1)$ be the subchain complex of Ker(λ) generated by all oriented graphs in which the valency of x_1 is 0.

For each j, $2 \leq j \leq m$, let $\mathcal{K}(1,j)$ be the subchain complex of Ker(λ) generated by all oriented graphs in which $\{x_1, x_j\}$ is the unique edge containing x_1.

Then $\text{Ker}(\lambda) = \mathcal{K}(1) \oplus \mathcal{K}(1,2) \oplus \mathcal{K}(1,3) \oplus \cdots \oplus \mathcal{K}(1,m)$.

$\mathcal{K}(1)$ is exactly the chain complex of oriented graphs based on $(x_2, x_3, \cdots, x_m)$, it is canonically isomorphic to $\mathcal{C}^{m-1}$, and for other j, $\mathcal{K}(1,j)$ is isomorphic to $\mathcal{K}(1)$ with the elements decreasing the degree by 1.

To describe the structure precisely, for any positive integer p, let $\mathcal{E}^p$ be the chain complex defined by: for degree 0 and -1, $\mathcal{E}^p_0 = \mathbf{R}$, $\mathcal{E}^p_{-1} = \mathbf{R}^p$; for other degree i, $\mathcal{E}^p_i = 0$. The boundary operator in $\mathcal{E}^p$ are all the zero-map. $\mathcal{E}^p_i \cong H^{-i}(X_p, \mathbf{R})$, for all i.

Then $\text{Ker}(\lambda) \cong \mathcal{E}^{m-1} \otimes \mathcal{K}(1) \cong \mathcal{E}^{m-1} \otimes \mathcal{C}^{m-1}$.

Thus $H_*(\mathcal{C}^m) \cong H_*(\mathcal{E}^{m-1} \otimes \mathcal{C}^{m-1}) \cong \mathcal{E}^{m-1} \otimes H_*(\mathcal{C}^{m-1})$.
By induction, we have

$$H_*(\mathcal{C}^m) \cong \mathcal{E}^{m-1} \otimes \mathcal{E}^{m-2} \otimes \cdots \otimes \mathcal{E}^1 ,$$

it is the isomorphism needed in Theorem 1.

Proof of Lemma 3.1.2

Choose a linear order $(E_1, E_2, \cdots, E_k)$ for the edges of Γ. If Γ satisfies the $(\star)$-condition, then, for the non-basic edge E_j, E_j does not meet x_1 and Γ/E_j also satisfies the $(\star)$-condition. Thus, for the non-basic edge E_j, $\tau(\partial^{(j)}[\Gamma]) = \tau([\Gamma/E_j]) = 0$, and for the basic edge E_l, $\partial^{(l)}[\Gamma]$ is defined as 0; this concludes that

$$\tau(\partial[\Gamma]) = \Sigma_{i=1}^{k}(-1)^j(\tau(\partial^{(j)}([\Gamma]))) = 0 .$$

By the valency assumption, $\tau([\Gamma]) = 0$, and hence, $(\partial \circ \tau + \tau \circ \partial)([\Gamma]) = \partial(\tau([\Gamma])) + \tau(\partial([\Gamma])) = 0$, this proves the first part of the main lemma.

For the second part, assume that the valency of x_1 in Γ is larger than 1, or, the unique edge containing x_1 is equal to $\{x_1, v\}$, for some inner vertex v.

(Case 1): Valency $(x_1) \geq 2$.

In this case, $\tau([\Gamma])$ is non-zero. Consider the orientation $(E_1, E_2, \cdots, E_k)$ for Γ. $\tau([E_1, E_2, \cdots, E_k]) =$ $(-1)^{k+1}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E_{k+1}'))$
(**Note:** ϕ is defined in the definition of τ). Thus

$$(\partial \circ \tau)(E_1, E_2, \cdots E_k) = (-1)^{k+1} \sum_{i=1}^{k+1} (-1)^j \partial^{(j)}(\phi(E_1), \cdots \phi(E_k)).$$

The last term in the summation above,

$$(-1)^{k+1} \cdot (-1)^{k+1} \partial^{(k+1)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E'_{k+1}))$$

is exactly equal to $(E_1, E_2, \cdots E_k)$, the oriented graph of Γ. For $1 \leq j \leq k$, we should check that

$$(-1)^{k+1} \cdot (-1)^{k+1} \partial^{(j)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E'_{k+1}))$$

is equal to $-\tau((-1)^j \partial^{(j)}(E_1, E_2, \cdots, E_k))$. As in the definition of $\partial^{(j)}$, let $\pi_j : V(\Gamma) \to V(\Gamma)/E_j$ denote the quotient map.

$\partial^{(j)}(E_1, E_2, \cdots E_k) = (\pi_j(E_1), \cdots, \pi_j(E_{j-1})), \pi_j(E_{j+1}), \cdots, \pi_j(E_k))$.

Thus $\tau(\partial^{(j)}(E_1, E_2, \cdots E_k) =$
$(-1)^k(\phi(\pi_j(E_1)), \cdots, \phi(\pi_j(E_{j-1})), \phi(\pi_j(E_{j+1})), \cdots, \phi(\pi_j(E_k)), \phi(\pi_j(E'_k)))$.

On the other hand, to study $\partial^{(j)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_{k+1}'))$,
let $\overline{\pi_j} : V(\Gamma) \cup \{a\} \longrightarrow (V(\Gamma) \cup \{a\})/\phi(E_j)$ denote the quotient map, where a is the new inner vertex in the definition of $\tau(\Gamma)$.

Then $\partial^{(j)}(\phi(E_1), \cdots, \phi(E_k), \phi(E_{k+1}'))$
$= (\overline{\pi_j}(\phi(E_1)), \cdots, \overline{\pi_j}(\phi(E_{j-1})), \overline{\pi_j}(\phi(E_{j+1})), \cdots, \overline{\pi_j}(\phi(E_k)), \overline{\pi_j}(\phi(E_{k+1}')))$.
It is straightforward to find that $\phi(\pi_j(E_l)) = \overline{\pi_j}(\phi(E_l))$, for $1 \leq l \leq k, l \neq j$, and $\phi(E'_k) = \overline{\pi_j}(\phi(E_{k+1}'))$, which imply the equality
$-\tau((-1)^i \partial_i(E_1, E_2, \cdots, E_k))$
$= (-1)^{k+1}(-1)^j \partial^{(j)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E_{k+1}'))$,
and hence, we have

$$(\partial \circ \tau)(E_1, E_2, \cdots, E_k) = (E_1, E_2, \cdots, E_k) - (\tau \circ \partial)(E_1, E_2, \cdots, E_k) .$$

(Case 2): Valency(x_1) = 1.

There is some edge $E_s = \{x_1, v\}$, for some s, $1 \leq s \leq k$ and for some inner vertex v.

In this situation v can not be a base point, or else, Γ satisfies the ($\star$)-condition.

By the definition of τ, $\tau([\Gamma]) = 0$. For the integer $j \neq s, 1 \leq j \leq k$, x_1 is also of valency 1 in $\partial^{(j)}(\Gamma)$. Thus $\tau(\partial^{(j)}(\Gamma)) = 0$, for $j \neq s$. And it is easy to see that for the particular boundary $\partial^{(s)}(\Gamma)$, its τ-value, $\tau(\partial^{(s)}(\Gamma))$, is equivalent to the original graph Γ. Together with the orientation, we have

$\tau(\partial(E_1, E_2, \cdots, E_k))$
$= \tau((-1)^s(\pi_s(E_1)), \cdots, \pi_s(E_{s-1}), \pi_s(E_{s+1}), \cdots, \pi_s(E_k))$
$= (-1)^k(-1)^s(\phi(\pi_s(E_1)), \cdots, \phi(\pi_s(E_{s-1})), \phi(\pi_s(E_{s+1})), \cdots, \phi(\pi_s(E_k)), \phi(\pi_s(E'_k)))$.

In the equivalence of $\tau(\partial^{(s)}(\Gamma))$ and Γ, the edge $\phi(\pi_s(E'_k))$ is correspondent to E_s. When changing the position of $\phi(\pi_s(E'_k))$ to the original position of E_s, we get an additional sign $(-1)^{k-s}$. Thus $\tau(\partial(E_1, E_2, \cdots, E_k))$
$= (-1)^{k+s}(-1)^{k-s}(\phi(\pi_s(E_1)), \cdots, \phi(\pi_s(E_{s-1})), \phi(\pi_s(E'_k)), \phi(\pi_s(E_{s+1})), \cdots, \phi(\pi_s(E_k)))$,
which is exactly equal to $(E_1, E_2, \cdots, E_k)$.

That is,

$$\tau(\partial(E_1, E_2, \cdots, E_k)) = (E_1, E_2, \cdots, E_k) \ .$$

This completes the proof of Lemma 3.1.2.

3.2 Proof of Proposition 2.3.5

Suppose Γ is a restricted based graph with loop number $l > 0$. $\Gamma^\diamond$ is the inner subgraph. In the subgraph $\Gamma^\diamond$, there is a minimal subgraph Γ° which has the same betti number as $\Gamma^\diamond$; this subgraph Γ° can be called the central subgraph of Γ. Γ° shall have no univalent vertex and is a deformation retract of $\Gamma^\diamond$.

For each base point x_j, $1 \leq j \leq m$, there is a unique simple path α_j from x_j to a vertex y_j of Γ° such that α_j only meets Γ° at the point y_j. The map $\psi_\Gamma : \{1, 2, \cdots m\} \longrightarrow V(\Gamma^\circ)$, sending j to the vertex y_j of Γ°, is a well-defined map. For any two number i and j, $1 \leq i < j \leq m$, y_i may be equal to y_j, or not. Thus the image set $\{y_1, y_2, \cdots, y_m\}$ of ψ_Γ is not necessary to consist of m distinct elements. (If $y_i = y_j$, the two paths α_i and α_j may intersect a few edges of Γ.)

Let $c(\Gamma)$ denote the number of distinct elements in $\{y_1, y_2, \cdots, y_m\}$. For each k, $1 \leq k \leq m$, let $\mathcal{R}^l(k)$ denote the subchain complex of $\mathcal{R}^l$ generated by the oriented restricted based graphs Γ with $c(\Gamma) \leq k$. Then we have the following filtration:

$$\mathcal{R}^l(1) \subset \mathcal{R}^l(2) \subset \cdots \subset \mathcal{R}^l(m-1) \subset \mathcal{R}^l$$

It is easy to see that $\mathcal{R}^l(m-1) = \mathcal{N}^l$. If $m = 1$, $\mathcal{N}^l$ is trivial; thus in the following we assume $m \geq 2$ and we shall show that the chain complexes $\mathcal{R}^l(1)$, $\mathcal{R}^l(2)/\mathcal{R}^l(1)$, $\mathcal{R}^l(3)/\mathcal{R}^l(2), \cdots, \mathcal{R}^l(m-1)/\mathcal{R}^l(m-2)$ all have trivial homologies. These conclude the result we need.

Lemma 3.2.1: Suppose $m \geq 2$. For the chain complex $\mathcal{R}^l(1)$, there is a chain homotopy linear map $\tau : \mathcal{R}^l(1)_q \longrightarrow \mathcal{R}^l(1)_{q+1}$ such that $\tau \circ \partial_I + \partial_I \circ \tau$ is equal to the identity map. And hence, $\mathcal{R}^l(1)$ has trivial homologies. ▮

Proof of Lemma 3.2.1:

Suppose Γ is a restricted based graph with orientation $(E_1, E_2, \cdots, E_k)$ and $c(\Gamma) = 1$; thus $\psi_\Gamma(1) = \psi_\Gamma(2) = \cdots = \psi_\Gamma(m) = y$.

Let $\tau(\Gamma)$ be the graph Γ with an additional inner vertex a and an additional inner edge $E'_{k+1} = \{a, y\}$, furthermore changing the edges $\{y, w\}$ in Γ° to the edges $\{a, w\}$. Precisely, for each edge E_j of Γ, let E'_j be the edge defined as follows:

(i) $E'_j = E_j$, if y is not in E_j,

(ii) $E'_j = E_j$, if E_j is not an edge of Γ°,

(iii) $E'_j = \{a, w\}$, if $E_j = \{y, w\}$ and it is an edge of Γ°.

And $\tau(\Gamma)$ is defined as the oriented restricted based graph $(-1)^{k+1}(E'_1, E'_2, \cdots, E'_k, E'_{k+1})$. If the valency of y in Γ is equal to 1 plus the valency of y in Γ°, then y becomes a bivalent vertex of $\tau(\Gamma)$; in this situation, $\tau(\Gamma)$ is not adequate in our theory and it is redefined as 0.

It is easy to check that the linear map τ satisfying the assumption of Proposition 3.1.3 and the associated chain map $\lambda = \tau \circ \partial_I + \partial_I \circ \tau$ is equal to the identity map of $\mathcal{R}^l(1)$. This completes the proof of the lemma.

Now, for some k, $2 \leq k \leq m-1$, the quotient chain complex $\mathcal{R}^l(k)/\mathcal{R}^l(k-1)$ splits into the direct sum of many chain complexes, each of them is of the following form:

Assume $P = \{A_1, A_2, \cdots, A_k\}$ is a partition of $\{1, 2, \cdots, m\}$. Let $\mathcal{R}(P)$ be the subchain complex of $\mathcal{R}^l(k)/\mathcal{R}^l(k-1)$ generated by the oriented restricted based graphs Γ for which $\psi_\Gamma(A_i) = \{z_i\}$, for some vertex z_i, $i = 1, 2, \cdots, k$, and $c(\Gamma) = k$. (Thus z_i, $i = 1, 2, \cdots, k$, are all distinct vertices.)

Lemma 3.2.2: The homologies of $\mathcal{R}(P)$ are all trivial.

The proof is just completely similar to that for $\mathcal{R}^l(1)$. Choose any A_i containing more than one integers, then the vertex z_i is good for the position of y in the proof of Lemma 3.2.1.

And this finishes the proof of Proposition 2.3.5. $\square$

4 The acyclic property of based graphs of order 5

In this section, we concentrate on the case of two base point ($m = 2$) and consider only the graphs of order 5. Our purpose is to show that $H_q(\mathcal{D}_p^{2,5}) = 0$, for all $q < 0$ and for all p, where $\mathcal{D}_p^{2,5}$ is a chain complex defined in Section 2.2; it is equivalent to $H_q(\mathcal{Q}^{2,5,l}) = 0$, for $q < 0$ and for all l. (There is some relation between p and l, stated also in Section 2.2.) This is much stronger than that of Conjecture 2.2.3.

In the following, we also denote the chain complexes $\mathcal{D}_p^{2,5}$ and $\mathcal{Q}^{2,5,l}$ simply by $\mathcal{D}_p$ and $\mathcal{Q}^l$, respectively; and, denote the vector spaces $\mathcal{D}_{p,q}^{2,5}$ and $\mathcal{Q}_q^{2,5,l}$ simply by $\mathcal{D}_{p,q}$ and $\mathcal{Q}_q^l$, respectively.

Finally, we have the following

Theorem 2: The chain complex $\mathcal{C}^{2,5}$ of oriented plane graphs of order 5, based on (x_1, x_2), has a subchain complex $\{K_p, p = -1, 0, 1, 2, \cdots\}$ with trivial homology and satisfying that every elements in K_p are linear combination of trivalent based graphs. ∎

Now we sketch the proof of **Theorem 2** and outline our study as follows:

Step 1. List all possible central subgraphs of based plane graphs of order 5.

Step 2. For each central graph, draw all the based plane graphs of order 5. And partition all the graphs into different spaces in the double sequence $\{\mathcal{S}_q^l\}$.

Step 3. Construct the chain complex $\{\mathcal{D}_q^l, \partial_I\}$, for all p and q.

Step 4. Write out explicitly the inner boundary operator ∂_I for each horizontal sequence, actually for each graph considered. And show that $H_q(\mathcal{Q}^{2,5,l}) = 0$, for $q < 0$ and for all l, in a straightforward way.

Step 5. Claim that the sequence $\{K_p\}$ under the **outer boundary operator** is an acyclic, where $\{K_p\}$ are the kernel of the top stages of all the horizontal sequences.

By **step 4** and **step 5**, we reach the conclusion that the chain complex of based plane graphs of order 5 is acyclic, and hence, we finish the proof of **Theorem 2**.

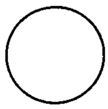
Now we explain the above steps one by one in the next subsections.

4.1 Listing all based plane graphs of order 5

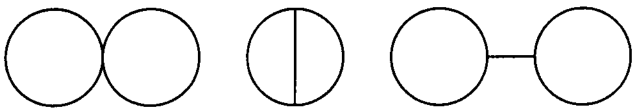
Now we use the central subgraph which is defined in **section** 1 to classify all the based plane graphs of order 5. We only give some examples of the connected central subgraphs as follows:

The connected central subgraphs:

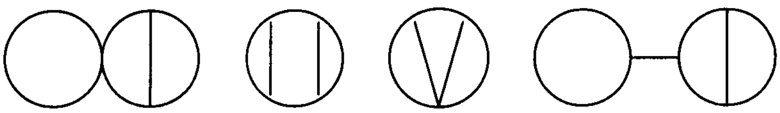
(1). Examples of loop 1 is listed as follows:



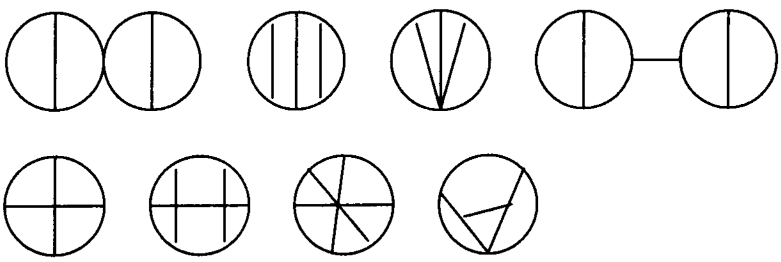
(2). Examples of loop 2 are listed as follows:



(3). Examples of the central subgraphs of loop 3:



(4). Examples of the central subgraphs of loop 4:

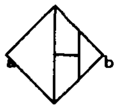


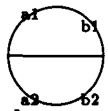
Remark 4.1.1: In the above examples of central subgraphs, we have some spacial graphs are called splitting graphs which becomes disconnected as take a vertex or an edge away.

For example:

The graphs with splitting central subgraphs will become an acyclic chain complex and then we can omit the computation of the graphs with such central subgraphs. For the detial, please see [Yu].

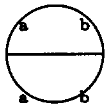
We will classify all the oriented based plane graphs of order 5. In order to simplify the notation of the based plane graphs by the central subgraphs, we denote the base points a and b locating on the left and right sides, respectively. Mark the ex-internal vertices by $a_1, a_2, \cdots, a_s$ which join to the base point a and mark the ex-internal vertices by $b_1, b_2, \cdots, b_t$ which join to the base point b. See the following example:

$\Gamma =$ 

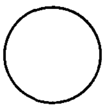
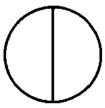
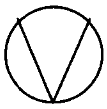
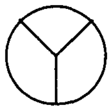
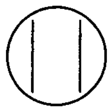
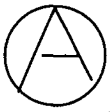
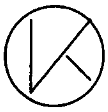
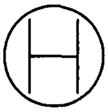
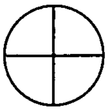
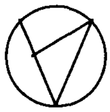
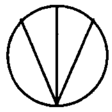
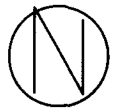
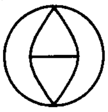
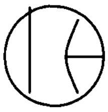
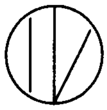
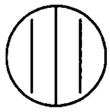
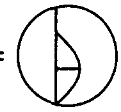
. The graph Γ can be denoted by the central subgraph 

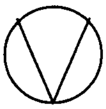
.

From the central subgraph, we can not change the based plane graph when we change the indexes of the labels of $\{a_i\}$ and $\{b_j\}$. From now on, we will only mark a and b on the central subgraphs to denote the based plane graphs. Thus the above example will become as follows:



Before the classification of the based palne graphs of order 5, we will give the notation of the central subgraphs as follows:

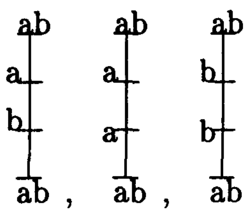
A=  , B=  , C=  , D=  , E=  , F=  , G=  , H=  , I=  , J=  , K=  , L=  , M=  , N=  , O=  , P=  , Q=  , R=  , S=  .

For example, we will use $\mathcal{S}^3_{-1}[D]$ to denote the subspace of $\mathcal{S}^3_{-1}$ with central subgraph D() and the same notation for the other central subgraphs.

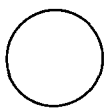
Now we will use the central subgraphs to classify all the based plane graphs of order 5 as follows:

(A). For the central subgraph  :

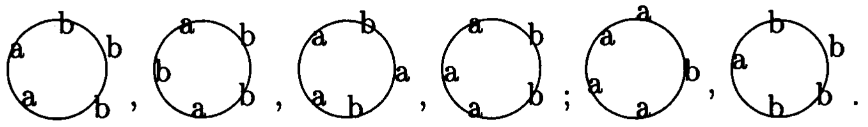
$\mathcal{S}^0_0[A]$ has the following generators:

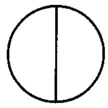


, , .

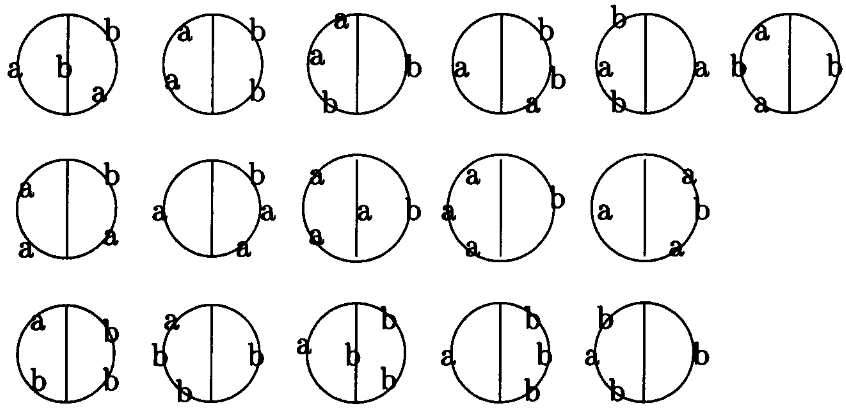
(B). For the central subgraph  :

$\mathcal{S}^1_0[B]$ has the following generators:

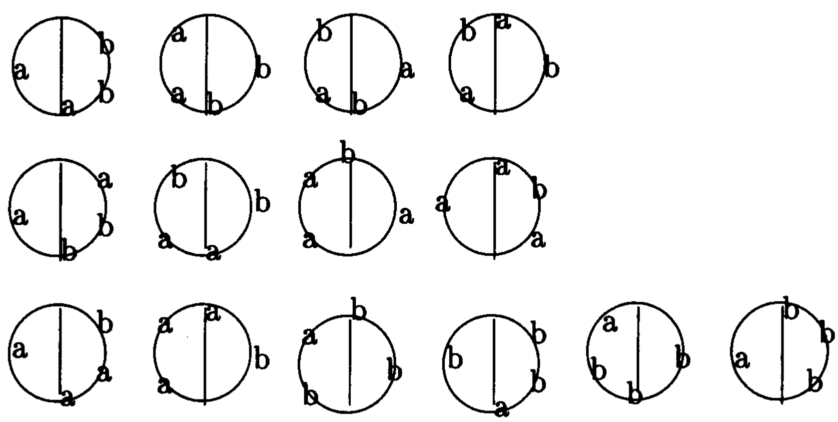


(C). For the central subgraph  :

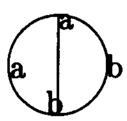
(C_1). $\mathcal{S}_0^2[C]$ has the following generators:

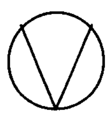


(C_2). $\mathcal{S}_{-1}^2[C]$ has the following generators:

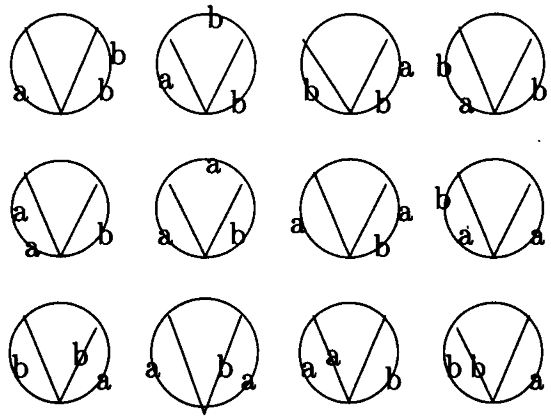


(C_3). $\mathcal{S}_{-2}^2[C]$ has the following generators:

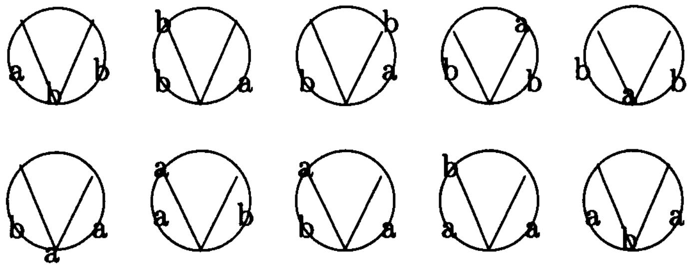


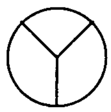
(D). For the central subgraph  :

(D_1). Let $\mathcal{S}_0^3[D]$ be a subspace of $\mathcal{S}_0^3$ and have the following generators:

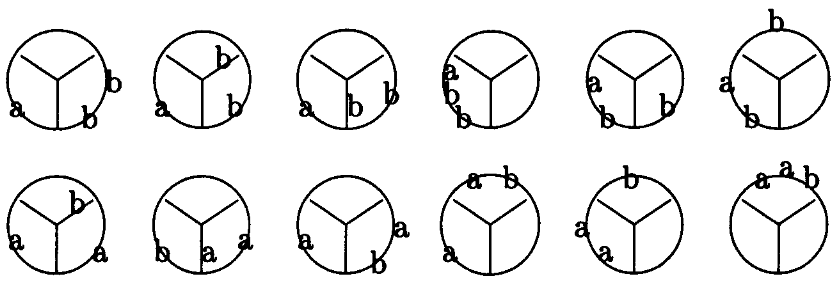


(D_2). Let $\mathcal{S}^3_{-1}[D]$ be a subspace of $\mathcal{S}^3_{-1}$ and have the following generators:

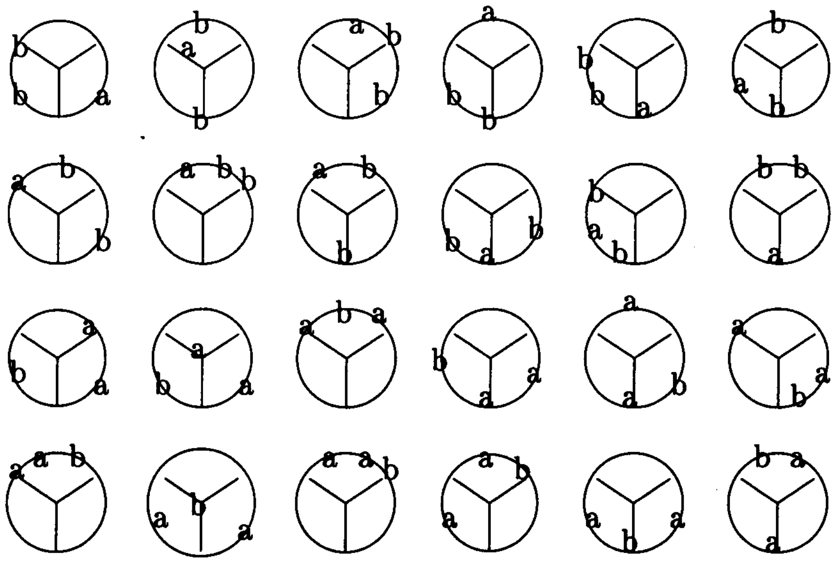


(E). For the central subgraph  :

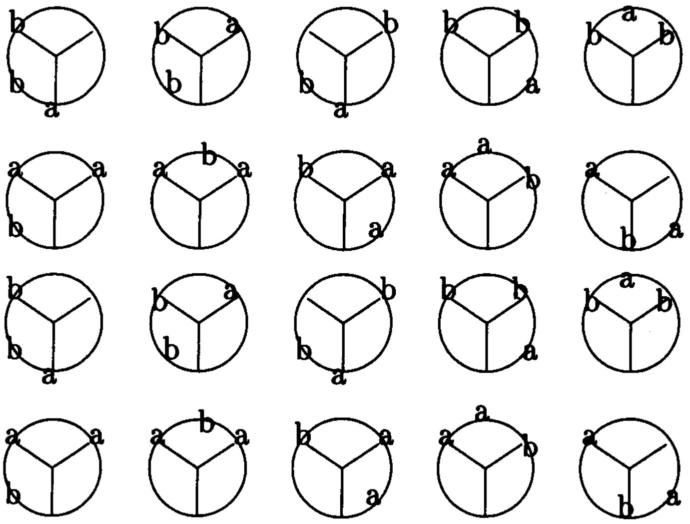
(E_1). Let $\mathcal{S}^3_0[E]$ be a subspace of $\mathcal{S}^3_0$ and have the following generators:

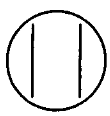


(E_2). Let $\mathcal{S}^3_{-1}[E]$ be a subspace of $\mathcal{S}^3_{-1}$ and have the following generators:

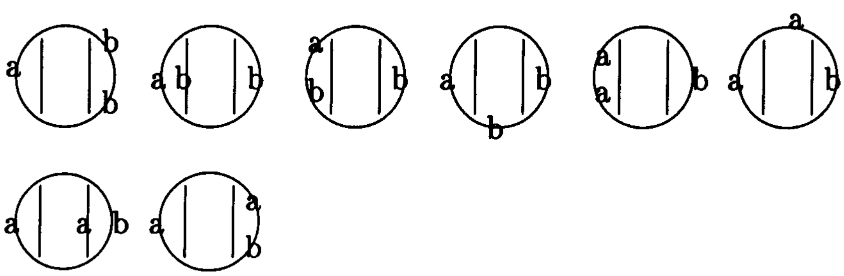


(E_3). Let $\mathcal{S}^3_{-2}[E]$ be a subspace of $\mathcal{S}^3_{-1}$ and have the following generators:

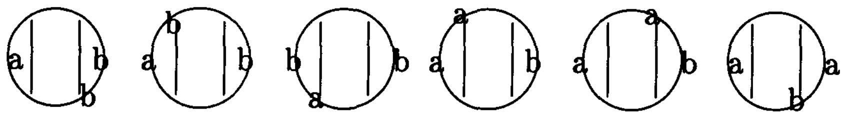


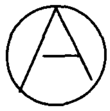
(F). For the central subgraph  :

(F_1). Let $\mathcal{S}^3_0[F]$ be a subspace of $\mathcal{S}^3_0$ and have the following generators:

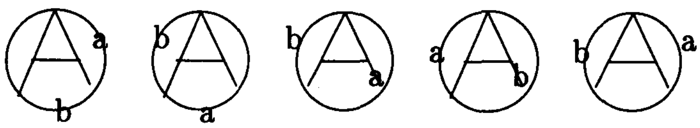


(F_2). Let $\mathcal{S}^3_{-1}[F]$ be a subspace of $\mathcal{S}^3_{-1}$ and have the following generators:

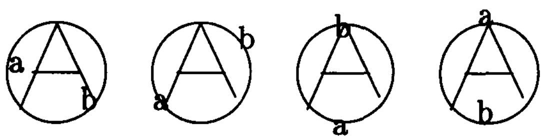


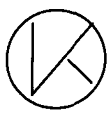
(G). For the central subgraph  :

(G_1). Let $\mathcal{S}^4_{-1}[G]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:

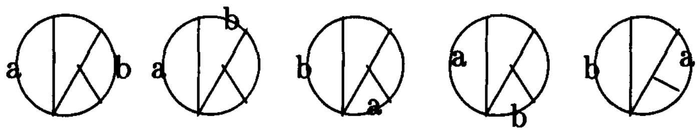


(G_2). Let $\mathcal{S}^4_{-2}[G]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:

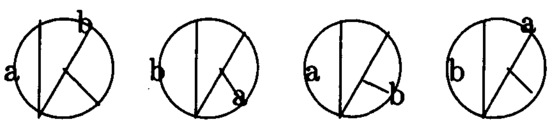


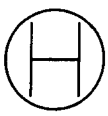
(H). For the central subgraph  :

(H_1). Let $\mathcal{S}^4_{-1}[H]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:

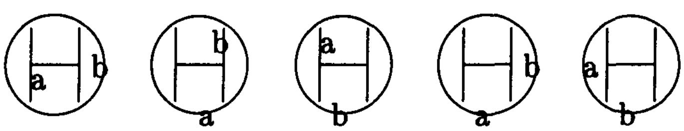


(H_2). Let $\mathcal{S}^4_{-2}[H]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:

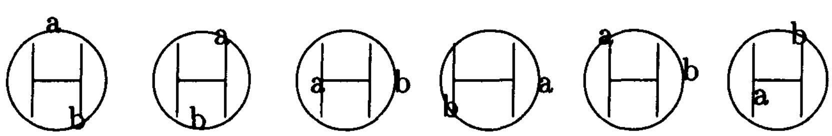


(I). For the central subgraph  :

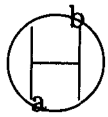
(I_1). Let $\mathcal{S}^4_0[I]$ be a subspace of $\mathcal{S}^4_0$ and have the following generators:

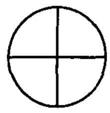


(I_2). Let $\mathcal{S}^4_{-1}[I]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:

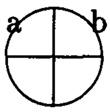


(I_3). Let $\mathcal{S}^4_{-2}[I]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:

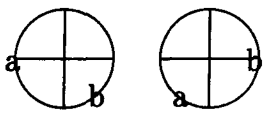


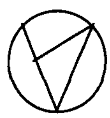
(J). For the central subgraph  :

(J_1). Let $\mathcal{S}^4_{-1}[J]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:

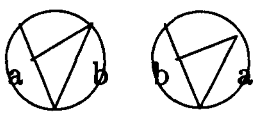


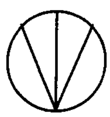
(J_2). Let $\mathcal{S}^4_{-2}[J]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:



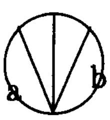
(K). For the central subgraph  :

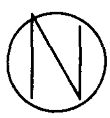
Let $\mathcal{S}^4_{-2}[K]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:



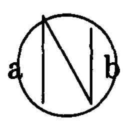
(L). For the central subgraph  :

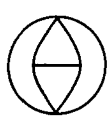
Let $\mathcal{S}^4_{-2}[L]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:



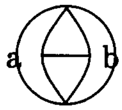
(M). For the central subgraph  :

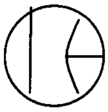
Let $\mathcal{S}^4_{-2}[M]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:



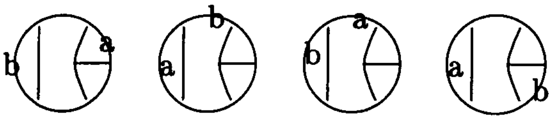
(N). For the central subgraph  :

Let $\mathcal{S}^4_{-2}[N]$ be a subspace of $\mathcal{S}^4_{-2}$ and have the following generators:

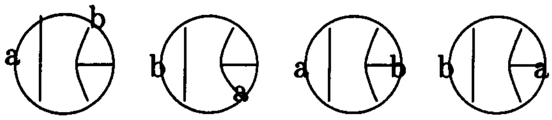


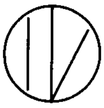
(O). For the central subgraph  :

(O_1). Let $\mathcal{S}^4_0[O]$ be a subspace of $\mathcal{S}^4_0$ and have the following generators:

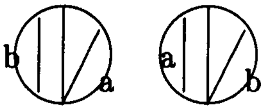


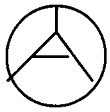
(O_2). Let $\mathcal{S}^4_{-1}[O]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:



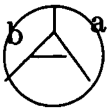
(P). For the central subgraph  :

Let $\mathcal{S}^4_{-1}[P]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:

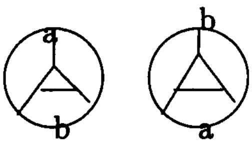


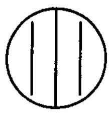
(Q). For the central subgraph  :

(Q_1). Let $\mathcal{S}^4_0[Q]$ be a subspace of $\mathcal{S}^4_0$ and have the following generators:

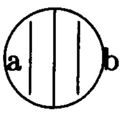


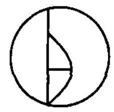
(Q_2). Let $\mathcal{S}^4_{-1}[Q]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:



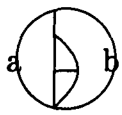
(R). For the central subgraph  :

Let $\mathcal{S}^4_0[R]$ be a subspace of $\mathcal{S}^4_0$ and have the following generators:



(S). For the central subgraph  :

Let $\mathcal{S}^4_{-1}[S]$ be a subspace of $\mathcal{S}^4_{-1}$ and have the following generators:



From the above classification, we have the relation of spaces $\{\mathcal{S}^l_q\}$ as follows:

1. $\mathcal{S}^0_0 = \mathcal{S}^0_0[A]$.
2. $\mathcal{S}^1_0 = \mathcal{S}^1_0[B]$.
3. $\mathcal{S}^2_0 = \mathcal{S}^2_0[C]$.
4. $\mathcal{S}^2_{-1} = \mathcal{S}^2_{-1}[C]$.
5. $\mathcal{S}^2_{-2} = \mathcal{S}^2_{-2}[C]$.
6. $\mathcal{S}^3_0 = \mathcal{S}^3_0[E] \oplus \mathcal{S}^3_0[F]$.
7. $\mathcal{S}^3_{-1} = \mathcal{S}^3_{-1}[D] \oplus \mathcal{S}^3_{-1}[E] \oplus \mathcal{S}^3_{-1}[F]$.
8. $\mathcal{S}^3_{-2} = \mathcal{S}^3_{-2}[D] \oplus \mathcal{S}^3_{-2}[E]$.
9. $\mathcal{S}^4_0 = \mathcal{S}^4_0[I] \oplus \mathcal{S}^4_0[O] \oplus \mathcal{S}^4_0[Q] \oplus \mathcal{S}^4_0[R]$.
10. $\mathcal{S}^4_{-1} = \mathcal{S}^4_{-1}[G] \oplus \mathcal{S}^4_{-1}[H] \oplus \mathcal{S}^4_{-1}[I] \oplus \mathcal{S}^4_{-1}[J] \oplus \mathcal{S}^4_{-1}[O] \oplus \mathcal{S}^4_{-1}[P] \oplus \mathcal{S}^4_{-1}[Q] \oplus \mathcal{S}^4_{-1}[S]$.
11. $\mathcal{S}^4_{-2} = \mathcal{S}^4_{-2}[G] \oplus \mathcal{S}^4_{-2}[H] \oplus \mathcal{S}^4_{-2}[I] \oplus \mathcal{S}^4_{-2}[J] \oplus \mathcal{S}^4_{-2}[K] \oplus \mathcal{S}^4_{-2}[L] \oplus \mathcal{S}^4_{-2}[M] \oplus \mathcal{S}^4_{-2}[N]$.

Combine the above classification, we obtain the following sequences:

$$\begin{array}{ccccccccc}
0 & \mapsto & \mathcal{S}^4_0 & \xrightarrow{\partial_{I_e}} & \mathcal{S}^4_{-1} & \xrightarrow{\partial_{I_f}} & \mathcal{S}^4_{-2} & \xrightarrow{\partial_I} & 0 \\
 & & \Big\downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{S}^3_0 & \xrightarrow{\partial_{I_c}} & \mathcal{S}^3_{-1} & \xrightarrow{\partial_{I_d}} & \mathcal{S}^3_{-2} & \xrightarrow{\partial_I} & 0 \\
 & & \Big\downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{S}^2_0 & \xrightarrow{\partial_{I_a}} & \mathcal{S}^2_{-1} & \xrightarrow{\partial_{I_b}} & \mathcal{S}^2_{-2} & \xrightarrow{\partial_I} & 0 \\
 & & \Big\downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{S}^1_0 & \xrightarrow{\partial_{I_g}} & 0 & & & & \\
 & & \Big\downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{S}^0_0 & \xrightarrow{\partial_{I_h}} & 0 & & & &
\end{array}$$

Remark 4.1.2:

(1). For the above sequences $\{\mathcal{S}_q^l, \partial_I\}$, we have $p = l - 1$ and then $\mathcal{S}_q^l = \mathcal{S}_{l-1,q} = \mathcal{S}_{p,q}$. We will compute the chain complex $\{\mathcal{S}_{p,q}, \partial_I\}$ and then complete the **Theorem 2**.

(2). For the sequences $\{\mathcal{S}_{p,q}, \partial_I\}$, p fixed. As $p = -1$, we have the boundaries of Γ in $\mathcal{S}_0^0$ under ∂_I are all nonorientable, thus we define $\partial_I \Gamma = 0$. For this reason, we have the sequence:

$$0 \mapsto \mathcal{S}_0^0 \xrightarrow{\partial_I} 0.$$

For the other sequences, we have the same reason to explain why the sequences stopped.

4.2 Construct the chain complex $\{\mathcal{D}^{2,5}, \partial_I\}$

For the application of the conjecture in the **subsection 2.2**, we want to compute the inner edge contraction and the homology groups of the chain complex of inner boundary operator ∂_I can be shown to be zero except top dimensional one.

We compute the homology of the sequences order 5 under the inner boundary operator. The inner boundary operator is to clean the graphs which are not trivalent and the outer boundary operator is to compute the remaining trivalent graphs. By the sequences, we can infer the conclusion we want.

By the structure of the chain complex $\{\mathcal{D}_{p,q}^{2,5}\}$ and the boundary operators ∂_I, ∂_O we obtain the following sequences:

$$
\begin{array}{ccccccccc}
0 & \mapsto & \mathcal{D}^{2,5}_{3,0} & \xrightarrow{\partial_{I_e}} & \mathcal{D}^{2,5}_{3,-1} & \xrightarrow{\partial_{I_f}} & \mathcal{D}^{2,5}_{3,-2} & \xrightarrow{\partial_I} & 0 \\
 & & \downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{D}^{2,5}_{2,0} & \xrightarrow{\partial_{I_c}} & \mathcal{D}^{2,5}_{2,-1} & \xrightarrow{\partial_{I_d}} & \mathcal{D}^{2,5}_{2,-2} & \xrightarrow{\partial_I} & 0 \\
 & & \downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{D}^{2,5}_{1,0} & \xrightarrow{\partial_{I_a}} & \mathcal{D}^{2,5}_{1,-1} & \xrightarrow{\partial_{I_b}} & \mathcal{D}^{2,5}_{1,-2} & \xrightarrow{\partial_I} & 0 \\
 & & \downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{D}^{2,5}_{0,0} & \xrightarrow{\partial_{I_g}} & \mathcal{D}^{2,5}_{0,-1} & \xrightarrow{\partial_I} & 0 & & \\
 & & \downarrow{\scriptstyle \partial_O} & & & & & & \\
0 & \mapsto & \mathcal{D}^{2,5}_{-1,0} & \xrightarrow{\partial_{I_h}} & 0 & & & &
\end{array}
$$

Propsition 4.2.1:
$H_q(\mathcal{D}^{2,5,l}) = H_q(\mathcal{Q}^{2,5,l}) = H_q(\mathcal{S}^{2,5,l}) = 0$, for $q < 0$ and for all l.
Proof:

(1). For $l = 0$, we have the graphs with two based edges or with one based-edge in the space $\mathcal{D}^{2,5,l}$, but not in the space $\mathcal{S}^{2,5,l}$ and $\mathcal{Q}^{2,5,l}$. Such graphs are all nonorientable in our consideration, and then we can omit them in our computation. So the equality of the homologies hold for $l = 0$.

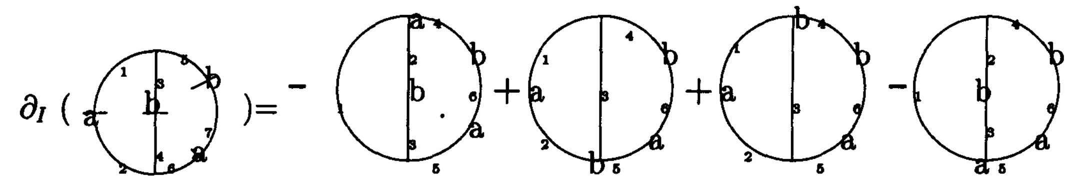
(2). As for $l = 1$, we have the graphs with just one based edge in the space $\mathcal{D}^{2,5,l}$, but not in the space $\mathcal{S}^{2,5,l}$ and $\mathcal{Q}^{2,5,l}$. Such graphs are some nonorientable, and smoe of them become an acyclic chain complex in our consideration(For the detial, please see [Yu]), and then we can omit them in our computation. So the equality of the homologies hold for $l = 1$.

(3). For $l \geq 2$, there are no based edges in graphs, and the inner graphs of all based plane graphs are connected. Thus from the statements in the **section 1 and 2**, we complete the equality of the proposition. □

From the proposition, we only compute the sequences $\{\mathcal{S}^{2,5}_{p,q}, \partial_I\}$ in **section 4.1** and then we can complete proof of **Theorem 2**.

4.3 Computing the row sequences

Before the computation of the row sequences, we will give some explanations for the boundary operator. Consider the following example:

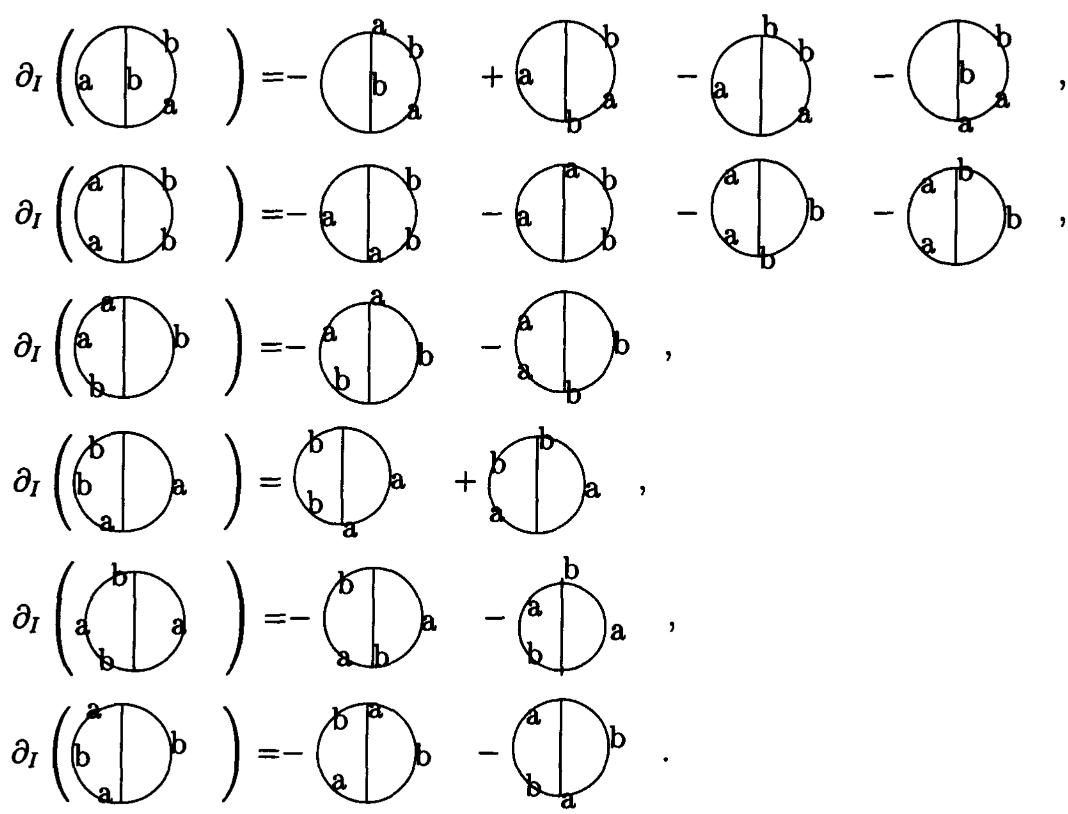


Since the above example is so difficult to describe, we will omit the labels of the edges and the short segments on the circle. For the following computation, all the equalities are with the same style as the above example and we will use the orientations of graphs by labeling the numbers from left to right and from up to down.

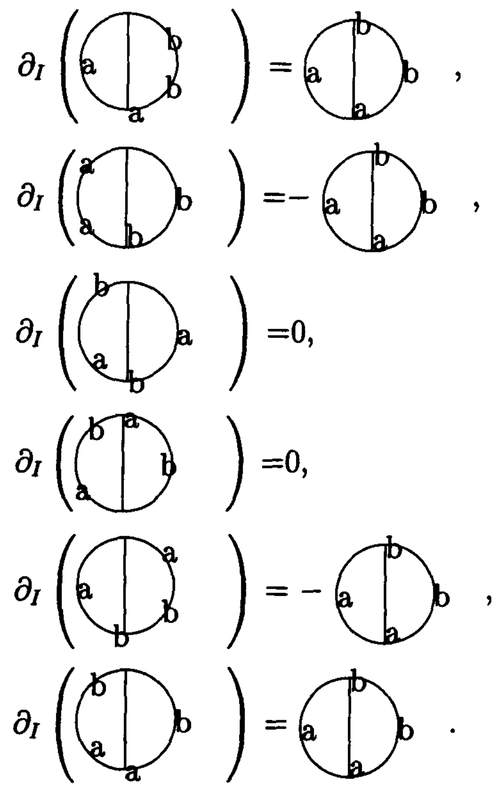
Now we compute the row sequences under the inner boundary operator and list some tables for our computation. For convenience, we only list the computation the following sequence:

$$0 \mapsto \mathcal{S}_{1,0} \xrightarrow{\partial_{Ia}} \mathcal{S}_{1,-1} \xrightarrow{\partial_{Ib}} \mathcal{S}_{1,-2} \xrightarrow{\partial_{I}} 0.$$

(A). $\partial_{Ia}(\mathcal{S}_{1,0})$ is showing in the following table:



(B). $\partial_{Ia}(\mathcal{S}_{1,-1})$ is showing in the following table:



Now we describe the matrix representations of boundaries ∂_{Ia} and ∂_{Ib}.

We compute all the graphs in $\partial_{Ia}(\mathcal{S}_{1,0})$ and $\partial_{Ib}(\mathcal{S}_{1,-1})$ as follows:

In $\partial_{Ia}(\mathcal{S}_{1,0})$, we take a graph from each **isomorphic class** and fix the **orientation** of the graph. And then we compare the orientation of each graphs which is isomorphic to the fixed orientation graph. The work for $\partial_{Ib}(\mathcal{S}_{1,-1})$ is just the same, thus we can obtain the two **matrix representations** of these two boundary operators. We pay attention to the **comparison** between the isomorphic graphs in $\partial_{Ia}(\mathcal{S}_{1,0})$ and $\mathcal{S}_{1,-1}$. This is an important thing we want to emphasize. From the computation of $\partial_{Ia}(\mathcal{S}_{1,0})$ and $\partial_{Ib}(\mathcal{S}_{1,-1})$ and the matrix representations, we define that the matrix of $\partial_{Ia}(\mathcal{S}_{1,0})$ as $M_{1,0}$ and that of $\partial_{Ib}(\mathcal{S}_{1,-1})$ as $M_{1,-1}$. The matrices are listed as the following:

$$M_{1,0} = \begin{pmatrix} 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ -1 & 0 & 0 & 0 & 0 & -2 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{1,-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

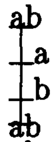
We have the multiplication of these two matrices to be zero matrix and conclude that $\partial_I^2 = 0$.

4.4 The acyclic property of based plane graphs

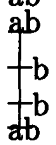
The main purpose of this subsection is to prove that the image of ∂_{Ia} is contained in the kernel of ∂_{Ib} and that the image of ∂_{Ic} is contained in the kernel of ∂_{Id} and similarly for ∂_{Ie} and ∂_{If}.

We compute the kernels of ∂_{Ia}, ∂_{Ic}, ∂_{Ie},∂_{Ig}, ∂_{Ih}, and denote the kernels $K_{1,0}$, $K_{2,0}$, $K_{3,0}$, $K_{0,0}$, $K_{-1,0}$, respectively. Thus we have the generators of them as follows:

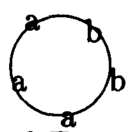
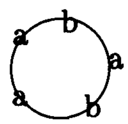
(1). $K_{-1,0}$ contains the following 3 generators:

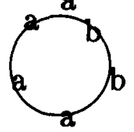
$\alpha_1 =$  ,

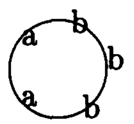
$\alpha_2 =$  ,

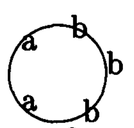
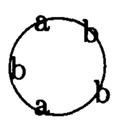
$\alpha_3 =$  .

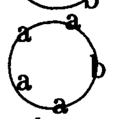
(2). $K_{0,0}$ contains the following 6 generators:

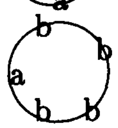
$\beta_1 =$  $+$  ,

$\beta_2 =$  ,

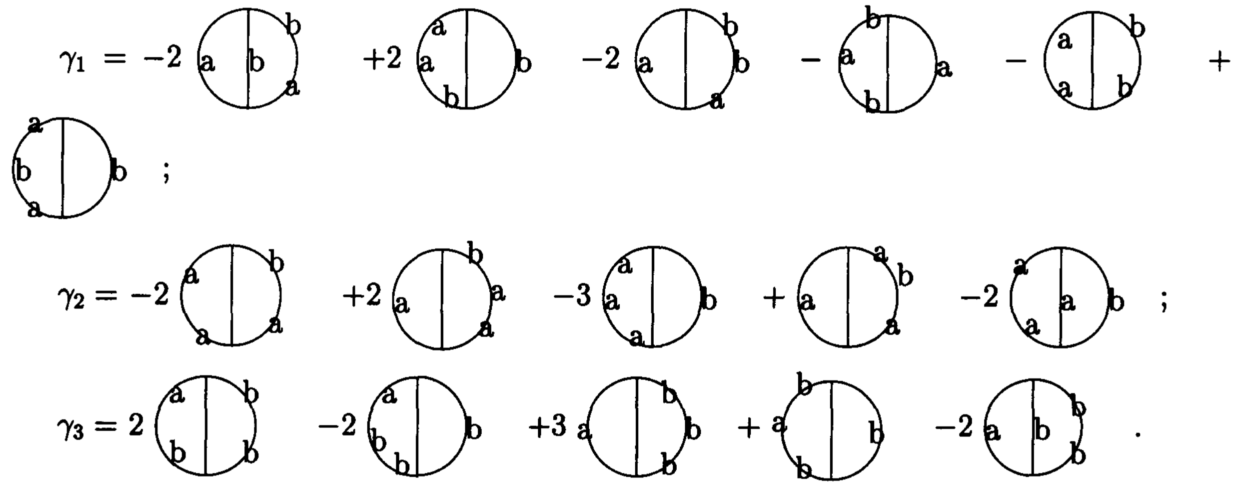
$\beta_3 =$  ,

$\beta_4 =$  $+$  ,

$\beta_5 =$  ,

$\beta_6 =$  .

(3). $K_{1,0}$ contains the following 3 generators:



(4). $K_{2,0}$ is trivial.

(5). $K_{3,0}$ is also trivial.

By the computation of the outer boundary operator for $\{K_{p,0}\}$, we have the result that the rank of $K_{1,0}$ is 3 and the rank of $K_{0,0}$ is 6 and that of $K_{-1,0}$ is 3. Thus the **Euler characteristic number** is zero. This is a good information for the acyclic property of the above sequence. We have computed the following sequence:

$$0 \mapsto K_{1,0} \xrightarrow{\partial_O} K_{0,0} \xrightarrow{\partial_O} K_{-1,0} \xrightarrow{\partial_O} 0.$$

By the relation of the image and kernel of ∂_O, and the number of the rank of $\{K_{p,0}\}$, we conclude that $H_{-1,0} = H_{0,0} = H_{1,0} = 0$. By the above consequences, we complete the proof of the **Theorem 2**. □

From the **subsections 4.1 to 4.4**, we have proved Theorem 2 in this section.

Appendix

In this appendix, we describe the historical background about the homology theory of based plane graph.

1 Knot invariants from the perturbative Chern-Simons theory

Knot invariant is a knot function which is invariant under the isotopy deformation.

The **Chern-Simons theory** is the most popular example of topological field theory in 3 dimensions. This theory associates some topological invariants and there are several ways to define the knot invariants, which are all closely related to one another. First of all there are the non-perturbative methods by Witten, Reshetikhin and Turaev: Witten [W] used fundamental properties of **quantum field theory**, in particular, the path integral formulation, and Reshetikhin and Turaev [RT] used **quantum groups**. These two definitions are equivalent to each other.

Around 1989, several efforts had been made with the perturbative approach. The first of them is given by Guadagnini, Martellini and Mintchev [GMM] in the case that a knot K in the 3-dimensional space $M = \mathbf{R}^3$, using propagators and the trivalent graphs which are composed of base points on the K, inner points in $\mathbf{R}^3$ and all of the inner points are trivalent, all the base points are univalent. This approach was then elaborated by Bar-Natan [B1] [B2] to all order. The case that the space M is a general closed 3-manifold without any link, was treated by Axelrod and Singer [AS]. A common feature of all the work is the trivalent graphs expansion that is familiar in **perturbative quantum field theory**. Invariants are defined at every order in the expansion, where the order of a graph is defined to be the number of edges minus the number of inner points of the graph; each is a formal sum of several terms corresponding to the graphs of the given order and each term is an integral over the **configuration space** of the graph. As in the paper, Γ denotes a graph, V denotes the set of all vertices of Γ, and U denotes the subset of V consisting of the univalent vertices, that is, the base points. The configuration space is defined as $C_K(\Gamma) = \{f : V \mapsto \mathbf{R}^3 \mid f \text{ is a one-to-one map and } f|_U \text{ is an order preserving map from } U \text{ to } K\}$. (The univalent vertices are supposed to have a cyclic order.) Here, the map $f : V \mapsto \mathbf{R}^3$ represents the graphs $f(\Gamma)$ which is equivalent to Γ, thus $C_K(\Gamma)$ is the set of all the graphs which is equivalent to Γ. For such a graph Γ, K is called the support of the graph.

Meanwhile, Vassiliev studied the topology of the complement of knot space in smooth function space and used the Vassiliev complex to present its topology construction. The homology group of the Vassiliev complex was shown to produce the knot invariants. The subject of Vassiliev knot invariant, also known as finite type invariant, was developed rapidly. The starting point of Vassiliev [V] was the space of all immersions of S^1 in S^3. In this space, a knot type is a cell whose faces are singular knots with a finite number of transversal double points. Any knot invariant can be extended to the kind of singular knots. It is said to be a finite type invariant of order $\leq n$, if it vanishes on all singular knots with more than n double points. Let V^n be the space of invariants of order $\leq n$, Bar-Natan found that V^n/V^{n-1} embeds in the dual of the space $\mathcal{A}_n$ of BN diagrams of order n, where the BN diagrams are proposed by Bar-Natan in papers [B1], [B2]. Kontsevich showed that these two spaces are in fact isomorphic.

In the following, $\mathcal{A}$ denotes the direct sum of all the BN diagram space $\mathcal{A}_n$, $n = 0, 1, 2, \cdots$, it is an algebra under the connected sum multiplication; and this is the famous "algebra of diagram", or "algebra of uni-trivalent graph".

In 1991, from the Perturbative Chern-Simons theory, Kontsevich and Bar-Natan independently achieved a universal Vassiliev invariant integral over the configuration space of Feynman graphs of any order with value in the algebra of uni-trivalent graph $\mathcal{A}$, that is,

$$Z(K) = \sum_{\Gamma} \frac{I(\Gamma)}{|\Gamma|}[\Gamma] ,$$

in which $I(\Gamma)$ is the configuration space integral of Γ and $|\Gamma|$ is the number of elements in its automorphism group.

To get the configuration space integral $I(\Gamma)$ of Γ, we give each edge an orientation, then there is a canonical map $\Psi : C_K(\Gamma) \mapsto \prod_k S^2$ defined by $\Psi(f) = (\frac{f(w_i)-f(v_i)}{|f(w_i)-f(v_i)|})_{i=1}^k$, where v_i and w_i is the two vertices of the i-th edge of Γ. Let ω_0 be the normalized Gaussian form on S^2 and ω be the product of ω_0 which is the normalized volume form on $\prod_k S^2$. Then we define the configuration space integral of Γ as $I(\Gamma) = \int_{C_K(\Gamma)} \Psi^*(\omega)$. Although the formula was written out, there was no rigorous proof.

Fortunately, in 1993, Kontsevich by adopting the iterated integral theory of Kuo-Tsai Chen, and the Cauchy form in the integral, arrived the Kontsevich integral $Z(K)$, and was soon proved to be the "Universal Vassiliev invariant", also known as universal Quantum Invariant. And so, the framework on Quantum Invariant initiated by Jones since 1984 had been combined.

Nevertheless, a breakthrough on the theory of the configuration space was made in 1994. Bott and Taubes [BT] used the construction of compactification of the configuration space due to W. Fulton and MacPherson [FP], to get a compact manifold with corners and to prove finiteness of the configuration space integral. In order to show that the contributions of graphs of a given order summed up to an invariant, we must compute the variations of these integrals under a small change of the embedding of K; this procedure was proved to be quite difficult and lengthy.

By the standard argument of Stokes's Theorem, the variation of the integral $I(\Gamma)$ can be expressed in terms of integrals of $\Psi^*(\omega)$ restricted to the boundaries of the configuration space of Γ. Bott and Taubes [BT] showed that the variations could be split into two parts, the "diagrammatic" variation and the "anomalous" variation. The **diagrammatic variation** can immediately be read from the trivalent graphs which correspond to the differential of Kontsevich graph complex, obtained by collapsing the edges. The **anomalous variation** is more difficult to compute, but is well-behaved of proportional to the variation of the first order contribution, the "**self-linking number**". The constant of the proportionality is however unknown in general, except the independence of the embedding. Thus the anomalous boundaries couldn't be neglected besides the principal diagrammatic boundary considered by Kontsevich and Bar-Natan.

2 Anomalous boundary and its universal configuration space

The anomalous boundaries contain graphs concentrated on a certain point of the knot. These graphs are called infinitesimal graphs on knots. By the smoothing condition, the infinitesimal part of knot can be considered as a straight line, that is, the tangent line at the point . Therefore, as the whole graph collapses into a point P on the support, it can be considered a graph lying on the tangent line, that is, with the line as the support.

When the knot deforms in the 3-space, the tangent direction of knots varies on a portion of S^2. Thus Bott and Taubes constructed a universal configuration space of infinitesimal graphs, which we describe as follows:

For a unit vector $x \in S^2$, l_x denotes the line $\{tx, t \in \mathbb{R}\}$ with increasing order. Let $C_{l_x}(\Gamma)$ be the configuration space of Γ with the support l_x, then define $W_x = C_{l_x}(\Gamma)/T.D.$, where $T.D.$ denotes the translation and dilatation relations. Also define $W(\Gamma) = \cup_{x \in S^2} W_x(\Gamma)$, it is a fibre bundle over S^2 with fibre $W_x(\Gamma)$. $W(\Gamma)$ is the universal space of infinitesimal graphs equivalent to Γ. The dimension of $W(\Gamma)$ is the same as that of $C_K(\Gamma)$ and we may also consider the configuration space integral over $W(\Gamma)$. $W(\Gamma)$ is called the configuration space of the anomalous boundary of Γ. And $I(W(\Gamma))$ denote the integral over the space $W(\Gamma)$, it is called the anomalous integral of Γ

Altschuler and Freidel [AF] proved the following results about the anomalous integrals $I(W(\Gamma))$:

(a). If Γ is even order, then $I(W(\Gamma)) = 0$.

(b). If Γ is not primitive, then $I(W(\Gamma)) = 0$, where "primitive" means that Γ is connected when we take away the support of Γ.

(c). $I(W(\ \ ))=2$, where " " is the one chord graph.

Therefore, to solve the case of order 3, we need to compute the anomalous integral $I(W(\ \ ))$. As far as infinitesimal graphs are concerned, we should compute the two integrals, $I(W(\ \ ))$ and $I(W(\ \ ))$. It was guessed that $I(W(\ \ ))\neq 0$. But the first author used the degree theory proving that $I(W(\ \ ))$ and $I(W(\ \ ))$ were both equal to $\frac{1}{192}$. It results in zero when the two integrals are summed up together. Thus, for the case of order 3, there is no graph of non-zero $I(W(\Gamma))$, that is, the zero-anomaly in order 3, explained below.

3 "Anomaly" and Universal Vassiliev Invariants

As in the construction of universal Vassiliev invariant, we combine the graphs multiplying with the associated anomalous integrals,

$$\alpha=\sum_{\Gamma:Primitive}\frac{I(W(\Gamma))}{2|\Gamma|}[\Gamma]=0\ ,$$

it is the "anomaly", also an element in $\mathcal{A}$. In the future work, we shall show that Conjecture 2.4 implies the zero-anomaly, that is, $\alpha=0$ in $\mathcal{A}$.

With the help of the anomaly α, Altschuler and Freidel [AF] adjusted the configuration space integral $Z(K)$ and obtained a framed knot invariant. They reached a conclusion that for a knot K and a framing ν of K, $\hat{Z}(K,\nu)=Z(K)(exp(\alpha\cdot\tau(K,\nu)))$ is a framed knot invariant, where $\tau(K,\nu)$ is the total torsion, given by $\tau(K,\nu)=\frac{1}{2\pi}\int ds\dot{\phi}(s)\frac{(\dot{\phi}(s),\nu(x),\dot{\nu}(s))}{|\dot{\phi}(s)\wedge\nu(s)|^2}$ and ϕ is the embedding of the knot K. (The framing ν is a normal vector field on ϕ.) $\hat{Z}(K,\nu)$ is also a Universal Vassiliev invariant.

Altschuler and Freidel's computation strongly depends on the theory developed by Bar-Natan, we describe their theory in the following.

A key step of the computations was deriving the logarithm of $Z(K)$.

$$log(Z(K))=\sum_{[\Gamma]}\frac{I_K(\Gamma)}{|\Gamma|}C([\Gamma])\ ,$$

where $C:\mathcal{A}_n\mapsto\mathcal{A}_n$ is an endomorphism of vector space proposed by Bar-Natan, satisfying the following equality

$$C([\Gamma])=\begin{cases}[\Gamma], & \text{if }\Gamma\text{ is a primitive graphs,}\\ 0, & \text{if }\Gamma\text{ is not prime,}\end{cases}$$

where, a graph Γ is **prime** if Γ cannot be expressed as a product of two or more non-trivial subgraphs; and a graph Γ is **primitive** if the resulting graph is connected after taking out the support of Γ.

Together with the excellent analysis of Bott and Taubes on the variation of the integrals, Altschuler and Freidel got the following beautiful variation formula for the logarithm of $Z(K)$.

$$d[log(Z(K))] = \frac{dI(\theta)}{2} \sum_{\Gamma:Primitive} \frac{I(W(\Gamma))}{|\Gamma|}[\Gamma] \text{ ,}$$

which was exactly equal to a multiple of the anomaly α. ($I(\theta)$ is the self-linking integral.)

Thus, if the anomaly α is equal to zero, then $Z(K)$ is a knot invariant. By combining this result of zero-anomaly with the consequence of T. Q. T. Le and J. Murakami [LM], we have the important result that the universal Vassiliev invariants derived from the configuration space integral and the Kontsevich integral are equal.

In 1999, S. Poirier[P] tried to extend his study of the framed knot invariants of Altschuler and Freidel to the tangles. He, in turn put forth his theory of the **limit configuration space integral**, and obtained the equality $Z^l(\times) = exp(\frac{\Delta\alpha-\alpha_1-\alpha_2}{2})$, where α_1, α_2 denote the anomaly on the first and the second string, respectively, and Δ denotes the duplication map which maps a graph with the support of a line into the sum of the graphs with the support of two parallel lines. That is, $\Delta(\Gamma)$ is the sum of all of the $[\Gamma']$, where $[\Gamma']$ are obtained by all possible distribution of the base points of Γ on the support of the two parallel lines.

In light of the result of S. Poirier equality $Z^l(\times) = exp(\frac{\Delta\alpha-\alpha_1-\alpha_2}{2})$, if we can prove the zero-anomaly, we are sure to obtain the result $Z^l(\times) = exp(\frac{H}{2})$, in which H refers to one chord graph with the support of two parallel lines, and then achieve the equality of the configuration space integral and the Kontsevich integral. On the other hand, if we can show the equality $Z^l(\times) = exp(\frac{H}{2})$ directly, then we also get the zero-anomaly and etc.

In the secondly coming paper, we reduces the computation of anomaly and $Z^l(\times)$ to the computation of homology of based plane graphs in this paper.

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