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PROGRESS REPORT

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We give a short review of what we need from [?], in preparation for next section. Let $\pi : X \to S$ be a smooth proper morphism of relative dimension 1 between smooth varieties in characteristic 0. We write $K_{X/S}$ or $\omega_{X/S}$ interchangeably for the dualizing sheaf. One has an exact sequence

$$0 \to T_{X/S} \to T_X \xrightarrow{d\pi} \pi^* T_S \to 0. \tag{1.1}$$

As in [?], one defines the subsheaf $\pi^{-1}T_S \subset \pi^* T_S$ and its preimage $T_\pi = d\pi^{-1} T_\pi \subset T_X$, defining the exact sequence

$$0 \to T_{X/S} \to T_\pi \xrightarrow{d\pi} \pi^{-1} T_S \to 0. \tag{1.2}$$

Let E be a vector bundle on X, and $\lambda_E = \det R\pi_* E$ be its determinant bundle. The Atiyah algebra $\mathcal{A}_E$ is the subalgebra of the sheaf of first order differential operators on E with symbolic part in $(\mathrm{id} \otimes T_X) \cong T_X$. The relative Atiyah algebra $\mathcal{A}_{E/S} \subset \mathcal{A}_E$ consists of those differential operators with symbolic part in $T_{X/S}$, and $\mathcal{A}_{E,\pi} \subset \mathcal{A}_E$ with symbolic part in T_π. Let ${}^{tr}\mathcal{A}_E^{-1}$ be the subquotient of the sheaf defined in [?]

$$E \boxtimes_{\mathcal{O}_S} (E^* \otimes \omega_{X/S})(2\Delta) / E \boxtimes_{\mathcal{O}_S} (E^* \otimes \omega_{X/S})(-\Delta)$$

where $\Delta \subset X \times_S X$ denotes the diagonal, which fits into an exact sequence

$$0 \to \omega_{X/S} \to {}^{tr}\mathcal{A}_E^{-1} \xrightarrow{\mathrm{res}} \mathcal{A}_{E/S} \to 0. \tag{1.3}$$

The trace complex is defined by

$${}^{tr}\mathcal{A}_E^{\bullet} : \mathcal{O}_X \xrightarrow{d_{X/S}} {}^{tr}\mathcal{A}_E^{-1} \xrightarrow{\mathrm{res}} \mathcal{A}_{E,\pi} \tag{1.4}$$

with $\mathcal{A}_{E,\pi}$ in degree 0. One has

Proposition 1.1. *${}^{tr}\mathcal{A}_E^{\bullet}$ carries an algebra structure for which $R^0\pi_*({}^{tr}\mathcal{A}_E^{\bullet})$ is canonically isomorphic to $\mathcal{A}_{\lambda_E}$ ([?], 2.3.1, see also [?]).*

For the purpose of this paper it is more convient to define the trace complex concentrated only on $i = -1$ and $i = 0$ of the original trace complex. This modified trace complex is still denoted by ${}^{tr}\mathcal{A}_E^{\bullet}$ whose 0-th direct image is easily seen to be the same as that of the orginal one. Henceforth we shall use the modified trace complex. One has now an exact sequence

$$0 \to \omega_{X/S}[1] \to {}^{tr}\mathcal{A}^\bullet_E \to \mathcal{A}^\bullet_{E,\pi} \to 0, \tag{1.5}$$

where the complex $\mathcal{A}^\bullet_{E,\pi}$ is defined by

$$\mathcal{A}^\bullet_{E,\pi} : \mathcal{A}_{E/S} \to \mathcal{A}_{E,\pi}, \tag{1.6}$$

and thus is quasi-isomorphic to $\pi^{-1}T_S$.

Notations 1.2. Let $\pi : X \to S$ be as before, and $g : S \to M$ be a smooth morphism where M is a smooth variety. Denote by

$$\mathcal{A}^\bullet_{E,\pi/M} \subset \mathcal{A}^\bullet_{E,\pi}$$

the inverse image of $\pi^{-1}T_{S/M}$ by the symbolic map $\mathcal{A}^\bullet_{E,\pi} \to \pi^{-1}T_S$, and by

$${}^{tr}\mathcal{A}^\bullet_{E/M} \subset {}^{tr}\mathcal{A}^\bullet_E$$

the inverse image of $\mathcal{A}^\bullet_{E,\pi/M}$ via the map ${}^{tr}\mathcal{A}^\bullet_E \to \mathcal{A}^\bullet_{E,\pi}$.

In summary one has

Proposition 1.3. *The exact sequences*

$$0 \to \omega_{X/S}[1] \to {}^{tr}\mathcal{A}^\bullet_E \to \mathcal{A}^\bullet_{E,\pi} \to 0$$
$$0 \to \omega_{X/S}[1] \to {}^{tr}\mathcal{A}^\bullet_{E/M} \to \mathcal{A}^\bullet_{E,\pi/M} \to 0$$

have 0*-th direct images (via* π*) isomorphic to*

$$0 \to \mathcal{O}_S \to \mathcal{D}^{\leq 1}_S(\lambda_E) \to T_S \to 0$$
$$0 \to \mathcal{O}_S \to \mathcal{D}^{\leq 1}_{S/M}(\lambda_E) \to T_{S/M} \to 0.$$

For use in later sections, we review a description of ${}^{tr}\mathcal{A}^\bullet_E$ in terms of local coordinates, [?], p. 660. Let t be a local coordinate (along the fiber), and a trivilization $\mathcal{O}^n_X \cong E$; s a local coordinate on S. Note t naturally induces local coordinates (t_1, t_2) around the diagonal of $X \times_S X$. One has isomorphisms

$$\mathcal{O}_X \oplus \mathrm{Mat}_n(\mathcal{O}_X) \oplus \mathcal{O}_X \cong {}^{tr}\mathcal{A}^{-1}_E; \tag{1.7}$$
$$(\chi, B, \nu) \to \left[\frac{\chi(t_1)}{(t_2 - t_1)^2} + \frac{B(t_1)}{(t_2 - t_1)} + \nu(t_1)\right] dt_2;$$
$$T_\pi \oplus \mathrm{Mat}_n(\mathcal{O}_X) \cong {}^{tr}\mathcal{A}^0_E = \mathcal{A}_{E,\pi}; (\tau, A) \to \tau(t,s)\partial_t + \mu(s)\partial_s + A$$

For different choices of coordinates and trivializations, there are formulas for transition functions, and also for the algebra structure (Lie brackets); we refer the details to [?], p. 660.

2. First Order Differential Operators of the Determinant Bundle on the Moduli

The main result of this section is Theorem ?? which enables us to take care of (??). First we set up the notation. Let $p : C \to B$ be a smooth family of genus g ($g \geq 2$) curves between smooth varieties. We are free to shrink B if necessary. Denote by $L \to C$ a line bundle, by $\tilde{g} : \tilde{S} \to B$ the family of moduli spaces of stable vector bundles of fixed rank $r \geq 2$ and fixed determinant $L|_{C_b}$ ($b \in B$), by $\tilde{\pi} : \tilde{X} = C \times_B \tilde{S} \to \tilde{S}$ and by $\tilde{E}$ a universal bundle on $\tilde{X}$ (assuming it exists).

We shall be working with a slightly more general situation. Let S, $S \subset \tilde{S}$, be a smooth variety such that $g = \tilde{g}|_S : S \to B$ is surjective; denote by $\pi : X \to S$ the pullback of $\tilde{\pi} : \tilde{X} \to \tilde{S}$ via $S \hookrightarrow \tilde{S}$, by $E \to X$ the pullback of $\tilde{E} \to \tilde{X}$.

For the sake of completeness, later in this section we will relax the situation so that bundles above are allowed to be *semistable*. This part will find an application to the genus $g = 2$ case, cf. Theorem ??.

Definition 2.1. In the notation of Section 5, define

$$\mathcal{E}nd(E)^{-1} := \mathrm{res}^{-1}(\mathcal{E}nd(E)) \subset^{tr} \mathcal{A}_E^{-1} \quad \text{and}$$
$$\mathcal{E}nd^0(E)^{-1} := \mathrm{res}^{-1}(\mathcal{E}nd^0(E)) \subset \mathcal{E}nd(E)^{-1}$$

(where $\mathcal{E}nd(E) = \mathcal{E}nd^0(E) \oplus \mathcal{O}_X$ with its trace free part $\mathcal{E}nd^0(E)$ and the trivial bundle $\mathcal{O}_X$) giving exact sequences

$$0 \to \omega_{X/S} \to \mathcal{E}nd^0(E)^{-1} \xrightarrow{\mathrm{res}} \mathcal{E}nd^0(E) \to 0;$$
$$0 \to \omega_{X/S} \to \mathcal{E}nd(E)^{-1} \xrightarrow{\mathrm{res}} \mathcal{E}nd(E) \to 0.$$

Notations 2.2. $F := \mathcal{E}nd^0(E)$, $F^{-1} := \mathcal{E}nd^0(E)^{-1}$ and λ_F the associated determinant bundle on S ($\lambda_F \cong K_{\tilde{S}/B}|_S$ canonically).

Consider the natural morphism

$$\mathrm{Sym}^2(\mathcal{E}nd^0(E)^{-1}) \otimes_{\mathcal{O}_X} T_{X/S} \xrightarrow{q} \mathrm{Sym}^2(\mathcal{E}nd^0(E)) \otimes_{\mathcal{O}_X} T_{X/S} \to 0$$

induced by $\mathcal{E}nd^0(E)^{-1} \to \mathcal{E}nd^0(E)$. Denote the kernel of q by $\mathcal{K}$. It is easily seen a canonical isomorphism $\iota : \mathcal{K} \cong \mathcal{E}nd^0(E)^{-1}$ (cf. (??)).

Definition 2.3.

$$q^{-1}(\mathrm{id} \otimes T_{X/S}) := S(\mathrm{Sym}^2(\mathcal{E}nd^0(E)^{-1}) \otimes_{\mathcal{O}_X} T_{X/S})$$

where $id \in \mathrm{Sym}^2(\mathcal{E}nd^0(E))$ is the identity element.

It follows the exact sequence

(2.1)
$$0 \to \mathcal{E}nd^0(E)^{-1} \xrightarrow{\iota} S(\mathrm{Sym}^2(\mathcal{E}nd^0(E)^{-1}) \otimes_{\mathcal{O}_X} T_{X/S}) \to T_{X/S} \to 0.$$

Write

$$\text{(2.2)} \qquad \mathrm{KS}_S : T_S \to R^1\pi_*\mathcal{A}^0_{E/S}, \quad \mathcal{A}^0_{E/S} = \mathcal{A}_{E/S}/\mathcal{O}_X;$$
$$\mathrm{KS}_B : g^*T_B \to R^1\pi_* T_{X/S}$$

for the Kodaira-Spencer maps.

Remark 2.4. One way to see KS_S of (??) is via the natural map $\mathcal{A}^\bullet_{E,\pi} \to \mathcal{A}^0_{E/S}[1]$ with (??); similarly KS_B is via $(T_{C/B} \to T_p) \to T_{C/B}[1]$ combined with its pullback via $g : S \to B$. The diagram (??) via natural maps $T_S \to g^*T_B$ and $R^1\pi_*\mathcal{A}^0_{E/S} \to R^1\pi_*T_{X/S}$, for KS maps commutes.

The main theorem of this section is the following.

Theorem 2.5. *i) Suppose* KS_B *is injective and* $\dim S_b = \dim \tilde{S}_b$, $\forall b \in B$. *There are canonical isomorphisms*

$$\phi : R^1\pi_*(\mathcal{E}nd^0(E)^{-1}) \cong R^0\pi_*(\mathcal{E}nd(F)^{-1} \to \mathcal{A}_{F,\pi/B})$$
$$\cong R^0\pi_*^{tr}\mathcal{A}^\bullet_{F/B}.$$

ii) Suppose KS_S *is an isomorphism. Fix a section* $s : B \to C$ *and a coordinate* x *(along fiber of* $C \to B$*) around* s*. There is a canonical isomorphism (dependent only on the choice of* x*)*

$$\phi = \phi_{-2r} : R^1\pi_*S(\mathrm{Sym}^2(\mathcal{E}nd^0(E)^{-1})\otimes_{\mathcal{O}_X} T_{X/S}) \cong R^0\pi_*({}^{tr}\mathcal{A}_F^{-1} \to \mathcal{A}_{F,\pi}).$$

(The R.H.S. of i), ii) are canonically identified with $\mathcal{D}^{\leq 1}_{S/B}(\lambda_F)$, $\mathcal{D}^{\leq 1}_S(\lambda_F)$ *respectively, cf. Proposition ??.)*

References

[1] A. Beilinson, V. Schechtman: Determinant bundles and Virasoro algebras, Commun. Math. Phys. **118** (1988), 651-701.

[2] H. Esnault, I-H. Tsai: Determinant bundle in a family of curves, after A. Beilinson and V. Schechtman, Commun. Math. Phys. **211** (2000), 359-363.