行政院國家科學委員會專題研究計畫 <u>期中進度報告</u>

殘留應力彈性方程式的 Carleman 型估計及其應用(1/2)

計畫類別：個別型計畫
計畫編號：NSC91-2115-M-002-019-
執行期間：91 年 08 月 01 日至 92 年 07 月 31 日
執行單位：國立臺灣大學數學系暨研究所

計畫主持人：王振男

報告類型：精簡報告
報告附件：國外研究心得報告
處理方式：本計畫可公開查詢

中　華　民　國 92 年 5 月 28 日

Uniqueness in inverse problems for an elasticity system with residual stress by single measurement*

Jenn-Nan Wang

Abstract

In this paper we consider an elasticity system with residual stress. The constitutive equation of this elasticity system differs from that of the isotropic elasticity system by $R + (\nabla u)R$, where R is the residual stress tensor. This system is not isotropic due to the existence of the residual stress R. Thus, it is not possible to reduce the principal part of the system to uncoupled wave operators as we have for the isotropic elasticity system. Here we investigate inverse problems of identifying the force term or the density by a single measurement of lateral boundary. We establish uniqueness results by Carleman estimates when the residual stress is small.

We consider a linear elasticity system with non-vanishing residual stress in this article. The residual stress is modelled by a symmetric second-rank tensor $R(x) = (r_{jk}(x)) \in C^1(Q)$ satisfying

$$\nabla \cdot R = 0 \quad \text{in} \quad Q,$$

where $\nabla \cdot R$ is a vector with components given by

$$(\nabla \cdot R)_j = \sum_k \partial_k r_{jk},$$

and

$$R\nu = \sum_k r_{jk}\nu_k = 0 \quad \text{on} \quad \Gamma,$$

where $\nu = {}^t(\nu_1, \nu_2, \nu_3)$ is the unit outer normal vector to $\partial\Omega$. Let $u(t,x) = {}^t(u_1, u_2, u_3) : Q \to \mathbb{R}^3$ be the displacement vector, then the first Piola-Kirchhoff stress is

$$\begin{aligned} S(u) = \; & R + (\nabla u)R + \tilde{\lambda}(\mathrm{tr}\epsilon)I + 2\tilde{\mu}\epsilon + \beta_1(\mathrm{tr}\epsilon)(\mathrm{tr}R)I + \beta_2(\mathrm{tr}R)\epsilon \\ & + \beta_3((\mathrm{tr}\epsilon)R + \mathrm{tr}(\epsilon R)I) + \beta_4(\epsilon R + R\epsilon), \end{aligned} \tag{0.1}$$

*This is a joint work with Ching-Lung Lin, Department of Mathematics, National Cheng-Kung University.

where $\tilde{\lambda}(x), \tilde{\mu}(x)$ are the Lamé moduli, $\beta_1(x)\cdots\beta_4(x)$ are material parameters, and

$$\epsilon = \frac{1}{2}(\nabla u + {}^t\nabla u)$$

is the strain tensor and $(\nabla u)_{jk} = (\partial_k u_j)$ (see [14]). In this paper we assume that $\beta_3 = \beta_4 = 0$, i.e.

$$S(u) = \lambda(\mathrm{tr}\epsilon)I + 2\mu\epsilon + R + (\nabla u)R, \tag{0.2}$$

where

$$\lambda = \tilde{\lambda} + \beta_1(\mathrm{tr}R), \quad \mu = \tilde{\mu} + \frac{1}{2}\beta_2(\mathrm{tr}R).$$

We interrupt the exposition here to say a few words on the residual stress model. The constitutive equation (0.2) is close to the one considered by Robertson in [17] where he used the form

$$S(u) = \tilde{\lambda}(\mathrm{tr}\epsilon)I + 2\tilde{\mu}\epsilon + R + (\nabla u)R$$

to investigate the boundary determination of the residual stress by the Dirichlet-to-Neumann map. Hoger [4] also considered an elasticity system with residual stress where she used the constitutive equation

$$S(u) = R + (\nabla u)R - \frac{1}{2}(\epsilon R + R\epsilon) + \tilde{\lambda}(\mathrm{tr}\epsilon)I + 2\tilde{\mu}\epsilon$$

in her study. Based on Hartig's law, Man [14] argued that the constitutive equation of a realistic isotropic medium with residual stress should be given by (0.1) which describes a prestropic polycrystalline aggregate whose constituting crystallites are randomly oriented. Here we choose the constitutive equation (0.2) for studying related inverse problems based on two reasons. On one hand, it is close to the realistic model as pointed out by Man. On the other hand, some basic properties for the elasticity system with the constitutive equation (0.2) have been established, especially Carleman estimates which lead to the uniqueness and stability of the Cauchy problem [10]. Those Carleman estimates play an important role in the study of some related inverse problems.

Henceforth, we denote

$$\begin{aligned} \mathcal{L}u &= \nabla \cdot S(u) \\ &= (\lambda+\mu)\nabla(\nabla\cdot u) + \mu\Delta u - (\nabla\cdot u)\nabla\lambda - (\nabla u + {}^t(\nabla u))\nabla\mu \\ &\quad -\nabla\cdot((\nabla u)R)). \end{aligned} \tag{0.3}$$

Now let y be a solution to the following initial boundary value problem

$$\begin{cases} \rho\partial_t^2 y - \mathcal{L}y = 0 & \text{in} \quad Q, \\ y(t,x) = \varphi(t,x) & \text{on} \quad \Gamma, \\ y(0,x) = a(x) & \text{in} \quad \Omega, \end{cases} \tag{0.4}$$

where $\rho(x) > 0$ is the density of the medium. In this paper, we are concerned with the inverse problem of determining the density $\rho(x)$ by measuring the traction of y on Γ.

Problem 0.1. *Assume that coefficients $\tilde{\lambda}$, $\tilde{\mu}$, β_1, β_2 and R are given. Can one uniquely determine $\rho(x)$ in Ω by measuring $S(y)\nu|_\Gamma$? In other words, let y and $\tilde{y}$ be solutions of* (0.4) *associated with densities ρ and $\tilde{\rho}$. Does $S(y)\nu|_\Gamma = S(\tilde{y})\nu|_\Gamma$ imply $\rho(x) = \tilde{\rho}(x)$ in Ω?*

Using the standard technique, see Klibanov [12] for example, we can see that Problem 0.1 is closely related to an inverse source problem. To be precise, let $u(t,x)$ be a solution solving

$$\begin{cases} \rho\partial_t^2 u - \mathcal{L}u = f(x)g(t,x) & \text{in} \quad Q, \\ u(t,x) = 0 & \text{on} \quad \Gamma, \\ u(0,x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{0.5}$$

where $f(x)$ is a scalar function in Ω and $g(t,x) = {}^t(g_1(t,x), g_2(t,x), g_3(t,x))$ is a vector function in Q. Then we consider the following inverse problem.

Problem 0.2. *Let an appropriate $T > 0$ be given. Does $S(u)\nu|_\Gamma = 0$ imply $f(x) = 0$ in Ω?*

Notice that in (0.4) and (0.5), we do not provide $\partial_t y(0,x)$ and $\partial_t u(0,x)$ since they can be also determined in the related inverse problem (see [8], [12]).

The proofs of uniqueness in Problem 0.1 and 0.2 rely on Carleman estimates. The basic idea originated from Bukhgein and Klibanov's paper [2]. After their paper, the uniqueness to similar inverse problems for hyperbolic equations based on Carleman estimates have been investigated, for example Bukhgeim [1], Isakov [7], [8], Khaĭdarov [11], Klibanov [12], Yamamoto [18] and Kubo [13].

The aforementioned results all dealt with a single hyperbolic equation. There were only a few attempts on systems of equations. For the isotropic elasticity system, an attempt has been made by Isakov [9] where he proved the Carleman estimate and established the uniqueness for the inverse source problem. It should be noted that in [9] Isakov transformed the principal part of the isotropic elasticity system to a composition of two scalar wave operators. For Maxwell's system, we mention Yamamoto's result in [19].

Our work is motivated by [6] in which Ikehata, Nakamura and Yamomoto considered the isotropic elasticity system with variable coefficients. They used a different way to diagonalize the system. Namely, they introduced an auxiliary function $\nabla \cdot u$ and transformed the principal part of the elasticity system to a diagonal system with wave operators as its diagonal components. The elasticity system we consider here is not isotropic due to the existence of the residual stress. Therefore, the principal part can not be reduced to uncoupled wave operators as we have for the isotropic elasticity system. Nevertheless, by introducing two auxiliary functions $\nabla \cdot u$ and $\nabla \times u$, we can transform the principal part to uncoupled wave operators plus second order operators in x variables acting only on u with coefficients involving first derivatives of the residual stress. When the residual stress is assumed to be small, to take care of the additional second order derivatives of u, we merely need a Carleman estimate for the Laplacian (see similar arguments in [10]).

One the other hand, for the determination of the density in Problem 0.1, we only require one *single* measurement provided that the initial displacement satisfies an appropriate condition. We would like to point out that for the inverse problem of identifying the density in [6], *three* measurements are needed in the three dimensional case. We also want to compare our result with a result by Isakov [9] where he proved the uniqueness in determining the density by using *four* measurements in three dimensional case.

Finally, we would like to make some remarks on other related results in the parameters identification problem for the elasticity system. The first general result in this direction was proved by Nakamura and Uhlmann [15] in which they showed that two Lamé coefficients are uniquely determined by the static Dirichlet-to-Neumann map. In the dynamic setting, Rachele [16] proved that the finite-time Dirichlet-to-Neumann map uniquely determines the speeds of compressional and shear waves. Rachele's result implies that if one of the parameters, namely, density function and Lamé coefficients, is known, then other two parameters can be uniquely determined by the boundary map. It should be noted that results in [15] and [16] require *infinitely* many boundary measurements. On the practical side, the unique determination of Lamé coefficients or Lamé coefficients plus density by *finitely* many boundary measurements has not been solved yet. Starting from this paper, we hope to pursue other interesting inverse problems of identifying parameters in the elasticity system by finitely many boundary measurements, even including the identification of residual stress.

References

[1] A.L. Bukhgeim, *Introduction to the theory of inverse problems*, Nauka, Novosibrisk, 1988.

[2] A.L. Bukhgeim and M.V. Klibanov, *Global uniqueness of a class of multidimensional inverse problems*, English translation, Soviet Math. Dokl., 24(1981), 244-247. Nauka, Novosibrisk, 1988.

[3] M. Eller and V. Isakov, *Carleman estimates with two large parameters and applications*, Proceedings of AMS summer research conference in Boulder, Colorado, July 1999, Contemporary Math., 268(2000), 117-137.

[4] A. Hoger, *On the determination of residual stress in an elastic body*, J. Elasticity, 16(1986), 303-324.

[5] L. Hörmander, "The Analysis of Linear Partial Differential Operators I-IV", Springer-Verlag, Berlin, 1983.

[6] M. Ikehata, G. Nakamura, M. Yamamoto *Uniqueness in inverse problems for the isotropic Lame system*, J. Math. Sci. Univ. Tokyo, 1998(5), 627-692.

[7] V. Isakov, *Uniqueness of the continuation across a time-like hyperplane and related inverse problems for hyperbolic equations*, Comm. PDE, 14(1989), 465-478.

[8] V. Isakov, "Inverse Source Problems", Mathematical Surveys and Monographs No.34, AMS, Providence, 1990.

[9] V. Isakov, *A nonhyperbolic Cauchy problem for* $\Box_b \Box_c$ *and its applications to elasticity theory*, Comm. Pure Appl. Math., 1986(39), 747-767.

[10] V. Isakov, G. Nakamura, J.N. Wang *Uniqueness and stability in the Cauchy problem for the elasticity system with residual stress*, Preprint.

[11] A. Khaĭdarov, *Carleman estimates and inverse problems for second order hyperbolic equations*, Math. USSR Sbornik, 58(1987), 267-277.

[12] M. Klibanov, *Inverse problems and Carleman estimates*, Inverse Problems, 8(1992), 575-596.

[13] M. Kubo, *Identification of the potential term of the wave equation*, Preceedings of the Japan Acad. 71(1995), 174-176.

[14] C.S. Man, *Hartig's law and linear elasticity with initial stress*, Inverse Prob., 14(1998), 313-319.

[15] G. Nakamura and G. Uhlmann, *Global uniqueness for an inverse boundary problem arising in elasticity*, 118(1994), 457-474.

[16] L. Rachele, *An inverse problem in elastodynamics: uniqueness of the wave speeds in the interior*, J. Diff. Equ., 162(2000), 300-325.

[17] R. Robertson, *Boundary identifiability of residual stress via the Dirichlet to Neumann map*, Inverse Problems, 13(1997), 1107-1119.

[18] M. Yamamoto, *Uniqueness and stability in multidimensional hyperbolic inverse problems*, J. Math. Pures Appl., 78(1999), 65-98.

[19] M. Yamamoto, *On an inverse problem of determining source terms in Maxwell's equations with a single measurement*, Inverse problems, tomography, and image processing (Newark, DE, 1997), 241–256, Plenum, New York, 1998.