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Diagnosabilities of Multiprocessor Systems

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Abstract

In this paper, we compute diagnosabilities of multiprocessor systems under two diagnosis models: the PMC model and the comparison model. In each model, we further consider two different diagnosis strategies: the precise diagnosis strategy proposed by Preparata *et al.* and the pessimistic diagnosis strategy proposed by Friedman. The main result of this paper is to determine diagnosabilities of regular systems with certain conditions, which include several widely used multiprocessor systems such as hypercubes and its variants.

Keywords. Diagnosis, diagnosis by comparison, hypercube, multiprocessor system, pessimistic diagnosis strategy, PMC model, precise diagnosis strategy.

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1 Introduction

Fault diagnosis is an important step in the design of multiprocessor systems and VLSI/WSI-oriented computing systems. And automatic fault diagnosis has been considered an integral part of the process of achieving fault tolerance. A diagnosis strategy means a process to diagnose faults, and it is *precise* (respectively, *pessimistic*) if no fault-free processor is mistaken as a faulty one (respectively, a fault-free processor may be mistaken as a faulty one). In order to diagnose faults, a number of tests are performed among processors and the collection of all test results is referred to as a *syndrome*.

Suppose that S is a system with at most t faulty processors. Based on a precise diagnosis strategy, S is *t-diagnosable* if given any syndrome, all faulty processors can be determined [23]. The maximum t for which S is t-diagnosable is called the *diagnosability* of S [3]. On the other hand, based on a pessimistic diagnosis strategy, S is *t/s-diagnosable* if given any syndrome, all faulty processors can be confined to a set of at most s processors, where $t \leq s$ [14]. The maximum t for which S is t/t-diagnosable is also called the *diagnosability* of S [19].

Preparata, Metzem, and Chien [23] first proposed a model, called the *PMC model*, for fault diagnosis in a multiprocessor system. Under the PMC model, all tests are performed between two adjacent processors, and it was assumed that a test result is reliable (respectively, unreliable) if the processor that initiates the test is fault-free (respectively, faulty). The PMC model was also adopted in [3], [12], [15], [16], [18], [19] and [29].

Malek [22] proposed another model, called the comparison model, under which each test is initiated by a unique arbitrator. The arbitrator feeds a pair of processors with the same task and input and then compares their outputs. It is assumed that the outputs are identical if they are fault-free, and distinct otherwise. Only a fault-free arbitrator can guarantee a reliable test result. Later, Maeng and Malek [21] modified Malek's model so that multiple arbitrators were allowed and each arbitrator can test any two of its adjacent processors. Maeng and Malek's model is referred to as the *MM model*. Sengupta and Dahbura [26] further suggested a modification of the MM model, called the *MM* model*, in which any processor has to test another two processors if the former is adjacent to the later two. The MM* model was also adopted in [2], [13] and [30].

Under the PMC model with a precise (respectively, pessimistic) strategy, an n-dimensional hypercube has diagnosability n [3] (respectively, $2n-2$ [19]); an n-dimensional enhanced hypercube has diagnosability $n+1$ (respectively, $2n$) [29]; an n-dimensional Möbius cube has diagnosability n (respectively, $2n-2$) [12]; an n-dimensional star graph has diagnosability $n-1$ (respectively, $2n-4$) [18]. On the other hand, under the MM* model with a precise strategy, an n-dimensional hypercube has diagnosability n [30]; an n-dimensional enhanced hypercube has diagnosability $n+1$ [30]; an n-dimensional crossed cube has diagnosability n [13]; a k-ary n-dimensional butterfly graph has diagnosability $2k-2$ if $k \geq 3$ and $n \geq 3$ [2].

In this paper, we establish sufficient conditions for computing diagnosabilities of regular systems. Our results are valid for both the PMC and the MM* models with both the precise and the pessimistic strategies. As consequences, diagnosabilities of many well-known multiprocessor systems can be obtained. These include hypercubes, enhanced hypercubes, twisted cubes, crossed cubes, Möbius cubes, cube-connected cycles, tori, star graphs, *etc.* Some of these are established in several papers as described in the previous paragraph.

In the next section, we introduce definitions and notations which are used throughout this paper. We then derive in Section 3 the diagnosabilities of regular systems with certain conditions under different models and strategies. Consequently, the diagnosabilities of several widely used multiprocessor systems are determined in Section 4. Finally, in Section 5, we conclude the paper with some remarks.

2 Preliminaries

In the study of multiprocessor systems, the topology of a system is often adequately represented by a graph $G = (V, E)$, where each node $u \in V$ denotes a processor and each edge $(u, v) \in E$ denotes a link between nodes u and v. Previously, when the PMC model was adopted, a self-diagnosable system was often represented by a directed graph in which an arc directed from node u to node v means that u can test v. On the other hand, when the MM* model was adopted, a self-diagnosable system was often represented by a multigraph in which an edge (u, v) labeled with w means that w is an arbitrator for u and v, i.e., w can test both u and v. Since multiple arbitrators for the same pair of nodes are allowed, the representing graph can be a multigraph.

Throughout this paper we use a graph $G = (V, E)$ to represent a self-diagnosable system. For a node u of G, denote by $N(u)$ the set of all its neighboring nodes, i.e., $N(u) = \{v \in V : v \text{ is adjacent to } u\}$. For a subset S of V, let $N(S) = \cup_{v \in S} N(v)$.

Definition 1 *Under the PMC model, a syndrome σ for system G is defined as follows. For any two distinct nodes u and v with $v \in N(u)$,*

$$\sigma(u, v) = \begin{cases} 0, & \text{if } v \text{ is tested by } u \text{ to be fault-free;} \\ 1, & \text{if } v \text{ is tested by } u \text{ to be faulty.} \end{cases}$$

Definition 2 *Under the MM* model, a syndrome σ for system G is defined as follows. For any three distinct nodes u, v and w with $u, v \in N(w)$,*

$$\sigma(u, v; w) = \begin{cases} 0, & \text{if the test results of } u \text{ and } v \text{ by } w \text{ are identical;} \\ 1, & \text{if the test results of } u \text{ and } v \text{ by } w \text{ are distinct.} \end{cases}$$

Notice that the test result initiated by a faulty processor is unreliable, and more than one syndrome may be produced for G with faulty nodes. For each subset $F \subseteq V$, let $\Omega(F)$ represent the set of syndromes that can be produced if F is the set of all faulty nodes. When G has faulty nodes, a syndrome σ is randomly generated for the purpose of fault diagnosis. We call F an *allowable fault set with respect to* σ under the PMC model (respectively, the MM* model) if (1) and (2) hold (respectively, (1*) and (2*) hold).

(1) $\sigma(u, v) = 0$ for $u \in V - F$ and $v \in V - F$.

(2) $\sigma(u, v) = 1$ for $u \in V - F$ and $v \in F$.

(1*) $\sigma(u, v; w) = 0$ for $u \in V - F$, $v \in V - F$ and $w \in V - F$.

(2*) $\sigma(u, v; w) = 1$ for ($u \in F$ or $v \in F$) and $w \in V - F$.

It is easy to see that F is an allowable fault set with respect to σ if and only if $\sigma \in \Omega(F)$. Also, the set of all faulty nodes in G is an allowable fault set with respect to σ.

Two subsets F_1 and F_2 of V are *distinguishable* if $\Omega(F_1) \cap \Omega(F_2) = \emptyset$, and *indistinguishable* otherwise. When F_1 and F_2 are distinguishable, for each syndrome σ in $\Omega(F_1) \cup \Omega(F_2)$, exactly one of F_1 and F_2 is an allowable fault set with respect to σ. On the other hand, when F_1 and F_2 are indistinguishable, they are allowable fault sets with respect to each syndrome in $\Omega(F_1) \cap \Omega(F_2)$.

Lemma 1 *Under the precise diagnosis strategy, a system $G = (V, E)$ is t-diagnosable if for every syndrome σ any two subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ are distinguishable.*

Proof. Suppose F^* is the set of all faulty nodes in G and $|F^*| \leq t$. It suffices to show that for the syndrome σ, F^* can be uniquely identified. In other words, F^* is the only set in the family

$$\mathcal{C} = \{F \subseteq V : F \text{ is an allowable fault set with respect to } \sigma \text{ and } |F| \leq t\}.$$

Suppose to the contrary that there exists another set $F' \in \mathcal{C}$. Then $\sigma \in \Omega(F^*) \cap \Omega(F')$, which is a contradiction to the assumption that F^* and F' are distinguishable. □

The following lemma follows from the definition easily, and so its proof is omitted.

Lemma 2 **([9])** *If F_1 and F_2 are two allowable fault sets with respect to σ, then so is $F_1 \cup F_2$.*

Lemma 3 *Under the pessimistic diagnosis strategy, a system $G = (V, E)$ is t/t-diagnosable if for every syndrome σ any two subsets F_1 and F_2 of V with $|F_1| \leq t$, $|F_2| \leq t$ and $|F_1 \cup F_2| > t$ are distinguishable.*

Proof. Suppose F^* is the set of all faulty nodes in G and $|F^*| \leq t$. It suffices to show that for the syndrome σ, a superset S of F^* with $|S| \leq t$ can be determined. Let

$$\mathcal{C} = \{F \subseteq V : F \text{ is an allowable fault set with respect to } \sigma \text{ and } |F| \leq t\}.$$

Choose a set $S \in C$ with the maximum cardinality. It suffices to show that $F^* \subseteq S$. Suppose to the contrary that F^* is not a subset of S. In this case, $|S| < |F^* \cup S|$. Since F^* is an allowable fault set with respect to σ, Lemma 2 assures that $F^* \cup S$ is also an allowable fault set with respect to σ. Then, $|S| < |F^* \cup S|$ and the maximality of $|S|$ imply $F^* \cup S \notin \mathcal{C}$, and so $|F^* \cup S| > t$. By the assumption, F^* and S are distinguishable, which is a contradiction because $\sigma \in \Omega(F^*)$ and $\sigma \in \Omega(S)$. $\square$

The following characterization is useful for the distinguishability of two sets under the MM* model. The *symmetric difference* of two set A and B is the set $A\Delta B = (A\cup B)-(A\cap B)$.

Lemma 4 ([26]) *Suppose $G = (V, E)$ is a system under the MM* model. Two distinct subsets F_1 and F_2 of V are distinguishable if and only if there is a node $v \in V - (F_1 \cup F_2)$ such that at least one of the following conditions holds.*

(1) $|N(v) \cap (F_1 - F_2)| \geq 2$.

(2) $|N(v) \cap (F_2 - F_1)| \geq 2$.

(3) $|N(v) - (F_1 \cup F_2)| \geq 1$ and $|N(v) \cap (F_1 \Delta F_2)| \geq 1$.

3 Diagnosabilities of regular systems

This section determines diagnosablities of regular systems with certain conditions, which include many well-know multiprocessor systems such as hypercubes and its variants. Our results are for systems under the PMC model and the MM* model, each using both the precise and pessimistic diagnosis strategies.

3.1 Precise diagnosis strategy

In this subsection, we consider results for regular systems with the precise diagnosis strategy. See Theorems 6 and 7. In this case, we need the following lemma. A graph is called *r-regular* if its every node has the same degree r. A graph is *triangle-free* if it does not contain a complete graph of three nodes as a subgraph.

Lemma 5 *Suppose $r \geq 2$ and $G = (V, E)$ is an r-regular graph satisfying the following two conditions.*

(a) *G is triangle-free.*

(b) *$N(u) \neq N(v)$ for every two distinct nodes u and v of G.*

Then, for any two distinct subsets F_1 and F_2 of V with $|F_1| \leq r$ and $|F_2| \leq r$, there exists a node $w \in F_1 \Delta F_2$ adjacent to some node $x \notin F_1 \cup F_2$.

Proof. Suppose to the contrary that $N(w) \subseteq F_1 \cup F_2$ for all $w \in F_1 \Delta F_2$. As $F_1 \neq F_2$, we may choose $u \in F_1 \Delta F_2$. In this case, $N(u) \subseteq F_1 \cup F_2$. By the facts that $|N(u)| = r$ and $|F_1 \cap F_2| < \max\{|F_1|, |F_2|\} \leq r$, we know that u has a neighbor $v \in F_1 \Delta F_2$. Again, we have $N(v) \subseteq F_1 \cup F_2$. Since G is triangle-free, $N(u) \cap N(v) = \emptyset$. Therefore,

$$2r = |N(u)| + |N(v)| = |N(u) \cup N(v)| \leq |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2r.$$

Consequently, all inequalities are equalities and so $N(v) = (F_1 \cup F_2) - N(u)$ and $F_1 \cap F_2 = \emptyset$. For $r \geq 2$, u has another neighbor $v' \neq v$. Since $F_1 \cap F_2 = \emptyset$, we have $v' \in F_1 \Delta F_2$. By a similar argument as above, we have $N(v') = (F_1 \cup F_2) - N(u)$ and so $N(v) = N(v')$, a contradiction to condition (b). $\square$

Theorem 6 *Suppose $r \geq 2$ and G is an r-regular graph. Then G is r-diagnosable under the PMC model using the precise diagnosis strategy if the following conditions hold.*

(a) *G is triangle-free.*

(b) *$N(u) \neq N(v)$ for every two distinct nodes u and v of G.*

Proof. Assume to the contrary that G is not r-diagnosable. Then, by Lemma 1, there exist a syndrome σ and two indistinguishable allowable fault sets F_1 and F_2 with respect to σ, where $|F_1| \leq r$ and $|F_2| \leq r$. By Lemma 5, there exists a node $w \in F_1 \Delta F_2$ such that $N(w) - (F_1 \cup F_2) \neq \emptyset$. Without loss of generality, we may assume that $w \in F_1 - F_2$. Choose a node $p \in N(w) - (F_1 \cup F_2)$. If $\sigma(p, w) = 0$ (respectively, $\sigma(p, w) = 1$), then F_1 (respectively, F_2) is not an allowable fault set with respect to σ, a contradiction (refer to the paragraph after Definition 2). $\square$

For the discussion of the diagnosability under the MM* model using the precise diagnosis strategy, we need the following two graphs. The first one is the graph G_8 obtained from a 8-cycle joining the 4 pairs of the farest vertices. More precisely, $V(G_8) = \{x_1, x_2, \ldots, x_8\}$ and $E(G_8) = \{(x_i, x_{i+1}) : 1 \le i \le 7\} \cup \{(x_8, x_1)\} \cup \{(x_j, x_{j+4}) : 1 \le j \le 4\})$. See Figure 1 for G_8.

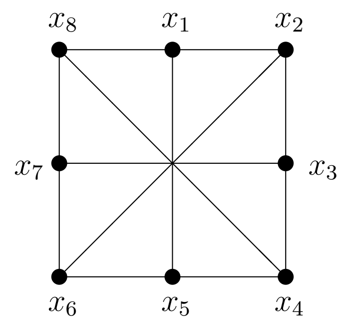

Figure 1: The graph G_8.

The second one is the graph $G_{n,n}$ obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. More formally, $V(G_{n,n}) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and $E(G_{n,n}) = \{(x_i, y_j) : 1 \le i \le n, 1 \le j \le n \text{ and } i \ne j\}$. See Figure 2 for $G_{n,n}$.

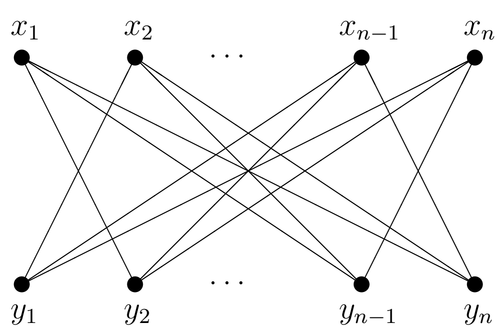

Figure 2: The graph $G_{n,n}$.

Theorem 7 *Suppose $r \ge 3$ and $G = (V, E)$ is an r-regular graph, which is not isomorphic to G_8 or $G_{r+1,r+1}$. Then G is r-diagnosable under the MM* model using the precise diagnosis strategy if the following conditions hold.*

(a) *G is triangle-free.*

(b) *$N(u) \ne N(v)$ for every two distinct nodes u and v of G.*

Proof. Suppose to the contrary that G is not r-diagnosable. Then, by Lemma 1, there exist a syndrome σ and two indistinguishable allowable fault sets F_1 and F_2 with respect to σ, where $|F_1| \leq r$, $|F_2| \leq r$ and $F_1 \neq F_2$. Let $F' = F_1 \Delta F_2$, which is nonempty. According to Lemma 5, F' has at least one node w adjacent to some node $x \notin F_1 \cup F_2$. Denote F_3' and F_3 the sets of all such nodes w and x, respectively. Then F_3' and F_3 both are nonempty. Since F_1 and F_2 are indistinguishable, none of the conditions in Lemma 4 holds. It follows that for any node $v \in F_3$,

(i) $|N(v) \cap (F_1 - F_2)| \leq 1$,

(ii) $|N(v) \cap (F_2 - F_1)| \leq 1$, and

(iii) $N(v) \subseteq F_1 \cup F_2$.

Notice that (iii) follows from Lemma 4 (3) and the fact $N(v) \cap F' \neq \emptyset$ for $v \in F_3$. Then, F_3 is an independent set with $N(F_3) \subseteq F_1 \cup F_2$. We also have $|N(v) \cap F_1 \cap F_2| \geq r-2$, which implies $|F_1 \cap F_2| \geq r-2$ and so

$$r = |N(v)| \leq |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq r + r - (r-2) = r + 2.$$

Let $F_3 = \{v_1, v_2, \ldots, v_s\}$ and consider the following cases.

Case 1. $|F_1 \cup F_2| = r$.

For each $v_i \in F_3$, since $N(v_i) \subseteq F_1 \cup F_2$ and $|N(v_i)| = r = |F_1 \cup F_2|$, we have $N(v_i) = F_1 \cup F_2$. Choose a node $w \in F_3'$. As G is triangle-free, w has no neighbors in $F_1 \cup F_2$, and so has all neighbors in F_3. Then, $s \geq r \geq 3$. Therefore, F_3 has at least two distinct nodes with the same neighbors, a contradiction to condition (b).

Case 2. $|F_1 \cup F_2| = r + 1$. In this case, both $F_1 - F_2$ and $F_2 - F_1$ are nonempty.

For each $v_i \in F_3$, since $N(v_i) \subseteq F_1 \cup F_2$ and $|N(v_i)| = r < r + 1 = |F_1 \cup F_2|$, we have $N(v_i) = (F_1 \cup F_2) - \{w_i\}$ for some $w_i \in F_1 \cup F_2$. Notice that $w_i \neq w_j$ when $i \neq j$.

Choose a node $w \in F_3'$, say $w \in F_1 - F_2$, adjacent to some node $v_i \in F_3$. As G is triangle-free, w has no neighbor in $N(v_i) = (F_1 \cup F_2) - \{w_i\}$, and so has at least $r-1$ neighbors in F_3, say $\{v_1, v_2, \ldots, v_{r-1}\} \subseteq N(w)$. Then, $s \geq r - 1 \geq 2$.

Suppose $F_1 - F_2$ has another node w' other than w. Then, (i) implies $w_1 = w' = w_2$ and so $N(v_1) = N(v_2)$, a contradiction to condition (b). So $F_1 - F_2 = \{w\}$. Suppose w is adjacent to some $w' \in F_1 \cup F_2$. Then condition (a) implies $w_1 = w' = w_2$ and so $N(v_1) = N(v_2)$, a contradiction to condition (b). So $N(w) \subseteq F_3$, say $N(w) = \{v_1, v_2, \ldots, v_r\}$. It is then the case that $F_1 \cup F_2 = \{w_1, w_2, \ldots, w_r, w\}$.

Choose $w_i \in F_2 - F_1$ with $1 \le i \le r$, say $i = 1$. As $w_1 \in F_3'$, the same arguments for w show that $N(w_1) \subseteq F_3$. Condition (b) implies that $N(w_1) \neq N(w)$ and so $s \ge r + 1$ On the other hand, the fact that $w_i \neq w_j$ for $i \neq j$ and condition (b) imply that $s \le |F_1 \cup F_2| = r + 1$. Therefore, $s = r + 1$ and so G is isomorphic to $G_{r+1,r+1}$, which is impossible.

Case 3. $|F_1 \cup F_2| = r + 2$.

In this case, $r - 2 \le |F_1 \cap F_2| = |F_1| + |F_2| - |F_1 \cup F_2| \le r + r - (r + 2) = r - 2$. Thus, $|F_1 \cap F_2| = r - 2$ and $|F_1| = |F_2| = r$. These imply that the inequalities in (i) and (ii) are equalities and $F_1 \cap F_2 \subseteq N(v_i)$ for each $v_i \in F_3$, say $N(v_i) = (F_1 \cap F_2) \cup \{w_i, x_i\}$, where $w_i \in F_1 - F_2$ and $x_i \in F_2 - F_1$. Notice that the nodes w_i (respectively, x_i) are not necessarily distinct, but the sets $\{w_i, x_i\}$ are distinct. Let $F_1 - F_2 = \{w_i, w_i'\}$ and $F_2 - F_1 = \{x_i, x_i'\}$.

For each i, since G is triangle-free, w_i (respectively, x_i) has at most two neighbors in $F_1 \cup F_2$ (i.e., w_i' and x_i') and hence at least $r - 2$ neighbors in F_3. If $|N(w_i) \cap F_3| = |N(x_i) \cap F_3| = r - 2$, then $w_i', x_i' \in N(w_i) \cap N(x_i)$ and so $N(w_i) \cap F_3 \neq N(x_i) \cap F_3$ by condition (b). Hence $s \ge r - 1 \ge 2$. Also since each node in $F_1 \cap F_2$ is adjacent to all nodes in F_3, we have $r \ge s$. As $\{w_1, x_1\} \neq \{w_2, x_2\}$, either $w_1 \neq w_2$ or $x_1 \neq x_2$. By symmetric, we may assume $w_1 \neq w_2$. As w_1 and w_2 are both in $F_1 - F_2$, by (i), $N(w_1) \cap F_3$ and $N(w_2) \cap F_3$ are disjoint. Hence $s \ge 2r - 4$.

Suppose $r \ge 4$. Then, $r \ge s \ge 2r - 4 \ge r$ and $|F_1 \cap F_2| = r - 2 \ge 2$. Therefore, $r = s$ and the set $F_1 \cap F_2$ contains two distinct nodes a and b with $N(a) = N(b) = F_3$, a contradict to condition (b). Hence $r = 3$ and so $3 = r \ge s \ge 2r - 4 = 2$. Suppose $s = 2$. Then both w_1 and w_2 are adjacent to exactly one node in F_3, and hence two nodes in $F_1 \cup F_2$. That is, $w_1w_2, w_1x_1', w_2x_2' \in E$. As G is triangle free, $x_1' \neq x_2'$ which means $x_1 = x_2'$ and $x_2 = x_1'$.

By interchanging the role of w_i and x_i in the above arguments, we also have $x_1x_2 \in E(G)$. Hence $N(w_1) = N(x_1) = \{v_1, w_2, x_2\}$, a contradiction to condition (b). Thus $s = 3$. In this case, we have $|F_1 \cap F_2| = 1$, $|F_3| = 3$ and $|F_1 - F_2| = |F_2 - F_1| = 2$. Also each node in F_3 has exactly one neighbor in $F_1 - F_2$, $F_1 \cap F_2$ and $F_2 - F_1$, respectively. Then G is isomorphic to G_8 as shown in Figure 3, a contradiction to the assumption. $\square$

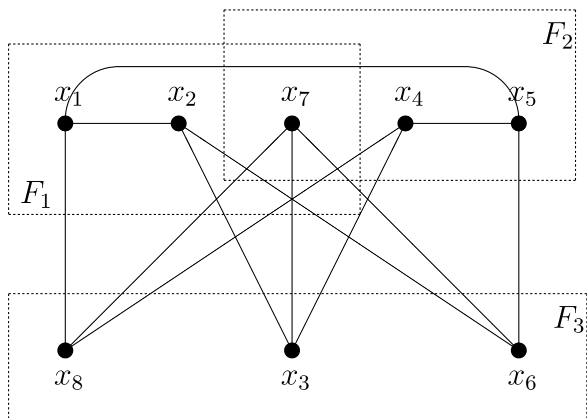

Figure 3: G is isomorphic to G_8 for $r = 3$.

3.2 Pessimistic diagnosis strategy

In this subsection, we consider results for regular systems with pessimistic diagnosis strategy. See Theorems 9 and 10. In this case, we need the following lemma. First, define the graph G_5 shown in Figure 4.

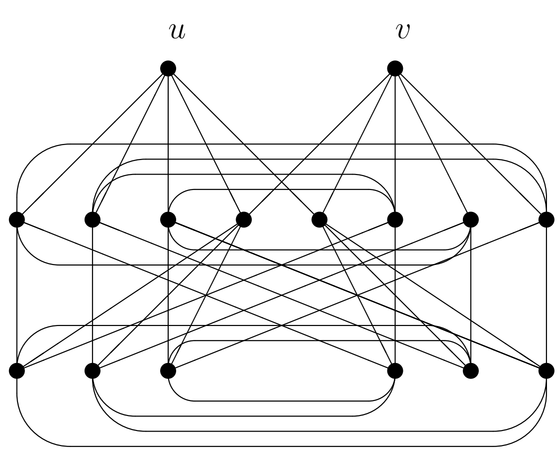

Figure 4: The graph G_5.

Lemma 8 *Suppose $r \geq 5$ and $G = (V, E)$ is an r-regular graph, which is not isomorphic to G_5 and satisfies the following two conditions.*

(a) *G is triangle-free.*

(b) *$|N(u) \cap N(v)| \leq 2$ for every two distinct nodes u and v of G.*

Then, for any two distinct subsets F_1 and F_2 of V with $|F_1| \leq 2r - 2$ and $|F_2| \leq 2r - 2$ but $|F_1 \cup F_2| > 2r - 2$, there exists a node $w \in F_1 \Delta F_2$ adjacent to some node $x \neq F_1 \cup F_2$.

Proof. Suppose to the contrary that $N(w) \subseteq F_1 \cup F_2$ for all $w \in F_1 \Delta F_2$. Let $F' = F_1 \Delta F_2$. By the assumptions, $F_1 \cap F_2$ is a proper subset of F_1 and F_2, and so $|F'| \geq 2$. We may choose two distinct vertices u and v from F'. Let

$$\begin{array}{lll} A_0 = \{u, v\}, & A'_0 = A_0 \cap F' = A_0, & A''_0 = A_0 \cap F_1 \cap F_2 = \emptyset, \\ A_1 = N(A'_0) - A_0, & A'_1 = A_1 \cap F', & A''_1 = A_1 \cap F_1 \cap F_2, \\ A_2 = N(A'_1) - (A_0 \cup A_1), & A'_2 = A_2 \cap F', & A''_2 = A_2 \cap F_1 \cap F_2; \end{array}$$

and $\alpha_i = |A_i|$, $\alpha'_i = |A'_i|$, $\alpha''_i = |A''_i|$ for $0 \leq i \leq 2$. Notice that $A_i \subseteq F_1 \cup F_2$ is the disjoint union of A'_i and A''_i and so $\alpha_i = \alpha'_i + \alpha''_i$ for each i.

Suppose u and v are adjacent. Since G is triangle-free, $\alpha_1 = 2r - 2$. As $\alpha''_1 \leq |F_1 \cap F_2| < |F_1| \leq 2r - 2$, we have $\alpha'_1 \geq 1$. So, F' has at least 3 nodes. Since G is triangle-free, two of these 3 nodes are non-adjacent. Hence, without loss of generality, we may assume that u is not adjacent to v. Let $\delta = |N(u) \cap N(v) \cap F'|$. By condition (b), $\delta \leq 2$. Since $\alpha_1 = |N(u)| + |N(v)| - |N(u) \cap N(v)|$ and $\alpha''_1 \geq |N(u) \cap N(v) \cap F_1 \cap F_2|$, we have

$$\alpha_1 + \alpha''_1 \geq |N(u)| + |N(v)| - |N(u) \cap N(v) \cap F'| = 2r - \delta. \tag{1}$$

Without loss of generality, we may assume $|N(u) \cap F'| \geq |N(v) \cap F'|$. Then $|(N(u) - N(v)) \cap F'| \geq \lceil |(N(u) \Delta N(v)) \cap F'|/2 \rceil = \lceil (\alpha'_1 - \delta)/2 \rceil$. Since each node in $N(u) \cap N(v) \cap F'$ has two neighbors in A_0 and no neighbor in A_1, it has $r - 2$ neighbors in A_2. On the other hand, each $w \in (N(u) - N(v)) \cap F'$ has one neighbor in A_0 and at most two neighbors in A_1 (for otherwise, $|N(w) \cap N(v)| \geq 3$, a contradiction to condition (b)). Hence $|N(w) \cap A_2| \geq r - 3$.

By condition (b), each node in A_2 has at most two neighbors in $N(u) \cap F'$. Consequently,

$$\begin{aligned}\alpha_2 &\geq \lceil(\delta(r-2) + \textstyle\sum_{w \in (N(u)-N(v)) \cap F'} |N(w) \cap A_2|)/2\rceil \\ &\geq \lceil(\delta(r-2) + \lceil(\alpha_1' - \delta)/2\rceil \times (r-3))/2\rceil \\ &= \lceil\delta/2 + \lceil(\alpha_1' + \delta)/2\rceil \times (r-3)/2\rceil. \end{aligned} \tag{2}$$

Also, $|F_1 \cup F_2| \geq \alpha_0 + \alpha_1 + \alpha_2 = 2 + \alpha_1 + \alpha_2$ and $|F_1 \cap F_2| \geq \alpha_1'' + \alpha_2''$ imply

$$4r - 4 \geq |F_1| + |F_2| = |F_1 \cup F_2| + |F_1 \cap F_2| \geq 2 + \alpha_1 + \alpha_2 + \alpha_1'' + \alpha_2''. \tag{3}$$

Since $\alpha_1 \geq 2r - 2$ and $\alpha_1'' = \alpha_1 - \alpha_1'$, formula (3) further induces

$$\alpha_2 \leq \alpha_1' - \alpha_2'' - 2. \tag{4}$$

Case 1. $r \geq 7$. By formula (2), $\alpha_2 \geq \alpha_1'$. Then, by formula (3), $4r - 4 \geq 2 + 2\alpha_1 \geq 2 + 2(2r-2) = 4r - 2$, a contradiction.

Case 2. $r = 6$. If $\alpha_1' \geq 9$, then $\lceil(\alpha_1' + \delta)/2\rceil \geq 5$ and so $\alpha_2 \geq (\delta + 15)/2$ by formula (2). By formulas (1) and (3), $\alpha_2 \leq 6 + \delta$. Consequently, $\delta \geq 3$, a contradiction. Hence $\alpha_1' \leq 8$. By formulas (2) and (4), $\alpha_1' - \alpha_2'' - 2 \geq \alpha_2 \geq (5\delta + 3\alpha_1')/4$, which implies $\delta = \alpha_2'' = 0$, $\alpha_1' = 8$ and $\alpha_2 = 6$. Again, by formula (2), $6 \geq \lceil(\sum_{w \in (N(u)-N(v)) \cap F'} |N(w) \cap A_2|)/2\rceil \geq \lceil\lceil\alpha_1'/2\rceil \times (r-3)/2\rceil = 6$, which implies $|(N(u) - N(v)) \cap F'| = \lceil\alpha_1'/2\rceil = 4$ and $|N(w) \cap A_2| = r - 3 = 3$ for each $w \in (N(u) - N(v)) \cap F'$.

Since G is triangle-free, $|N(w) \cap N(v)| = |N(w) \cap A_1| = r - |\{u\}| - |N(w) \cap A_2| = 2$. For any two nodes $w_1, w_2 \in (N(u) - N(v)) \cap F'$, since $u \in N(w_1) \cap N(w_2)$ we have $|N(w_1) \cap N(w_2) \cap A_2| \leq 1$ by condition (b). In fact, $|N(w_1) \cap N(w_2) \cap A_2| = 1$ for otherwise as $\alpha_2 = 6$ a third node $w_3 \in N(u) - N(v)$ has two common neighbors with w_1 or w_2, which is impossible. It follows that $N(w_1) \cap N(w_2) \cap N(v) = \emptyset$. Then $6 = |N(v)| \geq |N(u) \cap F'| \cdot |N(w) \cap N(v)| = 4 \times 2 = 8$, a contradiction.

Case 3. $r = 5$. We have $\alpha_1 \geq 8$ by formula (1). If $\alpha_1' \geq 9$, then $\alpha_2 \geq \delta + 5$ by formula (2) and $16 \geq 12 + \alpha_2 + \alpha_2'' - \delta$ by formulas (1) and (3). Both imply $16 \geq 17 + \alpha_2''$, a contradiction. If $\alpha_1' \leq 8$, then by formulas (2) and (4), $\alpha_1' - \alpha_2'' - 2 \geq \alpha_2 \geq \delta + \alpha_1'/2$, which further implies $\alpha_1' \geq 2\alpha_2'' + 2\delta + 4$.

Case 3.1. $\delta = 2$. Then $\alpha_1' = 8$, $\alpha_2'' = 0$ and $\alpha_2 = 6$. Also, $|N(u) \cap N(v)| = 2$ and so $\alpha_1 = 2r - 2 = 8$. Hence the graph is isomorphic to G_5, a contradiction.

Case 3.2. $\delta = 1$. We have $6 \leq \alpha_1' \leq 8$. If $\alpha_1' = 6$, then $\alpha_2 \geq 5$ by formula (2) and $\alpha_2 \leq 4$ by formula (4), a contradiction. If $\alpha_1' = 8$, then $\alpha_2 \geq 6$ by formula (2). By formulas (1) and (3), $16 \geq 17 + \alpha_2''$, a contradiction. Hence $\alpha_1' = 7$. Since $\alpha_1 = |N(u) + |N(v)| - |N(u) \cap N(v)| \geq 2r - 2 = 8$, we have $\alpha_1'' \geq 1$. By formulas (2) and (4), $\alpha_2 = 5$ and $\alpha_2'' = 0$. Then, by formula (3), $16 \geq 2 + \alpha_1 + 5 + 1 + 0$ imping $8 \geq \alpha_1$ and so $\alpha_1 = 8$. This further implies $|N(u) \cap N(v)| = 2$ and $|N(u) - N(v)| = 3$. Also, $|N(w) \cap A_2| \geq r - 3 = 2$ for each $w \in (N(u) - N(v)) \cap F'$.

Since $\alpha_2 = 5 = \lceil \delta/2 + \lceil (\alpha_1' + \delta)/2 \rceil \times (r-3)/2 \rceil$, formula (2) assures $|(N(u) - N(v)) \cap F'| = \lceil (\alpha_1' - 1)/2 \rceil = 3$ and $\sum_{w \in (N(u)-N(v)) \cap F'} |N(w) \cap A_2| = 6$ or 7. Three possible node adjacencies between A_2 and $N(u) \cap F'$ are shown in Figure 5. Since G is triangle-free, $|N(w) \cap (N(u) \cap N(v))| = \emptyset$, i.e., $| \cup_{w \in (N(u)-N(v)) \cap F'} N(w) \cap N(v)| \leq |N(v) - N(u)| = 3$. It is not difficult to check that there exist two nodes w_1, $w_2 \in (N(u) - N(v)) \cap F'$ satisfying at least one of the following two conditions:

(C1) $|N(w_1) \cap N(w_2) \cap A_2| = 1$ and $|N(w_1) \cap N(w_2) \cap (N(v) - N(u))| \geq 1$;

(C2) $|N(w_1) \cap N(w_2) \cap (N(v) - N(u))| \geq 2$.

Both imply $|N(w_1) \cap N(w_2)| \geq 3$, a contradiction.

Case 3.3. $\delta = 0$. We have $4 \leq \alpha_1' \leq 8$. By formulas (1) and (3), $\alpha_2 + \alpha_2'' \leq 4$. By formulas (2) and (4), $\alpha_1' - \alpha_2'' - 2 \geq \alpha_2 \geq \lceil (\sum_{w \in (N(u)-N(v)) \cap F'} |N(w) \cap A_2|)/2 \rceil \geq \lceil \alpha_1'/2 \rceil$. Table I shows possible values of α_1', $|(N(u) - N(v)) \cap F'|$ and α_2. When $4 \leq \alpha_1' \leq 7$, we have $\alpha_1 \leq 8$ and hence $\alpha_1 = 8$ by formula (3). Similar to the discussion in Case 3.2, there exist two nodes w_1, $w_2 \in (N(u) - N(v)) \cap F'$ satisfying condition (C1) or (C2), which is a contradiction. When $\alpha_1' = 8$, $|(N(u) - N(v)) \cap F'| = \alpha_2 = 4$. Since each node of A_2 is adjacent

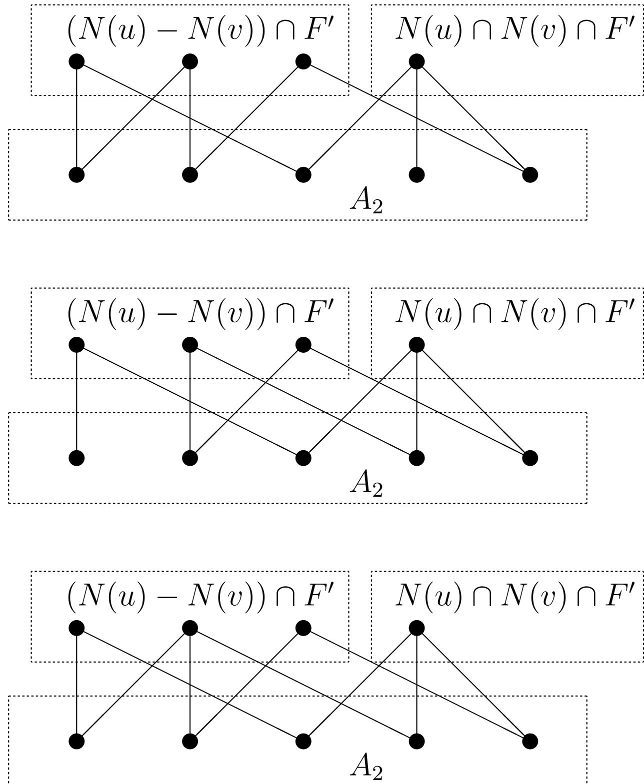

Figure 5: Three possible node adjacencies between A_2 and $N(u) \cap F'$ for $\alpha_2 = 5$, $|(N(u) - N(v)) \cap F'| = 3$ and $\delta = 1$.

to at most two nodes of $N(u)$ and $|N(w) \cap A_2| \geq r - 3 = 2$ for each $w \in (N(u) - N(v)) \cap F'$, we have $|N(w) \cap A_2| = r - 3 = 2$. It is not difficult to check that there exist two nodes w_1, $w_2 \in (N(u) - N(v)) \cap F'$ satisfying condition (C1) or (C2), a contradiction. $\square$

Table I: Possible values of α'_1, $|(N(u) - N(v)) \cap F'|$ and α_2 for $r = 5$ and $\delta = 0$.

α'_1	4	5	6		7	8
$\lvert(N(u) - N(v)) \cap F'\rvert$	2	3	3	4	4	4
α_2	2	3	3 or 4	4	4	4

Theorem 9 *Suppose $r \geq 5$ and G is an r-regular graph, which is not isomorphic to G_5. Then G is $(2r-2)/(2r-2)$-diagnosable under the PMC model using the pessimistic strategy if the following two conditions hold.*

(a) *G is triangle-free.*

(b) $|N(u) \cap N(v)| \leq 2$ *for every two distinct nodes u and v of G.*

Proof. Assume to the contrary that G is not $(2r-2)/(2r-2)$-diagnosable. Then, by Lemma 3, there exist a syndrome σ and two indistinguishable allowable fault sets F_1 and F_2 with respect to σ, where $|F_1| \leq 2r-2$ and $|F_2| \leq 2r-2$ but $|F_1 \cup F_2| > 2r-2$. By Lemma 8, there exists a node $w \in F_1 \Delta F_2$ adjacent to some $x \notin F_1 \cup F_2$. Without loss of generality, we may assume that $w \in F_1 - F_2$. If $\sigma(x, w) = 0$ (respectively, $\sigma(x, w) = 1$), then F_1 (respectively, F_2) is not an allowable fault set with respect to σ, a contradiction. □

Theorem 10 *Suppose $r \geq 6$ and $G = (V, E)$ is an r-regular graph. Then G is $(2r-2)/(2r-2)$-diagnosable under the MM* model using the pessimistic strategy if the following two conditions hold.*

(a) *G is triangle-free.*

(b) *$|N(u) \cap N(v)| \leq 2$ for every two distinct nodes u and v of G.*

Proof. Suppose to the contrary that G is not $(2r-2)/(2r-2)$-diagnosable. Then, by Lemma 3, there exist a syndrome σ and two indistinguishable allowable fault sets F_1 and F_2 with respect to σ, where $|F_1| \leq 2r-2$ and $|F_2| \leq 2r-2$ but $|F_1 \cup F_2| > 2r-2$. Let $F' = F_1 \Delta F_2$, which is non-empty. According to Lemma 8, F' has at least one node w adjacent to some node $x \notin F_1 \cup F_2$. Denote F_3' and F_3 the sets of all such nodes w and x, respectively. Then F_3' and F_3 both are nonempty. Since F_1 and F_2 are indistinguishable, none of the conditions in Lemma 4 holds. It follows that for any node $v \in F_3$,

(i) $|N(v) \cap (F_1 - F_2)| \leq 1$,

(ii) $|N(v) \cap (F_2 - F_1)| \leq 1$, and

(iii) $N(v) \subseteq F_1 \cup F_2$ $(v \in N(F')$.

Notice that (iii) follows from Lemma 4 (3) and the fact that $N(v) \cap F' \neq \emptyset$ for $v \in F_3$. Consequently, F_3 is an independent set with $N(F_3) \subseteq F_1 \cup F_2$. We also have $|N(v) \cap F_1 \cap F_2| \geq r-2$, which implies $|F_1 \cap F_2| \geq r-2$ and so

$$|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq (2r-2) + (2r-2) - (r-2) = 3r-2.$$

We first claim that for each node $w \in F_3'$, $|N(w) \cap F_3| \leq 2$. Assume to the contrary that w is adjacent to three distinct nodes p_1, p_2 and p_3 in F_3. Conditions (i) to (iii) imply $|N(p_i) \cap (F_1 \cap F_2)| \geq r-2$ for $1 \leq i \leq 3$; and condition (b) implies $|N(p_i) \cap N(p_j) \cap (F_1 \cap F_2)| \leq 1$ for $i \neq j$. Thus, $|F_1 \cap F_2| \geq (r-2)+(r-3)+(r-4) = 3r-9$, and so $|F_1 \cup F_2| = |F_1|+|F_2|-|F_1 \cap F_2| \leq (2r-2)+(2r-2)-(3r-9) = r+5$. On the other hand, $|F_1 \cup F_2| \geq |(N(p_1) \cup N(p_2) \cup N(p_3)) \cap (F_1 \cup F_2)| \geq r+(r-2)+(r-4) \geq 3r-6 > r+5$ as $r \geq 6$, a contradiction.

Case 1. $|F_3| \geq 2$ and $|N(p) \cap F_3'| = 1$ for each node $p \in F_3$.

Choose $w \in F_3'$ and $p_1 \in N(w) \cap F_3$. Also choose $p_2 \in (N(w) \cap F_3) - \{p_1\}$ if $|N(w) \cap F_3| = 2$, and $p_2 \in F_3 - \{p_1\}$ else. By condition (b), $|F_1 \cap F_2| \geq |N(\{p_1, p_2\}) \cap (F_1 \cap F_2)| \geq (r-1)+(r-3) = 2r-4$. So, $|F_1 \cup F_2| = |F_1|+|F_2|-|F_1 \cap F_2| \leq (2r-2)+(2r-2)-(2r-4) = 2r$.

On the other hand, by condition (a), $N(w) \cap N(p_1) = \emptyset$. If w is (respectively, is not) adjacent to p_2, by condition (a) (respectively, condition (b)), $N(w) \cap N(p_2) = \emptyset$ (respectively, $|N(w) \cap N(p_2)| \leq 2$). In either case, $|N(w) - (N(\{p_1, p_2\}) \cup F_3)| \geq r-3$. Hence $|F_1 \cup F_2| \geq |N(\{p_1, p_2\})| + |N(w) - (N(\{p_1, p_2\}) \cup F_3)| \geq r+(r-2)+(r-3) \geq 2r+1$ as $r \geq 6$, a contradiction to $|F_1 \cup F_2| \leq 2r$.

Case 2. $|F_3| \geq 2$ and $|N(p_1) \cap F_3'| \geq 2$ for some node $p_1 \in F_3$.

Assume that p_1 is adjacent to two distinct nodes w_1 and w_2 in F_3'. Further, assume that $|N(w_1) \cap F_3| \geq |N(w_2) \cap F_3|$. Choose $p_2 \in (N(w_1) \cap F_3) - \{p_1\}$ if $|N(w_1) \cap F_3| = 2$, or $p_2 \in (N(w_2) \cap F_3) - \{p_1\}$ if $|N(w_2) \cap F_3| = 2$, or $p_2 \in F_3 - \{p_1\}$ else. By conditions (i), (ii), (iii) and (b), $|F_1 \cap F_2| \geq |N(\{p_1, p_2\}) \cap (F_1 \cap F_2)| \geq (r-2)+(r-4) = 2r-6$. Hence $|F_1 \cup F_2| = |F_1|+|F_2|-|F_1 \cap F_2| \leq (2r-2)+(2r-2)-(2r-6) = 2r+2$.

On the other hand, condition (b) assures $|N(\{p_1, p_2\})| \geq r+(r-2) = 2r-2$. If w_1 is adjacent to p_2, then by conditions (a) and (b), $|N(w_1) - (N(\{p_1, p_2\}) \cup F_3)| \geq r-2$ and $|N(w_2) - (N(w_1) \cup N(\{p_1, p_2\}) \cup F_3)| \geq r-5$. Similarly, if w_1 is not adjacent to p_2 (as $N(w_1) \cap F_3 = 1$), then $|N(w_1) - (N(\{p_1, p_2\}) \cup F_3)| \geq r-3$ and $|N(w_2) - (N(w_1) \cup N(\{p_1, p_2\}) \cup F_3)| \geq r-4$. It follows that $|N(w_1) \cup N(w_2) - (N(\{p_1, p_2\}) \cup F_3)| \geq 2r-7$.

Hence, $|F_1 \cup F_2| \geq |N(\{p_1, p_2\})| + |N(w_1) \cup N(w_2) - (N(\{p_1, p_2\}) \cup F_3)| \geq 4r - 9 \geq 2r + 3$ as $r \geq 6$, a contradiction to $|F_1 \cup F_2| \leq 2r + 2$.

Case 3. $|F_3| = 1$, say $F_3 = \{p\}$.

Since $|F'| \geq 2$, we may choose two nodes u and v from F' such that $p \in N(u)$. Let $A = \{u, v\}$. Notice that $A \cup (N(A) - F_3) \cup N(F_3) \subseteq F_1 \cup F_2$. By conditions (a) and (b), $|A \cup (N(A) - F_3)| \geq 2r - 1$ and $|N(F_3) - (A \cup N(A))| \geq r - 3$. Then, $|F_1 \cup F_2| \geq (2r - 1) + (r - 3) = 3r - 4$. We assume that $|F_1 \cup F_2| = 3r - 4 + s$, where $s \geq 0$. Hence, $|F_1 \cap F_2| = |F_1| + |F_2| - |F_1 \cup F_2| \leq (2r-2) + (2r-2) - (3r-4+s) = r - s$. Also, $|N(F_3) \cap (F_1 \cap F_2)| \geq r - 2$. By condition (a), $|N(u) \cap F'| \geq |N(u)| - (|F_1 \cap F_2| - |N(F_3) \cap (F_1 \cap F_2)|) - |F_3| \geq r - ((r-s) - (r-2)) - 1 = r - 3 + s$. Let $F_4 = N(u) \cap F'$ and $\alpha = |N(F_4) - A \cup N(A) \cup N(F_3)|$. By conditions (a) and (b), for each node $x \in F_4$, $|N(x) - A \cup N(A)| \geq r - 3$. Then, by condition (b), $\alpha \geq (|F_4|(r-3) - |N(F_3) - A|)/2 \geq ((r-3+s)(r-3) - (r-1))/2 \geq 2 + 3s/2$ as $r \geq 6$. Hence, $|F_1 \cup F_2| \geq |A \cup (N(A) - F_3) \cup N(F_3)| + \alpha \geq (2r-1) + (r-3) + (2 + 3s/2) = 3r - 2 + 3s/2$, a contradiction to $|F_1 \cup F_2| = 3r - 4 + s$. □

4 Diagnosabilities of multiprocessor systems

The purpose of this section is to apply the results of Section 3 to eight popular multiprocessor systems. To introduce these systems, we need the following notations. Define $[m] = \{0, 1, \ldots, m-1\}$ and $[m]^n = \{x_{n-1}x_{n-2}\ldots x_0 : x_i \in [m] \text{ for } i \in [n]\}$, where m and n are positive integers. Let $x = x_{n-1}x_{n-2}\ldots x_0 \in [m]^n$ and $y = y_{n-1}y_{n-2}\ldots y_0 \in [m]^n$. The *Hamming distance* of x and y, denoted by $H(x, y)$, is the number of indices i such that $x_i \neq y_i$.

Example 1 Hypercube Q_n [27]

A *hypercube* of n dimensions can be expressed by a graph $Q_n = (V, E)$ with $V = [2]^n$ and $E = \{(x, y) : H(x, y) = 1\}$.

Example 2 Enhanced hypercube $EQ_{n,s}$ [28]

An enhanced hypercube is just a hypercube augmented with certain extra links. More precisely, an (n, s)-*enhanced hypercube* can be expressed by a graph $EQ_{n,s} = (V, E)$ with $V = [2]^n$ and $E = \{(x, y) : H(x, y) = 1$ or $y = x_{n-1}x_{n-2} \ldots x_{s+1}\bar{x}_s\bar{x}_{s-1} \ldots \bar{x}_0$ for some $0 \leq s \leq n-1\}$, where $\bar{x}_i = 1 - x_i$ for $0 \leq i \leq s$.

Example 3 Twisted cube TQ_n [11]

Assume that n is odd. Define $P_j(x) = (x_j + x_{j-1} + \ldots + x_0) \bmod 2$, where $0 \leq j \leq n-1$. A *twisted cube* of n dimensions can be expressed by a graph $TQ_n = (V, E)$ with $V = [2]^n$ and E consisting of all (x, y)'s that satisfy the following two conditions for some $0 \leq k \leq (n-1)/2$:

(1) $x_{2k}x_{2k-1} = \bar{y}_{2k}y_{2k-1}$ or ($x_{2k}x_{2k-1} = y_{2k}\bar{y}_{2k-1}$ and $P_{2k-2}(x) = 1$) or ($x_{2k}x_{2k-1} = \bar{y}_{2k}\bar{y}_{2k-1}$ and $P_{2k-2}(x) = 0$);

(2) $x_{2j}x_{2j-1} = y_{2j}y_{2j-1}$ for all $j \neq k$,

where x_0x_{-1} is regarded as x_0 when $k = 0$.

Example 4 Möbius cube MQ_n[8]

A *Möbius cube* of n dimensions can be expressed by a graph $MQ_n = (V, E)$ with $V = [2]^n$ and E containing those (x, y)'s with $y = x_{n-1}x_{n-2} \ldots x_{i+2}\, 0\, \bar{x}_i x_{i-1} \ldots x_0$ or $y = x_{n-1}x_{n-2} \ldots x_{i+2}\, 1\, \bar{x}_i\bar{x}_{i-1} \ldots \bar{x}_0$ for some $0 \leq i \leq n-2$. Besides, E contains $(x, \bar{x}_{n-1}x_{n-2} \ldots x_0)$ or $(x, \bar{x}_{n-1}\bar{x}_{n-2} \ldots \bar{x}_0)$ but not both.

Example 5 Crossed cube CQ_n [10]

A *crossed cube* of n dimensions can be expressed by a graph $CQ_n = (V, E)$ with $V = [2]^n$ and E consisting of all (x, y)'s that satisfy the following conditions for some $1 \leq m \leq n$:

(1) $x_{n-1}x_{n-2} \ldots x_m x_{m-1} = y_{n-1}y_{n-2} \ldots y_m\bar{y}_{m-1}$;

(2) $x_{m-2} = y_{m-2}$ if m is even;

(3) $(x_{2i+1}x_{2i}, y_{2i+1}y_{2i}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$ for $0 \leq i \leq \lfloor (m-1)/2 \rfloor - 1$.

Example 6 Cube-connected cycles CCC_n [24]

Cube-connected cycles can be obtained by replacing each node of a hypercube with a cycle. More precisely, *cube-connected cycles* of n dimensions can be expressed by a graph $CCC_n = (V, E)$ with $V = \{[x, i] : x \in [2]^n \text{ and } i \in [n]\}$ and $E = \{([x, i], [x, j]) : x \in [2]^n, i, j \in [n] \text{ and } j \equiv (i \pm 1) \bmod n\} \cup \{([x, i], [y, i]) : x, y \in [2]^n, i \in [n] \text{ and } y = x_{n-1}x_{n-2} \ldots x_{i+1}\bar{x}_i x_{i-1} \ldots x_0\}$.

Example 7 Torus $T_n(m)$ [6]

An *m-sided torus* of n dimensions can be expressed by a graph $T_n(m) = (V, E)$ with $V = [m]^n$ and $E = \{(x, y) : y_i \equiv (x_i \pm 1) \bmod m \text{ for some } i \in [n] \text{ and } x_j = y_j \text{ for all } j \neq i\}$.

Example 8 Star graph S_n [1]

A *star graph* of n dimensions can be expressed by a graph $S_n = (V, E)$ with V being the set of all permutations of $\{1, 2, \ldots, n\}$, and E consisting of all (u, v)'s such that $u = u_1u_2 \ldots u_k \ldots u_n$ and $v = u_ku_2 \ldots u_{k-1}u_1u_{k+1} \ldots u_n$ (i.e., swap u_1 and u_k) for some $2 \leq k \leq n$.

The diagnosabilities of these multiprocessor systems can be determined by the aid of Theorems 6, 7, 9 and 10. We first have to check if they satisfy the conditions in these theorems. As the checking is easy, we only summarize the results in Table II. Consequently, we have their diagnosabilities, as shown in Table III.

Table II: Properties of multiprocessor systems.

system	r-regular	triangle-free	$G_{r+1,r+1}$	G_8	G_5	$N(u) \neq N(v)$	$\|N(u) \cap N(v)\| \leq 2$
Q_n	$r = n$	yes	$\ncong$	$\ncong$	$\ncong$	yes if $n \geq 3$	yes if $n \geq 2$
$EQ_{n,s}$	$r = n + 1$	yes if $s \geq 2$	$\ncong$	$\ncong$	$\ncong$	yes if $n \geq 3$	yes if $n \geq 2, s \neq 2$
TQ_n	$r = n$	yes	$\ncong$	$\ncong$ if $n \neq 3$	$\ncong$	yes if $n \geq 3$	yes if $n \geq 2$
CQ_n	$r = n$	yes	$\ncong$	$\ncong$ if $n \neq 3$	$\ncong$	yes if $n \geq 3$	yes if $n \geq 2$
MQ_n	$r = n$	yes	$\ncong$	$\ncong$ if $n \neq 3$	$\ncong$	yes if $n \geq 3$	yes if $n \geq 2$
CCC_n	$r = 3$ if $n \geq 3$	yes if $n \neq 3$	$\ncong$	$\ncong$	$\ncong$	yes	yes
$T_n(m)$	$r = 2n$	yes if $m \neq 3$	$\ncong$	$\ncong$	$\ncong$	yes if $n \geq 3$	yes if $n \geq 2$
S_n	$r = n - 1$	yes	$\ncong$	$\ncong$	$\ncong$	yes	yes

$\ncong$: not isomorphic.
$N(u) \neq N(v)$: $N(u) \neq N(v)$ for any two distinct nodes u and v in V.
$|N(u) \cap N(v)| \leq 2$: $|N(u) \cap N(v)| \leq 2$ for any two distinct nodes u and v in V.

Table III: Diagnosabilities of multiprocessor systems.

system	PMC		MM*	
	precise	pessimistic	precise	pessimistic
Q_n	n **[3]**	$2n-2/2n-2$ **[19]**	n **[30]**	$2n-2/2n-2$
$EQ_{n,s}$	$n+1$ **[29]**	$2n/2n$ **[29]**	$n+1$ **[30]**	$2n/2n$
TQ_n	n	$2n-2/2n-2$	n	$2n-2/2n-2$
CQ_n	n	$2n-2/2n-2$	n **[13]**	$2n-2/2n-2$
MQ_n	n **[12]**	$2n-2/2n-2$ **[12]**	n	$2n-2/2n-2$
CCC_n	$n+2$	$2n+2/2n+2$	$n+2$	$2n+2/2n+2$
$T_n(m)$	$2n$	$4n-2/4n-2$	$2n$	$4n-2/4n-2$
S_n	$n-1$ **[18]**	$2n-4/2n-4$ **[18]**	$n-1$	$2n-4/2n-4$

[i]: also obtained in [i].

5 Conclusion

The problem of determining diagnosabilities of multiprocessor systems has received much attention since Preparata *et al.* introduced in [23] the concepts of one-step diagnosis and sequential diagnosis. The *one-step diagnosis* requires that all faulty nodes are found out by decoding the syndrome, whereas the *sequential diagnosis* consists of several diagnosis and repair phases. In each phase, one or more faulty nodes will be determined and then repaired. The process is iterated until all faulty nodes are repaired. The *one-step diagnosability* of a multiprocessor system S is defined to be the maximum number of faulty nodes allowed in S such that the one-step diagnosis of S can be performed. The *sequential diagnosability* of S is defined similarly.

The diagnosis we considered in this paper belongs to the category of one-step diagnosis. With two diagnosis models (i.e., the PMC and the comparison models) and two diagnosis strategies (i.e., the precise and the pessimistic diagnosis strategies), we have computed one-step diagnosabilities of eight multiprocessor systems. Our results were obtained as a consequence of four sufficient conditions. Compared with most previous works which computed diagnosabilities for individual systems, the four sufficient conditions can derive diagnosabilities for a class of regular systems.

Fewer results were obtained for sequential diagnosis. In [25], the problem of computing the sequential diagnosability for a general graph was proved to be co-NP complete. In [20], lower bounds on sequential diagnosabilities of grids and hypercubes were suggested. One of our further research topics is to compute sequential diagnosabilities of multiprocessor systems for different diagnosis models and different diagnosis strategies.

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