行政院國家科學委員會專題研究計畫　成果報告

交互作用粒子系統的流力極限(7)

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行政院國家科學委員會專題研究計畫成果報告

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Hydrodynamic Limit of Interacting Particle Systems (7)

計畫編號：NSC 92 – 2115 – M – 002 – 010

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一、中文摘要

本計劃中我們探討 2 維格子點空間 Z^2 上對稱簡單互斥過程並導出了一個超指數估計。此估計增進了對前作一個位置被粒子佔據的時間的大離差估計中速率函數的了解，並有助於對數個位置被粒子佔據的時間差（總平均為零）的大離差估計的研究。

關鍵詞：對稱簡單互斥過程、粒子佔據的時間、大離差估計、超指數估計

Abstract

In this project we study the symmetric simple exclusion process (SEP) on two dimensional lattice space Z^2 and derive a super-exponential estimate. This estimate gives more understanding of the rate function of the occupation time large deviations. Moreover, it helps investigating the large deviations of occupation time difference of several sites.

Keywords: symmetric simple exclusion process, occupation time, large deviations estimate, super-exponential estimate

二、報告內容

（報告正文以英文撰寫於次頁起始）

Given $T > 0$, on the configuration space $\Omega = \{0,1\}^{\mathbb{Z}^2}$, consider the *speeded-up* symmetric simple exclusion process (SEP) generated by L_T given by

$$(L_T f)(\eta) \;=\; \frac{T}{2} \sum_{\substack{x,y\in\mathbb{Z}^2\\ |x-y|=1}} [f(\sigma^{x,y}\eta) - f(\eta)] \;,$$

where the summation is carried over all nearest neighbor sites x, y, $|x-y| = 1$, of $\mathbb{Z}^2$. In this formula, f is a local function and $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x,\, y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x\;. \end{cases}$$

For each $0 \le \alpha \le 1$, denote by ν_α the Bernoulli product measure on Ω with marginals given by

$$\nu_\alpha\{\eta,\, \eta(x) = 1\} \;=\; \alpha$$

for $x \in \mathbb{Z}^2$. Clearly, $\{\nu_\alpha,\, 0 \le \alpha \le 1\}$ is a one-parameter family of reversible invariant measures. For $0 \le \alpha \le 1$, denote by $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ the probability on the path space $D(\mathbb{R}_+, \Omega)$ corresponding to SEP starting from ν_α. From now on we fix an $\alpha \in (0,1)$.

The large deviations principle of the occupation time of the origin:

$$A_T \;=\; \int_0^1 \eta_s(0)\, ds$$

under $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ as $T \to \infty$ has been established in [1]. It states that the decay rate is of order $T/\log T$, and the rate function $\Upsilon_\alpha : [0,1] \to \mathbb{R}_+$ is given by

$$\Upsilon_\alpha(\beta) \;=\; \frac{\pi}{2}\Big\{ \sin^{-1}(2\beta - 1) - \sin^{-1}(2\alpha - 1)\Big\}^2 \;.$$

To prove the above large deviations principle, we introduced in [1] the so-called *polar empirical measure*. To define this we need some notation. Let $\mathbb{Z}^2_* = \mathbb{Z}^2 - \{0\}$ and for each $T > 1$ define the projection $\sigma_T : \mathbb{Z}^2 \to [0, \infty)$ by

$$\sigma_T(x) \;=\; \frac{\log |x|}{\log T}\;, \quad |x| = \sqrt{x_1^2 + x_2^2}\;, \quad \sigma_T(0) \;=\; 0\;.$$

Let $\mu^{1,T}$ be the polar empirical measure on $\mathbb{R}_+$ given by

$$\mu^{1,T}(\eta) \;=\; \frac{1}{2\pi \log T} \sum_{x\in\mathbb{Z}^2_*} \eta(x) \frac{1}{|x|^2} \delta_{\sigma_T(x)}\;, \tag{1}$$

where δ_v is the Dirac measure concentrated on v. For $H\colon \mathbb{R}_+ \to \mathbb{R}$ bounded and piecewise continuous, denote

$$\ll H, \mu^{1,T}(\eta) \gg \;=\; \frac{1}{2\pi log T} \sum_{x\in\mathbb{Z}^2_*} H(\sigma_T(x)) \frac{1}{|x|^2} \eta(x)\;. \tag{2}$$

Let $\bar\mu^T$ be the Radon measure on $\mathbb{R}_+$ defined by

$$\bar\mu^T \;=\; \int_0^1 \mu^{1,T}(\eta_s)\, ds\;. \tag{3}$$

For $c > 0$, denote by $\mathcal{M}_c$ the set of positive Radon measures μ on $\mathbb{R}_+$ such that $\mu([a,b]) \le (b-a)+c$ for every $0 \le a \le b < \infty$:

$$\mathcal{M}_c \;=\; \Big\{\mu(dr),\ \mu([a,b]) \le (b-a)+c \text{ for } 0 \le a \le b < \infty \Big\}\,.$$

The condition on the measure of intervals makes the set $\mathcal{M}_c$, which is endowed with the vague topology, a compact separable metric space. Let $\mathcal{M}_0$ be the subspace of $\mathcal{M}$ of all measures which are absolutely continuous with respect to the Lebesgue measure and whose density is bounded by 1. The subspace $\mathcal{M}_0$ is closed (thus compact) and a sequence μ_n in $\mathcal{M}_0$ converges to μ if and only if

$$\lim_{n\to\infty} \int H(r)\,\mu_n(dr) \;=\; \int H(r)\,\mu(dr)$$

holds for all continuous functions H with compact support in $(0,\infty)$.

Overestimating $\eta(x)$ by one, it is easy to show that the random measures $\bar\mu^T$ belong to $\mathcal{M}_c$ for T large enough. More precisely, there exists a finite universal constant C_0 such that

$$\bar\mu^T([a,b]) \;\le\; (b-a) \;+\; \frac{C_0}{\log T}$$

for all $0 \le a \le b < \infty$, $T > 1$. In particular, for each $c > 0$, there exists a finite $T(c)$ such that $\bar\mu^T$ belongs to $\mathcal{M}_c$ for all $T > T(c)$. The same statement remains in force for $\mu^{1,T}$ in place of $\bar\mu^T$. This property of the random measures $\bar\mu^T$ explains the introduction of the spaces $\mathcal{M}_c$. From now on we fix some $c > 0$.

For any α in $(0,1)$, let $C^2(\mathbb{R}_+,\alpha)$ be the space of twice continuously differentiable functions $\gamma : [0,\infty) \to (0,1)$ such that γ' has a compact support in $(0,\frac{1}{2})$ and such that $\gamma(r) = \alpha$ for $r \ge 1/2$. There exists therefore $0 < \beta < 1$ and $0 < \varepsilon < 1/4$ such that $\gamma(r) = \beta$ for $r \le \varepsilon$, and $\gamma(r) = \alpha$ for $r \ge \frac{1}{2} - \varepsilon$. For each γ in $C^2(\mathbb{R}_+,\alpha)$, let $\Gamma = \Gamma_{\gamma,\alpha}$ be the function given by

$$\Gamma(u) \;=\; \frac{1}{2}\log\frac{\gamma(u)\,[1-\alpha]}{[1-\gamma(u)]\,\alpha}\,. \tag{4}$$

Notice that Γ is a twice continuously differentiable function with compact support in $[0,1/2)$ and whose derivative has compact support on $(0,1/2)$. Denote by $\Sigma(\mathbb{R}_+)$ the space of functions $\{\Gamma'_{\gamma,\alpha},\ \gamma \in C^2(\mathbb{R}_+,\alpha)\}$. Notice that $\Sigma(\mathbb{R}_+)$ is a vector space (which is not the case of $C^2(\mathbb{R}_+,\alpha)$ or $\{\Gamma_{\gamma,\alpha},\ \gamma \in C^2(\mathbb{R}_+,\alpha)\}$) and that every function h in $\Sigma(\mathbb{R}_+)$ is continuously differentiable and has compact support in $(0,1/2)$. Moreover, given h in $\Sigma(\mathbb{R}_+)$ there exists one and only one γ in $C^2(\mathbb{R}_+,\alpha)$ such that $h = \Gamma'_{\gamma,\alpha}$. This means that the map from $C^2(\mathbb{R}_+,\alpha)$ to $\Sigma(\mathbb{R}_+)$ is one to one.

For $0 < \alpha < 1$, let $F(a) = a(1-a)$ and $I_\alpha : \mathcal{M}_c \to \mathbb{R}_+$ be given by

$$I_\alpha(\mu) \;=\; \pi \sup_{h\in\Sigma(\mathbb{R}_+)} \Big\{ - \ll h', m \gg \;-\; \ll h^2, F(m) \gg \Big\}$$

if $\mu(dr) = m(r)dr$ is absolutely continuous with respect to the Lebesgue measure and has a density such that $m(r) = \alpha$ for $r \ge 1/2$. In all other cases $I_\alpha(\mu) = \infty$. Note that Proposition 2.2, which states that in the large deviations regime the measure $\bar\mu^T(dr)$ is fixed on $[1/2,\infty)$ and equal to αdr, justifies the concentration of I_α on such measures. In [1] we show that the rate function I_α is lower semi-continuous, convex and such that

$$I_\alpha(\mu) \;=\; \frac{\pi}{4}\int_{\mathbb{R}_+} \frac{m'(r)^2}{m(r)[1-m(r)]}\,dr\;.$$

Furthermore, we derive the following large deviations principle for $\bar\mu^T$;

Theorem 2.1 *For every closed subset F of $\mathcal{M}_c$ and every open subset G of $\mathcal{M}_c$,*

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\big[\bar{\mu}^T \in F\big] \;\leq\; -\inf_{\mu\in F} I_\alpha(\mu)\ ,$$
$$\liminf_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\big[\bar{\mu}^T \in G\big] \;\geq\; -\inf_{\mu\in G} I_\alpha(\mu)\ .$$

Some observations about the law of large numbers type behavior of $\bar{\mu}^T$ are needed in order to prove Theorem 2.1. The first one, i.e. Proposition 2.2, is needed for the upper bound, while the second one, i.e. Proposition 2.3, is needed for the lower bound.

Proposition 2.2 *For r in $\mathbb{R}_+$ and $\varepsilon > 0$, let $\Phi_{r,\varepsilon}(r') = \varepsilon^{-1}\mathbf{1}\{[r, r+\varepsilon]\}(r')$. For every $r \geq 1/2$, $\varepsilon > 0$ and $\delta > 0$,*

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\Big[| \ll \Phi_{r,\varepsilon}, \bar{\mu}^T \gg -\alpha| > \delta\Big] \;=\; -\infty\ .$$

Proposition 2.3 *Fix γ in $C^2(\mathbb{R}_+, \alpha)$ and recall the definition of $\Gamma = \Gamma_{\gamma,\alpha}$ given by (4). As $T \uparrow \infty$, the measure $\bar{\mu}^T(\eta)$ converges in $\mathbb{P}_{T,\gamma}$-probability to the measure $\gamma(r)\, dr$.*

However, it is not clear from the above two observations what happens *at* $(1/2)^-$ for $\bar{\mu}^T(dr)$. Therefore, to fill in the missing information, this year in this project we prove a third Proposition, i.e. Proposition 2.5, which is based on the following Lemma.

Lemma 2.4 *For $r > 0$ and $\varepsilon > 0$, let $\Phi_{r,\varepsilon}(r') = \varepsilon^{-1}\mathbf{1}\{[r, r+\varepsilon]\}(r')$. For every $\varepsilon > 0$, $\delta > 0$, and sufficiently large $R > 1/2$,*

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\Big[| \ll \Phi_{\frac{1}{2}-\varepsilon,\varepsilon}, \bar{\mu}^T \gg - \ll \Phi_{R,\varepsilon}, \bar{\mu}^T \gg | > \delta\Big] \;=\; -\infty\ .$$

Proposition 2.5 *For every $\varepsilon > 0$ and $\delta > 0$,*

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\Big[| \ll \Phi_{\frac{1}{2}-\varepsilon,\varepsilon}, \bar{\mu}^T \gg -\alpha| > \delta\Big] \;=\; -\infty\ .$$

The proofs can be found in [2] and are omitted here.

3 Self-evaluation

The study of the empirical polar measures made in this project helps a lot in the investigation of other scaling limit problems of two dimensional exclusion processes.

4 References

1. Chang, C.-C., Landim, C., Lee, T.Y. (2004). Occupation time large deviations of two dimensional symmetric simple exclusion process. *Ann. Probab.* **32** 1B 661-691.

2. Chang, C.-C.: In preparation.