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A Note on the Stability and Uniqueness for Solutions to the Minimal Surface System

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Abstract. In this note, we show that the solution to the Dirichlet problem for the minimal surface system is unique in the space of distance-decreasing maps. This follows as a corollary of the following stability theorem: if a minimal submanifold Σ is the graph of a (strictly) distance-decreasing map, then Σ is (strictly) stable. We also give another criterion for the stability which covers the codimension one case. All theorems are proved in a more general setting, which concerns minimal maps between Riemannian manifolds. The complete statements of the results appear in Theorem 3.1, Theorem 3.2, and Theorem 4.1.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. Recall a C^2 vector-valued function $f = (f^1, \cdots, f^m) : \Omega \to \mathbb{R}^m$ is said to be a solution to the minimal surface system (see Osserman [OS] or Lawson-Osserman [LO]) if

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ij} \frac{\partial f^\alpha}{\partial x^j}) = 0 \text{ for each } \alpha = 1 \cdots m \tag{1.1}$$

where $g_{ij} = \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}$, $g = \det g_{ij}$ and g^{ij} is the (i, j) entry of the inverse matrix of (g_{ij}). The graph of f is called a non-parametric minimal submanifold. Equation (1.1) is indeed the Euler-Lagrange equation of the volume functional.

In the codimension one case, i.e. $m = 1$, a simple calculation shows $g^{ij} = \delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}$ and the equation is equivalent to the familiar one,

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$$div(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}) = 0. \quad (1.2)$$

It is well-known that the solution to (1.2) subject to the Dirichlet boundary condition is unique and stable(see for example, Lawson-Osserman [LO]).

However in the higher codimension case ($m > 1$), Lawson and Osserman [LO] discover a remarkable counterexample to the uniqueness and stability of solutions of (1.1) when $n = m = 2$. They construct two distinct non-parametric minimal surfaces with the same boundary. Lawson and Osserman then show an unstable non-parametric minimal surface with the same boundary exists as a result of the theorems of Morse-Tompkins [MT] and Shiffman [SH]. In the same paper, Lawson and Osserman also show the Dirichlet problem for the minimal surface system may not be solvable in higher codimension.

In this paper, we first derive a stability criterion for the minimal surface system in higher codimension. To describe the results, we define distance-decreasing maps.

Definition 1 *A map $f : \Omega \to \mathbb{R}^m$ is called distance-decreasing if the differential df satisfies $|df(v)| \leq |v|$ at each point of Ω. It is called strictly distance-decreasing if $|df(v)| < |v|$ at each point of Ω.*

We prove the following stability theorem.

Theorem A (see Theorem 3.1) *Suppose a nonparametric minimal submanifold Σ is the graph of a distance-decreasing map $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. Then Σ is stable. It is strictly stable if f is strictly distance-decreasing.*

This theorem generalizes the stability criterion in [LW]. It turns out the volume element is a convex function on the space of distance-decreasing linear transformations. The convexity is further exploited to derive a uniqueness criterion. Namely, we show the solution to the Dirichlet problem for the minimal surface system is unique in the space of distance-decreasing maps.

Theorem B (see Theorem 3.2) *Suppose that Σ_0 and Σ_1 are nonparametric minimal submanifolds which are the graph of $f_0 : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f_1 : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ respectively. If both f_0 and f_1 are distance-decreasing and $f_0 = f_1$ on $\partial\Omega$, then $\Sigma_0 = \Sigma_1$.*

We remark that solutions to the Dirichlet problem of minimal surface systems in higher dimension and codimension are constructed in

[WA1] and the solutions are graphs of distance-decreasing maps. For earlier uniqueness theorems for minimal surfaces, we refer to Meek's paper [ME].

We prove slightly more general stability and uniqueness theorems for minimal maps between Riemannian manifolds in this paper. It turns out the only extra assumption is on the sign of the target manifold curvature. In particular, Theorem 3.1 implies Theorem A while Theorem 3.2 implies Theorem B.

Another stability criterion for the minimal surface system, which covers the results in codimension one, is derived in section 4. The criterion is in terms of the rank of f. To describe the results, we first recall some notations. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. It induces a linear transformation $\wedge^2 L$, from the wedge product $\wedge^2\mathbb{R}^n$ to $\wedge^2\mathbb{R}^m$ by

$$(\wedge^2 L)(v \wedge w) = L(v) \wedge L(w).$$

With this we define

$$| \wedge^2 L| = \sup_{|v\wedge w|=1} |(\wedge^2 L)(v \wedge w)|.$$

In particular, $| \wedge^2 L| = 0$ if L is of rank one.

Theorem C (see Theorem 4.1) *Suppose a nonparametric minimal submanifold Σ is the graph of a map $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. Then Σ is stable if $| \wedge^2 df|(x) \leq \frac{1}{n-1}$.*

A more refined and more general version is proved in Theorem 4.1. A nonparametric minimal submanifold of codimension one has the rank of $df(x)$ at most one and $| \wedge^2 df|(x) = 0$. We prove the results for minimal maps between Riemannian manifolds as stated in Theorem 4.1.

2. A non-parametric variational formula for graphs

Suppose that (M, g) and (N, h) are two Riemannian manifolds. We fix a local coordinate system $\{x^i\}$ on M. Let f be a map from (M, g) to (N, h). The graph of f is an embedded submanifold of the product manifold $M \times N$, the induced metric is given by

$$\sum_{i,j=1}^{n} G_{ij} dx^i dx^j = \sum_{i,j=1}^{n} (g_{ij} + \langle\, df(\frac{\partial}{\partial x^i}), df(\frac{\partial}{\partial x^j})\,\rangle) dx^i dx^j,$$

and the induced volume form is

$$dv = \sqrt{\det G_{ij}}dx^1 \wedge \cdots \wedge dx^n.$$

Assume that there is a family of maps f_t, $0 \leq t \leq \epsilon$ from M to N with $f_0 = f$ on M and $f_t = f$ outside a compact subset of M. When the boundary of M is nonempty, we require that $f_t = f$ on ∂M. In the following, we compute the first and second variations of the volumes of the graphs. We compute

$$\frac{d\sqrt{\det G_{ij}(t)}}{dt} = \frac{1}{2}\sum_{i,j} G^{ij}(t)\dot{G}_{ij}(t)\sqrt{\det G_{ij}(t)},$$

where $G^{ij}(t)$ is the (i,j) entry of the inverse matrix of $(G_{ij}(t))$.

Denote the variation field $\frac{df_t}{dt}$ by $V(t)$. For simplicity, we omit the dependency of G_{ij} and V on t in the following calculation. Then

$$\begin{aligned}\dot{G}_{ij} &= \langle \nabla_V df_t(\frac{\partial}{\partial x^i}), df_t(\frac{\partial}{\partial x^j}) \rangle + \langle df_t(\frac{\partial}{\partial x^i}), \nabla_V df_t(\frac{\partial}{\partial x^j}) \rangle \\ &= \langle \nabla_{df_t(\frac{\partial}{\partial x^i})} V, df_t(\frac{\partial}{\partial x^j}) \rangle + \langle df_t(\frac{\partial}{\partial x^i}), \nabla_{df_t(\frac{\partial}{\partial x^j})} V \rangle.\end{aligned}$$

Here ∇ is the Riemannian connection on N and V and $df_t(\frac{\partial}{\partial x^i})$ are vector fields tangent to the N direction.

Hence the first variational formula is

$$\frac{dA_t}{dt} = \int_M \sum_{i,j} G^{ij} \langle \nabla_{df_t(\frac{\partial}{\partial x^i})} V, df_t(\frac{\partial}{\partial x^j}) \rangle \, dv_t. \tag{2.1}$$

Continuing the computation, we derive

$$\frac{d^2A_t}{dt^2} = \frac{1}{2}\int_M (\sum_{i,j} G^{ij}\ddot{G}_{ij} - \sum_{i,j,k,l} G^{ik}\dot{G}_{kl}G^{lj}\dot{G}_{ij})\, dv_t + \frac{1}{4}\int_M (\sum_{i,j} G^{ij}\dot{G}_{ij})^2 \, dv_t. \tag{2.2}$$

Now

$$\begin{aligned}\ddot{G}_{ij} &= \langle \nabla_V \nabla_{df_t(\frac{\partial}{\partial x^i})} V, df_t(\frac{\partial}{\partial x^j}) \rangle + \langle df_t(\frac{\partial}{\partial x^i}), \nabla_V \nabla_{df_t(\frac{\partial}{\partial x^j})} V \rangle \\ &\quad + 2\langle \nabla_V df_t(\frac{\partial}{\partial x^i}), \nabla_V df_t(\frac{\partial}{\partial x^j}) \rangle \\ &= \langle R(V, df_t(\frac{\partial}{\partial x^i}))V, df_t(\frac{\partial}{\partial x^j}) \rangle + \langle \nabla_{df_t(\frac{\partial}{\partial x^i})} \nabla_V V, df_t(\frac{\partial}{\partial x^j}) \rangle \\ &\quad + \langle R(V, df_t(\frac{\partial}{\partial x^j}))V, df_t(\frac{\partial}{\partial x^i}) \rangle + \langle df_t(\frac{\partial}{\partial x^i}), \nabla_{df_t(\frac{\partial}{\partial x^j})} \nabla_V V \rangle \\ &\quad + 2\langle \nabla_{df_t(\frac{\partial}{\partial x^i})} V, \nabla_{df_t(\frac{\partial}{\partial x^j})} V \rangle.\end{aligned}$$

Symmetrizing the indexes, the second variational formula becomes

$$
\begin{aligned}
\frac{d^2A_t}{dt^2} &= \int_M (\sum_{i,j} G^{ij}\langle \nabla_{df(\frac{\partial}{\partial x^i})}V, \nabla_{df(\frac{\partial}{\partial x^j})}V\rangle - \frac{1}{2}\sum_{i,j,k,l} G^{ik}\dot{G}_{kl}G^{lj}\dot{G}_{ij})\, dv_t \\
&\quad + \int_M \sum_{i,j} G^{ij}\langle R(V, df(\frac{\partial}{\partial x^j}))V, df(\frac{\partial}{\partial x^i})\rangle\, dv_t + \frac{1}{4}\int_M (\sum_{i,j} G^{ij}\dot{G}_{ij})^2\, dv_t \\
&\quad + \int_M \sum_{i,j} G^{ij}\langle \nabla_{df(\frac{\partial}{\partial x^i})}\nabla_V V, df(\frac{\partial}{\partial x^j})\rangle\, dv_t.
\end{aligned}
\tag{2.3}
$$

This formula will be used to prove the main theorems in the next section.

3. The stability and uniqueness of minimal maps

We recall a minimal submanifold is called *stable* if the second derivative of the volume functional with respect to any compact supported normal variation is non-negative. We prove the following lemma for minimal graphs.

Lemma 3.1 *Suppose that the graph of $f: M \to N$ is a minimal submanifold in $M \times N$. Then Σ is stable if and only if it is stable with respect to any compact supported deformation of maps from M to N.*

Proof. Suppose that a_i is an orthonormal basis of the principal directions of df with stretches $\lambda_i \geq 0$ and that $df(a_i) = \lambda_i b_i$. Assume that the rank of $df(x)$ is p. The orthonormal set $\{b_i\}_{i=1\cdots p}$ can be completed to form a local orthonormal basis $\{b_\alpha\}_{\alpha=1\cdots m}$ of the tangent space of N. In the basis chosen as above, the tangent space of Σ is spanned by $t_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i b_i)$, $1 \leq i \leq n$. Observe that $\lambda_i = 0$ for $p < i \leq n$. The normal space of Σ is spanned by $n_i = \frac{1}{\sqrt{1+\lambda_i^2}}(b_i - \lambda_i a_i)$, $1 \leq i \leq p$ and $n_\alpha = b_\alpha$ for $p < \alpha \leq m$. Assume that $\bar{V} = \sum_{\alpha=1}^m v_\alpha n_\alpha$ is a compact supported normal vector field along Σ. Then the compact supported vector field $V = \sum_i \sqrt{1+\lambda_i^2}\, v_i b_i + \sum_{\alpha>p} v_\alpha b_\alpha$ tangent to N satisfies $V^\perp = \bar{V}$, where $(\cdot)^\perp$ denotes the normal part of a vector, i.e. the projection onto the normal space of Σ. The second derivative of volume functional in the direction $V^\perp = \bar{V}$ is the same as in the direction V. The Lemma is thus proved.

The notion of a (strictly) distance-decreasing map in Definition 1 can be generalized to maps between Riemannian manifolds and we can prove the following theorem.

Theorem 3.1 *Suppose that M and N are two Riemannian manifolds, where the sectional curvature of N is non-positive. Assume that $f : M \to N$ is a distance-decreasing map and the graph of f, which is denoted by Σ, is minimal in $M \times N$. Then the minimal submanifold Σ is stable. It is strictly stable in the following cases: (i) N has negative sectional curvature, and f is not a constant map when M is compact without boundary. (ii) f is strictly distance-decreasing, and M is noncompact or with nonempty boundary.*

Proof. For a minimal submanifold, we have $\frac{dA_t}{dt}|_{t=0} = 0$ for any variation field and in particular

$$\int_M \sum_{i,j} G^{ij} \langle \nabla_{df(\frac{\partial}{\partial x^i})} \nabla_V V, df(\frac{\partial}{\partial x^j}) \rangle \, dv = 0.$$

In the basis chosen in the proof of Lemma 3.1, we derive from (2.3)

$$\begin{aligned} \frac{d^2 A_t}{dt^2}|_{t=0} \geq & \int_M \Big(\sum_i \frac{1}{1+\lambda_i^2} (|\nabla_{df(a_i)} V|^2 - \langle R(V, df(a_i)) df(a_i), V \rangle) \\ & - \frac{1}{2} \sum_{i,j} \frac{1}{1+\lambda_i^2} \frac{1}{1+\lambda_j^2} (\langle \nabla_{df(a_i)} V, df(a_j) \rangle + \langle \nabla_{df(a_j)} V, df(a_i) \rangle)^2 \Big) \, dv. \end{aligned}$$

Since the sectional curvature of N is non-positive, this becomes

$$\begin{aligned} \frac{d^2 A_t}{dt^2}|_{t=0} \geq & \int_M \Big(\sum_i \frac{1}{1+\lambda_i^2} |\nabla_{df(a_i)} V|^2 \\ & - \frac{1}{2} \sum_{i,j} \frac{1}{1+\lambda_i^2} \frac{1}{1+\lambda_j^2} (\lambda_j \langle \nabla_{df(a_i)} V, b_j \rangle + \lambda_i \langle \nabla_{df(a_j)} V, b_i \rangle)^2 \Big) \, dv \\ \geq & \int_M \Big(\sum_{i,j} \frac{1}{1+\lambda_i^2} \langle \nabla_{df(a_i)} V, b_j \rangle^2 \\ & - \sum_{i,j} \frac{1}{1+\lambda_i^2} \frac{1}{1+\lambda_j^2} (\lambda_j^2 \langle \nabla_{df(a_i)} V, b_j \rangle^2 + \lambda_i^2 \langle \nabla_{df(a_j)} V, b_i \rangle^2) \Big) \, dv \\ = & \int_M \sum_{i,j} \frac{\langle \nabla_{df(a_i)} V, b_j \rangle^2}{1+\lambda_i^2} \, \frac{1-\lambda_j^2}{1+\lambda_j^2} \, dv. \end{aligned} \tag{3.1}$$

When f is a distance-decreasing map, we have $\lambda_j \leq 1$ for $1 \leq j \leq n$. From the estimate in (3.1), it follows that $\frac{d^2 A_t}{dt^2}|_{t=0} \geq 0$. This implies that Σ is stable by Lemma 3.1. Suppose that f is strictly distance-decreasing, i.e. $\lambda_j < 1$ for $1 \leq j \leq n$. If $\frac{d^2 A_t}{dt^2}|_{t=0} = 0$, it implies that $\langle \nabla_{df(a_i)} V, b_j \rangle = 0$ for

$1 \le i, j \le n$ and $|\nabla_{df(a_i)} V|^2 = \sum_j \langle \nabla_{df(a_i)} V, b_j \rangle^2$. Hence $\nabla_{df(a_i)} V = 0$ for $1 \le i \le n$. That is, V is a parallel vector field. Because V vanishes outside a compact set and on the boundary of M, it implies that V is a zero vector in case (ii). This proves that Σ is strictly stable in case (ii). When the sectional curvature of N is negative and f is not a constant map, one always has $\frac{d^2 A_t}{dt^2}|_{t=0} > 0$ unless V is a zero vector. If f is constant and M is noncompact or without boundary, it is proved in case (ii). Therefore, Σ is strictly stable in case (i).

Remark 1 *In the case that M is compact without boundary and f is strictly distance-decreasing, one has the following conclusion: If $\frac{d^2 A_t}{dt^2}|_{t=0} = 0$, then V is a parallel vector field and $\langle R(V, df_0(a_i)) df_0(a_i), V \rangle = 0$ for $1 \le i \le n$.*

Using the second variational formula, one also can prove the uniqueness of minimal maps.

Theorem 3.2 *Suppose that M and N are two Riemannian manifolds, where the sectional curvature of N is non-positive. Let Σ_0 and Σ_1 be minimal submanifolds in $M \times N$, which are the graphs of distance-decreasing maps $f_0 : M \to N$ and $f_1 : M \to N$ respectively. Assume that f_0 and f_1 are homotopic, and are identical on the boundary of M and outside a compact set of M. Then $\Sigma_0 = \Sigma_1$ in the following cases: (i) the sectional curvature of N is negative, and f is not a constant map when M is compact without boundary, (ii) the boundary of M is nonempty, (iii) M is noncompact.*

Proof. Lift the homotopy map between f_0 and f_1 to the universal covering of N. Because the sectional curvature of N is non-positive, there exists a unique geodesic connecting the lifting $\widetilde{f}_0(x)$ and $\widetilde{f}_1(x)$. Denote the projection of this unique geodesic onto N by $\gamma_x(t)$ and define $f_t(x) = \gamma_x(t)$. Then $V = \dot\gamma_x(t)$ satisfies $\nabla_V V = 0$. Hence the same bound on $\frac{d^2 A_t}{dt^2}$ as in (3.1) holds for $0 \le t \le 1$. The vector field $df_t(\frac{\partial}{\partial x^i})$ is a Jacobi field along $\gamma_x(t)$, which is denoted by $J_{i,x}(t)$. A direct calculation gives

$$\frac{d^2}{dt^2}|J_{i,x}|^2 = 2\langle \ddot J_{i,x}, J_{i,x} \rangle + 2|\dot J_{i,x}|^2 = 2\langle R(V, J_{i,x})V, J_{i,x} \rangle + 2|\dot J_{i,x}|^2 \ge 0. \tag{3.2}$$

The last inequality follows from the fact that N has nonpositive sectional curvature. Because both f_0 and f_1 are distance-decreasing maps, one has $|J_{i,x}(0)|^2 \le |\frac{\partial}{\partial x^i}|^2$ and $|J_{i,x}(1)|^2 \le |\frac{\partial}{\partial x^i}|^2$. The inequality (3.2) then implies $|J_{i,x}(t)|^2 \le |\frac{\partial}{\partial x^i}|^2$. Hence f_t is also distance-decreasing and one concludes that $\frac{d^2 A_t}{dt^2} \ge 0$ from (3.1) for $0 \le t \le 1$. Because $\frac{dA_t}{dt}|_{t=0} = \frac{dA_t}{dt}|_{t=1} = 0$, the bound gives $\frac{dA_t}{dt}=0$ and $\frac{d^2A_t}{dt^2}=0$ for $0\le t\le 1$. To have $\frac{d^2A_t}{dt^2}|_{t=0}=0$, the following conditions must hold:

1. $\sum_i \frac{1}{1+\lambda_i^2}\langle \nabla_{df_0(a_i)}V, df_0(a_i)\rangle = 0$.
2. $|\nabla_{df_0(a_i)}V|^2=\sum_j\langle \nabla_{df_0(a_i)}V, b_j\rangle^2$ for $1\le i\le n$.
3. $\langle \nabla_{df_0(a_i)}V, df_0(a_j)\rangle = \langle \nabla_{df_0(a_j)}V, df_0(a_i)\rangle$ for $1\le i,j\le n$.
4. If $\lambda_j<1$, then $\langle \nabla_{df_0(a_i)}V, b_j\rangle = 0$ for $1\le i\le n$, which implies $\langle \nabla_{df_0(a_i)}V, df_0(a_j)\rangle = 0$.
5. $\langle R(V, df_0(a_i))df_0(a_i), V\rangle = 0$ for $1\le i\le n$.

When the sectional curvature of N is negative and f is not a constant map, condition 5 implies that $V=0$. Hence $f_0=f_1$ and $\Sigma_0=\Sigma_1$. The other cases in (i) can be covered in case (ii) and (iii) proved in the following.

Now suppose that the sectional curvature of N is non-positive, we shall conclude $\nabla_{df_0(a_i)}V=0$ for any $1\le i\le n$. Fix a point $x\in M$ and choose coordinates at x such that $a_i=\frac{\partial}{\partial x^i}$ for $1\le i\le n$. If $\lambda_i=1$, we have $|df_0(\frac{\partial}{\partial x^i})|^2=1$ and $|J_{i,x}(t)|^2$ achieves its maximum at $t=0$. Therefore, we have $\frac{d}{dt}|J_{i,x}(t)|^2=0$ and $\frac{d^2}{dt^2}|J_{i,x}(t)|^2\le 0$ at $t=0$. The bound on (3.2) then implies $\dot{J}_{i,x}(0)=0$. Hence $\nabla_{df_0(a_i)}V=\nabla_{df_0(\frac{\partial}{\partial x^i})}V=\nabla_V df_0(\frac{\partial}{\partial x^i})=0$. If $\lambda_i<1$, condition 3 and 4 give $\langle \nabla_{df_0(a_i)}V, df_0(a_j)\rangle = \langle \nabla_{df_0(a_j)}V, df_0(a_i)\rangle = 0$ for $1\le j\le n$. Hence $\langle \nabla_{df_0(a_i)}V, b_j\rangle=0$ if $\lambda_j\ne 0$. In case $\lambda_j=0$, one still has $\langle \nabla_{df_0(a_i)}V, b_j\rangle = 0$ from condition 4. Therefore, condition 2 gives $\nabla_{df_0(a_i)}V=0$ in the case $\lambda_i<1$. Thus $\nabla_{df_0(a_i)}V=0$ for any $1\le i\le n$.

When M is noncompact or has boundary, one has $V=0$ at some place. It then implies $V=0$ on M. Therefore, $f_0=f_1$ and $\Sigma_0=\Sigma_1$ in case (ii) and case (iii).

Remark 2 *When M is compact without boundary, we first note that the discussion in the proof holds for $0\le t\le 1$. If N has negative sectional curvature, then either $f_0=f_1$ or both f_0 and f_1 are constants. If N has non-positive sectional curvature, one can conclude that V is a parallel vector field on $f_t(M)$ for $0\le t\le 1$. Hence the graphs of f_t, $0\le t\le 1$, are minimal submanifolds of constant distance. Moreover, the Jacobi fields $J_{i,x}(t)=df_t(\frac{\partial}{\partial x^i})$, $i=1,\cdots,n$ are parallel along $\gamma_x(t)$. It implies that the induced metrics on the graphs of f_t are the same. We also have $\dot{J}_{i,x}(t)=0$ and $\ddot{J}_{i,x}(t)=0$. The Jacobi equation thus leads to $R(V, df_t(\frac{\partial}{\partial x^i}))V=0$ for $1\le i\le n$ and $0\le t\le 1$. Hence $\langle R(V,T)V,T\rangle=0$ for any vector T tangent to $f_t(M)$ in N. The results and further exploration are very similar to the case of harmonic maps as studied by Schoen and Yau in [SY].*

4. Another criterion for stability

In this section, we will derive another criterion for the stability of minimal maps. It is in terms of bounds on the rank of $df(x)$ and $|\wedge^2 df|(x)$ as defined in the introduction. The theorem generalizes the results for nonparametric minimal submanifolds of codimension one.

Theorem 4.1 *Let M and N be Riemannian manifolds and Σ be the graph of a map $f : M \to N$. Suppose the sectional curvature of N is non-positive and Σ is minimal in $M \times N$. Then Σ is stable if $|\wedge^2 df|(x) \leq \frac{1}{p-1}$ for any $x \in M$ where p is any integer greater than* $\max\{1, rank(df(x))\}$.

Proof. We will keep the term $\frac{1}{4}\int_M (\sum_{i,j} G^{ij}\dot{G}_{ij})^2\, dv$ in the second variational formula. In the basis chosen in the proof of Lemma 3.1, we derive from (2.3)

$$
\begin{aligned}
\frac{d^2 A_t}{dt^2}|_{t=0} = & \int_M (\sum_i \frac{1}{1+\lambda_i^2}(|\nabla_{df(a_i)}V|^2 - \langle\, R(V, df(a_i))df(a_i), V\,\rangle) \\
& - \frac{1}{2}\sum_{i,j}\frac{1}{1+\lambda_i^2}\frac{1}{1+\lambda_j^2}(\langle\, \nabla_{df(a_i)}V, df(a_j)\,\rangle + \langle\, \nabla_{df(a_j)}V, df(a_i)\,\rangle)^2\,)\, dv \\
& + \int_M (\sum_i \frac{1}{1+\lambda_i^2}\langle\, \nabla_{df(a_i)}V, df(a_i)\,\rangle)^2\, dv.
\end{aligned}
$$

Since the sectional curvature of N is non-positive, this becomes

$$
\begin{aligned}
\frac{d^2 A_t}{dt^2}|_{t=0} \geq & \int_M (\sum_i \frac{1}{1+\lambda_i^2}|\nabla_{df(a_i)}V|^2 + (\sum_i \frac{\lambda_i}{1+\lambda_i^2}\langle\, \nabla_{df(a_i)}V, b_i\,\rangle)^2 \\
& - \frac{1}{2}\sum_{i,j}\frac{1}{(1+\lambda_i^2)(1+\lambda_j^2)}(\lambda_j\langle\, \nabla_{df(a_i)}V, b_j\,\rangle + \lambda_i\langle\, \nabla_{df(a_j)}V, b_i\,\rangle)^2\,)\, dv \\
\geq & \int_M (\sum_{i,j}\frac{1}{1+\lambda_i^2}\langle\, \nabla_{df(a_i)}V, b_j\,\rangle^2 \\
& + \sum_{i,j}\frac{\lambda_i\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\, \nabla_{df(a_i)}V, b_i\,\rangle\langle\, \nabla_{df(a_j)}V, b_j\,\rangle \\
& - \frac{1}{2}\sum_{i,j}\frac{1}{(1+\lambda_i^2)(1+\lambda_j^2)}(\lambda_j\langle\, \nabla_{df(a_i)}V, b_j\,\rangle + \lambda_i\langle\, \nabla_{df(a_j)}V, b_i\,\rangle)^2\,)\, dv
\end{aligned}
\tag{4.1}
$$

We break the terms into $i=j$ and $i\neq j$, and obtain

$$\begin{aligned}
&\sum_{i,j}\frac{1}{1+\lambda_i^2}\langle\,\nabla_{df(a_i)}V,b_j\,\rangle^2\\
=&\sum_i\frac{1}{1+\lambda_i^2}\langle\,\nabla_{df(a_i)}V,b_i\,\rangle^2+\sum_{i\neq j}\frac{1}{1+\lambda_i^2}\langle\,\nabla_{df(a_i)}V,b_j\,\rangle^2,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j}\frac{\lambda_i\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\,\nabla_{df(a_i)}V,b_i\,\rangle\langle\,\nabla_{df(a_j)}V,b_j\,\rangle\\
=&\sum_i\frac{\lambda_i^2}{(1+\lambda_i^2)^2}\langle\,\nabla_{df(a_i)}V,b_i\,\rangle^2+\sum_{i\neq j}\frac{\lambda_i\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\,\nabla_{df(a_i)}V,b_i\,\rangle\langle\,\nabla_{df(a_j)}V,b_j\,\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j}\frac{1}{(1+\lambda_i^2)(1+\lambda_j^2)}(\lambda_j\langle\,\nabla_{df(a_i)}V,b_j\,\rangle+\lambda_i\langle\,\nabla_{df(a_j)}V,b_i\,\rangle)^2\\
=&\sum_i\frac{2\lambda_i^2}{(1+\lambda_i^2)^2}\langle\,\nabla_{df(a_i)}V,b_i\,\rangle^2+\sum_{i\neq j}\frac{\lambda_j^2}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\,\nabla_{df(a_i)}V,b_j\,\rangle^2\\
&+\sum_{i\neq j}\frac{\lambda_i\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\,\nabla_{df(a_i)}V,b_j\,\rangle\langle\,\nabla_{df(a_j)}V,b_i\,\rangle).
\end{aligned}$$

Plug these expressions into (4.1), and obtain

$$\begin{aligned}
\frac{d^2A_t}{dt^2}|_{t=0}\geq&\int_M(\sum_i\frac{1}{(1+\lambda_i^2)^2}\langle\,\nabla_{df(a_i)}V,b_i\,\rangle^2\\
&+\sum_{i\neq j}\frac{\lambda_i\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\,\nabla_{df(a_i)}V,b_i\,\rangle\langle\,\nabla_{df(a_j)}V,b_j\,\rangle\,)\,dv\\
&+\int_M(\sum_{i\neq j}\frac{1}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\,\nabla_{df(a_i)}V,b_j\,\rangle^2\\
&-\sum_{i\neq j}\frac{\lambda_i\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}\langle\,\nabla_{df(a_i)}V,b_j\,\rangle\langle\,\nabla_{df(a_j)}V,b_i\,\rangle\,)\,dv
\end{aligned}\tag{4.2}$$

The first two terms give

$$\sum_{i\neq j}\frac{\langle \nabla_{df(a_i)}V, b_i\rangle^2}{(p-1)(1+\lambda_i^2)^2}+\frac{\lambda_i\lambda_j\langle \nabla_{df(a_i)}V, b_i\rangle\langle \nabla_{df(a_j)}V, b_j\rangle}{(1+\lambda_i^2)(1+\lambda_j^2)}$$

$$\geq \sum_{i\neq j}\frac{\langle \nabla_{df(a_i)}V, b_i\rangle^2}{(p-1)(1+\lambda_i^2)^2}-\frac{|\langle \nabla_{df(a_i)}V, b_i\rangle||\langle \nabla_{df(a_j)}V, b_j\rangle|}{(p-1)(1+\lambda_i^2)(1+\lambda_j^2)}$$

$$=\frac{1}{p-1}\sum_{i<j}\frac{\langle \nabla_{df(a_i)}V, b_i\rangle^2}{(1+\lambda_i^2)^2}-2\frac{|\langle \nabla_{df(a_i)}V, b_i\rangle||\langle \nabla_{df(a_j)}V, b_j\rangle|}{(1+\lambda_i^2)(1+\lambda_j^2)}+\frac{\langle \nabla_{df(a_j)}V, b_j\rangle^2}{(1+\lambda_j^2)^2}$$

$$=\frac{1}{p-1}\sum_{i<j}\left(\frac{|\langle \nabla_{df(a_i)}V, b_i\rangle|}{1+\lambda_i^2}-\frac{|\langle \nabla_{df(a_j)}V, b_j\rangle|}{1+\lambda_j^2}\right)^2.$$

Symmetrizing the indexes, the last two terms can be written as

$$\sum_{i\neq j}\frac{\langle \nabla_{df(a_i)}V, b_j\rangle^2-2\lambda_i\lambda_j\langle \nabla_{df(a_i)}V, b_j\rangle\langle \nabla_{df(a_j)}V, b_i\rangle+\langle \nabla_{df(a_j)}V, b_i\rangle^2}{2(1+\lambda_i^2)(1+\lambda_j^2)}.$$

It is clearly non-negative when $\lambda_i\lambda_j\leq\frac{1}{p-1}\leq 1$ for $i\neq j$. Hence we have $\frac{d^2A_t}{dt^2}|_{t=0}\geq 0$ and the minimal submanifold is stable as claimed.

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