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## 行列叢、Virasoro 代數與 Hitchin 聯絡(2/3)

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# PROGRESS REPORT

## 1. Virasoro algebra, Hitchin connection and determinant bundles 2/3: Heat operators and cohomology of sheaves

$\mathcal{M}$ (resp. $\overline{\mathcal{M}_g}$) denotes the moduli stack of smooth (resp. stable) curves of genus $g \geq 2$, and $\mathcal{C} \to \mathcal{M}$ (resp. $\overline{\mathcal{C}} \to \overline{\mathcal{M}_g}$) denotes the universal curve respectively. By *moduli stack* in this article, we mean that we are working on the *fine moduli spaces*. In particular, we assume that both the moduli spaces $\mathcal{M}$, $\overline{\mathcal{M}_g}$ and the universal curves $\mathcal{C}$, $\overline{\mathcal{C}}$ are smooth. Fix a line bundle $\mathcal{N}$ of relative degree $d$ on $\overline{\mathcal{C}} \to \overline{\mathcal{M}_g}$, let $f : S \to \mathcal{M}$ be the family of moduli spaces of stable bundles of rank $r$ with fixed determinant $\mathcal{N}_b := \mathcal{N}|_{\mathcal{C}_b}$ ($b \in \mathcal{M}$) on $\mathcal{C}_b$. Let $\pi : X = \mathcal{C} \times_{\mathcal{M}} S \to S$ be the pull-back of $\mathcal{C} \to \mathcal{M}$ via $f : S \to \mathcal{M}$.

**Remark 1.1.** If $(d, r) = 1$, there exists a universal bundle $E$ on $X$. In general $S$ is a good quotient of a Hilbert quotient scheme, denoted by $\mathcal{R}^s$, such that there is a universal bundle $E$ on $X_{\mathcal{R}^s} := \mathcal{C} \times_{\mathcal{M}} \mathcal{R}^s$. $E$ may not descend to $X$, but objects such as $\mathcal{E}nd^0(E)$, $\mathcal{G}_E$, ${}^{tr}\mathcal{A}_E^{\bullet}$ (and other relevant constructions that will be discussed later in this section) do descend. Recall that a sheaf $\mathcal{F}$ on $X_{\mathcal{R}^s}$ descends to $X$ if the action, of scalar automorphisms of $E$ (relative to $\mathcal{R}^s$), on $\mathcal{F}$ is trivial, e.g. [14].

*Without the danger of confusion we will henceforth be working as if a universal bundle $E$ exists on $X$.*

Let $\Theta$ be the theta line bundle on $S$. For a sheaf $\mathcal{F}$ on $X$,

$$\lambda_{\mathcal{F}} = \bigotimes_{q \geq 0} \det R^q \pi_* \mathcal{F}^{(-1)^q}$$

denotes the Knudsen-Mumford determinant bundle on $S$. As usual $T_S$, $K_S$ denote tangent bundle, canonical bundle; $T_{S/\mathcal{M}}$, $K_{S/\mathcal{M}}$ denote the relative counterparts. Assume that $K_{S/\mathcal{M}} = \Theta^{-\lambda}$. Let $\mathcal{D}_S^{\leq i}(\Theta^k)$ be the sheaf of differential operators on $\Theta^k$ of order $\leq i$, and by $\epsilon$ the symbol map of differential operators. Let $\mathcal{W}(\Theta^k) := \mathcal{D}_S^{\leq 1}(\Theta^k) + \mathcal{D}_{S/\mathcal{M}}^{\leq 2}(\Theta^k)$, one has an exact sequence

$$0 \to \mathcal{D}_{S/\mathcal{M}}^{\leq 2}(\Theta^k) \to \mathcal{W}(\Theta^k) \xrightarrow{\sigma} f^* T_{\mathcal{M}} \to 0. \tag{1.1}$$

**Definition 1.2.** (cf. [11], 2.3.2.) A *heat operator* H on $\Theta^k$ is an $\mathcal{O}_S$-map $f^* T_{\mathcal{M}} \xrightarrow{\mathrm{H}} \mathcal{W}(\mathcal{F})$ which, while composed with $\sigma$ above, is the identity. A *projective heat operator* is $\mathrm{H} : T_{\mathcal{M}} \to f_*\mathcal{W}(\Theta^k)/\mathcal{O}_{\mathcal{M}}$ such that any local lifting is a heat operator. The *symbol map* of H is $\epsilon \circ \mathrm{H} : f^*T_{\mathcal{M}} \to S^2T_{S/\mathcal{M}}$. A heat operator H induces an $\mathcal{O}_{\mathcal{M}}$-map, denoted by the same H, $T_{\mathcal{M}} \to f_*\mathcal{W}(\mathcal{F})$ The heat operator and the preceding induced map will be used interchangeably throughout. A (projective) heat operator on $\Theta^k$ determines a (projective) connection $\nabla^{\mathrm{H}}$ on $f_*\Theta^k$ in a natural way [11].

We recall firstly a general result (forget moduli) of G. Faltings (cf. [8]). Let $Z \to S$ be smooth and $\mathcal{K} = K_{Z/S}$. Consider

$$0 \to \mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}) \to \mathcal{D}^{\leq 2}_{Z/S}(\mathcal{L}) \xrightarrow{\epsilon_2} S^2T_{Z/S} \to 0$$

$$0 \to \mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}) \to S^2\mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}) \xrightarrow{S^2\epsilon_1} S^2T_{Z/S} \to 0.$$

For any $\rho \in H^0(Z, S^2T_{Z/S})$, let $a(\mathcal{L},\rho)$, $b(\mathcal{L},\rho)$ be the obstruction classes to lift $\rho$ to $H^0(Z, \mathcal{D}^{\leq 2}_{Z/S}(\mathcal{L}))$ and to $H^0(Z, S^2\mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}))$ respectively. Then

**Proposition 1.3.** *Under $\mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}^k) \cong \mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L})$, one has*
*i) $b(\mathcal{L}^k,\rho) = kb(\mathcal{L},\rho)$ for any $k \in \mathbb{Q}$.*
*ii) $a(\mathcal{O}_Z,\rho) \in H^1(\mathcal{O}_Z) \oplus H^1(T_{Z/S})$ has zero projection in $H^1(\mathcal{O}_Z)$.*
*iii) There is a class $c(\mathcal{K},\rho) \in H^1(\mathcal{O}_Z)$, independent of $\mathcal{L}$, such that*

$$2a(\mathcal{L},\rho) = b(\mathcal{L},\rho) + {}^tb(\mathcal{L}^{-1}\otimes\mathcal{K},\rho) + c(\mathcal{K},\rho).$$

Now we come back to the construction of the projective connection. Recall

$$\begin{array}{ccc} X = \mathcal{C}\times_{\mathcal{M}} S & \xrightarrow{\pi} & S \\ \big\downarrow & & f\big\downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{M} \end{array}$$

For simplicity, we assume that there exists a universal bundle $E$ on $X$, let $\mathbb{E} = \mathcal{E}nd^0(E)$. We will arrive at a subsheaf $\mathcal{G}_E \subset {}^{tr}\mathcal{A}^{-1}_E$ (see [4] for details of ${}^{tr}\mathcal{A}^{-1}_E$) fitting into an exact sequence

$$0 \to \omega_{X/S} \to \mathcal{G}_E \xrightarrow{\mathrm{res}} \mathbb{E} \to 0, \tag{1.2}$$

which induces, by taking 2-th symmetric tensor, the exact sequence

$$0 \to \mathcal{G}_E \to S^2(\mathcal{G}_E)\otimes T_{X/S} \xrightarrow{\mathrm{Sym}^2(\mathrm{res})\otimes\mathrm{id}} S^2(\mathbb{E})\otimes T_{X/S} \to 0.$$

Define $S(\mathcal{G}_E) := (\mathrm{Sym}^2(\mathrm{res})\otimes\mathrm{id})^{-1}(\mathrm{id}\otimes T_{X/S})$, which fits into

$$0 \to \mathcal{G}_E \xrightarrow{\iota} S(\mathcal{G}_E) \xrightarrow{q} T_{X/S} \to 0, \tag{1.3}$$

where $q = \mathrm{Sym}^2(\mathrm{res})\otimes\mathrm{id}$. Let

$$0 \to R^1\pi_*(\omega_{X/S}) \to R^1\pi_*(\mathcal{G}_E) \xrightarrow{res} R^1\pi_*(\mathbb{E}) \to 0 \tag{1.4}$$

$$0 \to R^1\pi_*(\mathcal{G}_E) \xrightarrow{\iota} R^1\pi_*S(\mathcal{G}_E) \xrightarrow{q} R^1\pi_*(T_{X/S}) \to 0 \tag{1.5}$$

be the exact sequences induced by (1.2), (1.3). Let $\Delta \subset X \times_S X$ be the diagonal, consider the induced diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_E \boxtimes \mathcal{G}_E & \longrightarrow & \mathcal{G}_E \boxtimes \mathcal{G}_E(\Delta) & \longrightarrow & \mathcal{G}_E \boxtimes \mathcal{G}_E(\Delta)|_\Delta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{E} \boxtimes \mathbb{E} & \longrightarrow & \mathbb{E} \boxtimes \mathbb{E}(\Delta) & \longrightarrow & \mathbb{E} \boxtimes \mathbb{E}(\Delta)|_\Delta & \longrightarrow & 0 \end{array}$$

on $X \times_S X$ ($\mathbb{E} \boxtimes \mathbb{E}$ denotes $p_1^*\mathbb{E} \otimes p_2^*\mathbb{E}$). All vertical maps are induced by $\mathcal{G}_E \xrightarrow{res} \mathbb{E} \to 0$, thus (1.3) is a sub-sequence of the rightmost vertical map above. Let $\mathcal{F}_1 \subset \mathcal{G}_E \boxtimes \mathcal{G}_E(\Delta)$, $\mathcal{F}_2 \subset \mathbb{E} \boxtimes \mathbb{E}(\Delta)$ be subsheaves satisfying

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_E \boxtimes \mathcal{G}_E & \longrightarrow & \mathcal{F}_1 & \longrightarrow & S(\mathcal{G}_E) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & \mathbb{E} \boxtimes \mathbb{E} & \longrightarrow & \mathcal{F}_2 & \longrightarrow & T_{X/S} & \longrightarrow & 0 \end{array}$$

Taking direct images, considering the connecting maps, we have

$$\begin{array}{ccc} R^1\pi_*S(\mathcal{G}_E) & \xrightarrow{\tilde{\delta}} & S^2R^1\pi_*(\mathcal{G}_E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle S^2(res)} \\ R^1\pi_*(T_{X/S}) & \xrightarrow{\delta} & S^2R^1\pi_*(\mathbb{E}) \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccc} R^1\pi_*(\mathcal{G}_E) & \xrightarrow{\delta_1} & R^1\pi_*(\mathcal{G}_E) \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \mathrm{Sym}^2} \\ R^1\pi_*S(\mathcal{G}_E) & \xrightarrow{\tilde{\delta}} & S^2R^1\pi_*(\mathcal{G}_E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle S^2(res)} \\ R^1\pi_*(T_{X/S}) & \xrightarrow{\delta} & S^2R^1\pi_*(\mathbb{E}) \end{array}$$

where $\delta_1$ denote the map induced by $\tilde{\delta}$, the first vertical exact sequence is (1.5), the second vertical exact sequence is induced by taking 2-th symmetric tensor of (1.4) (note that $\mathcal{O}_S \cong R^1\pi_*(\omega_{X/S})$). For $\mathrm{Sym}^2$ see Remark 1.6.

**Lemma 1.4.** *The map $\delta_1 : R^1\pi_*(\mathcal{G}_E) \to R^1\pi_*(\mathcal{G}_E)$ is the identity map.*

*Proof.* See Lemma **??**. □

**Theorem 1.5.** *There exist canonical identifications*

$$\phi : R^1\pi_*(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}), \quad \phi : R^1\pi_*S(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{S}(\lambda_{\mathbb{E}})$$

*such that the following diagram is commutative*

$$\begin{tikzcd}
R^1\pi_*(\mathcal{G}_E) \arrow[r, "\phi"] \arrow[d, "\iota"'] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[d] \\
R^1\pi_*S(\mathcal{G}_E) \arrow[r, "\phi"] & \mathcal{D}^{\leq 1}_{S}(\lambda_{\mathbb{E}}).
\end{tikzcd}$$

*Moreover, $\phi$ induces a map $\mathcal{O}_S \to \mathcal{O}_S$ by multiplying $2r$.*

*Proof.* See Theorem **??** and Corollary **??** □

Once we have the above theorem, we get the commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r, "\phi^{-1}"] \arrow[d] & R^1\pi_*(\mathcal{G}_E) \arrow[r, "\delta_1"] \arrow[d, "\iota"'] & R^1\pi_*(\mathcal{G}_E) \arrow[r, "2r\cdot\phi"] \arrow[d, "\mathrm{Sym}^2"'] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[d] \\
\mathcal{D}^{\leq 1}_{S}(\lambda_{\mathbb{E}}) \arrow[r, "\phi^{-1}"] \arrow[d, "\sigma"'] & R^1\pi_*S(\mathcal{G}_E) \arrow[r, "\tilde{\delta}"] \arrow[d] & S^2R^1\pi_*(\mathcal{G}_E) \arrow[r, "S^2(\phi)"] \arrow[d, "q"'] & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[d, "S^2\epsilon_1"'] \\
f^*T_{\mathcal{M}} \arrow[r, "\cong"] & R^1\pi_*(T_{X/S}) \arrow[r, "\delta"] & S^2R^1\pi_*(\mathbb{E}) \arrow[r, "\cong"] & S^2T_{S/\mathcal{M}}.
\end{tikzcd}$$

**Remark 1.6.** i) For a precise definition of $\mathrm{Sym}^2$ above we refer to proof of Lemma **??**; ii) The insertion of $2r$ in $2r\cdot\phi$ in the 1st row is due to the last statement in Theorem 1.5 combined with the definition of $\mathrm{Sym}^2$.

The above diagram gives the following commutaive diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r] \arrow[d] & \mathcal{D}^{\leq 1}_{S}(\lambda_{\mathbb{E}}) \arrow[r, "\sigma"] \arrow[d, "\tilde{\delta}_{\mathrm{H}}"'] & f^*T_{\mathcal{M}} \arrow[r] \arrow[d, "\delta_{\mathrm{H}}"'] & 0 \\
0 \arrow[r] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r] & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r, "S^2\epsilon_1"] & S^2T_{S/\mathcal{M}} \arrow[r] & 0
\end{tikzcd}$$

where $\delta_{\mathrm{H}}$, $\tilde{\delta}_{\mathrm{H}}$ are defined in a clear way such that $\tilde{\delta}_{\mathrm{H}}$ induces an identity map on $\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}})$.

**Proposition 1.7.** *For any $\rho = \delta_{\mathrm{H}}(v) \in H^0(S, S^2T_{S/\mathcal{M}})$, where $v \in H^0(S, f^*T_{\mathcal{M}})$, one has*
*i) $2a(\mathcal{L}, \rho) = b(\mathcal{L}, \rho) + {}^tb(\mathcal{L}^{-1} \otimes K_{S/\mathcal{M}}, \rho)$.*
*ii) When $\mathcal{L} = K^{\mu}_{S/\mathcal{M}}$, where $\mu \in \mathbb{Q}$ and $\mu \neq 1$, one has*

$$a(K^{\mu}_{S/\mathcal{M}}, \rho) = \frac{2\mu - 1}{2\mu} b(K^{\mu}_{S/\mathcal{M}}, \rho).$$

*Proof.* To prove i), by Proposition 1.3 iii), it is enough to show that $c(\mathcal{K},\rho)=0$. The class is independent of $\mathcal{L}$. Thus, by taking $\mathcal{L}=\mathcal{O}_S$ and using Proposition 1.3 ii), it is enough to show that

$$^t b(\mathcal{K},\rho)\in H^1(\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{O}_S))=H^1(\mathcal{O}_S)\oplus H^1(T_{S/\mathcal{M}})$$

has trivial projection in $H^1(\mathcal{O}_S)$ where $\mathcal{K}=K_{S/\mathcal{M}}$. We have (noting $\mathcal{K}=\lambda_{\mathbb{E}}$)

$$\begin{array}{ccc}
\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}) & = & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}) \\
\big\downarrow & & \big\downarrow \\
\mathcal{D}^{\leq 1}_{S}(\mathcal{K}) & \xrightarrow{\tilde{\delta}_{\mathrm{H}}} & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}) \\
\sigma\big\downarrow & & S^2\epsilon_1\big\downarrow \\
f^*T_{\mathcal{M}} & \xrightarrow{\delta_{\mathrm{H}}} & S^2T_{S/\mathcal{M}}
\end{array}$$

Let $\{\mathcal{U}_i\}_{i\in I}$ be an affine covering of $S$ and $v_i\in T_S(\mathcal{U}_i)$ be local liftings of $v\in f^*T_{\mathcal{M}}(S)$. Let $\{d_i\in\mathcal{D}^{\leq 1}_{S}(\mathcal{K})(\mathcal{U}_i)\}_{i\in I}$ be such that $\epsilon_1(d_i)=v_i$ ($i\in I$). Then, by above diagram, $b(\mathcal{K},\rho)=\{d_i-d_j\in\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K})(\mathcal{U}_i\cap\mathcal{U}_j)\}$. Thus the class $^t b(\mathcal{K},\rho)\in H^1(\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{O}_S))=H^1(\mathcal{O}_S)\oplus H^1(T_{S/\mathcal{M}})$ is defined by the cocycle $\{{}^t d_i-{}^t d_j\}$, where $^t d_i:=A_i-v_i\in\mathcal{D}^{\leq 1}_{S}(\mathcal{O}_S)(\mathcal{U}_i)$. Thus the projection of $^t b(\mathcal{K},\rho)$ in $H^1(\mathcal{O}_S)$ is defined by $\{A_i-A_j\}$, which is a trivial class.

To show ii), we remark that for any nonzero $\mu\in\mathbb{Q}$, through canonical isomorphisms $\psi_\mu:\mathcal{D}^{\leq 1}(\mathcal{K})\cong\mathcal{D}^{\leq 1}(\mathcal{K}^\mu)$, the above diagram induces

$$\begin{array}{ccc}
\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}^\mu) & \xrightarrow{\bullet\mu} & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}^\mu) \\
\big\downarrow & & \big\downarrow \\
\mathcal{D}^{\leq 1}_{S}(\mathcal{K}^\mu) & \xrightarrow{\tilde{\delta}^\mu_{\mathrm{H}}} & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}^\mu) \\
\sigma\big\downarrow & & S^2\epsilon_1\big\downarrow \\
f^*T_{\mathcal{M}} & \xrightarrow{\delta_{\mathrm{H}}} & S^2T_{S/\mathcal{M}}
\end{array}$$

where $\tilde{\delta}^\mu_{\mathrm{H}}=S^2\psi_\mu\circ\tilde{\delta}_{\mathrm{H}}\circ\psi_\mu^{-1}$. Using the above diagram, we can compute $^t b(\mathcal{K}^{1-\mu},\rho)$. Let $\{d_i\in\mathcal{D}^{\leq 1}_{S}(\mathcal{K}^{1-\mu})(\mathcal{U}_i)\}_{i\in I}$ be such that $\epsilon_1(d_i)=v_i$ ($i\in I$). Then $b(\mathcal{K}^{1-\mu},\rho)=(1-\mu)\{d_i-d_j\}$, which implies that

$$^t b(\mathcal{K}^{1-\mu},\rho)=(1-\mu)\{{}^t d_i-{}^t d_j\}.$$

On the other hand, $\{-{}^t d_i \in \mathcal{D}_S^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$ are local liftings of $v$, which means that $b(\mathcal{K}^\mu, \rho) = -\mu\{{}^t d_i - {}^t d_j\} = \frac{\mu}{\mu-1}\,{}^t b(\mathcal{K}^{1-\mu}, \rho)$. Thus

$$a(\mathcal{K}^\mu, \rho) = \frac{2\mu - 1}{2\mu} b(\mathcal{K}^\mu, \rho).$$

□

**Theorem 1.8.** *Let $\{\mathcal{U}_i\}_{i\in I}$ be an affine open covering of $S$. Then, for any $v \in f^* T_{\mathcal{M}}(S)$, there are*

$$d_S^i \in \mathcal{D}_S^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i), \quad d_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i), \quad D_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 2}(\mathcal{K}^\mu)(\mathcal{U}_i),$$

*where $\mathcal{K} := K_{S/\mathcal{M}}$, such that*

$$\left\{ \mathrm{H}(v)_i := d_S^i - d_{S/\mathcal{M}}^i + \frac{2}{1-2\mu} D_{S/\mathcal{M}}^i \in \mathcal{D}_S^{\leq 2}(\mathcal{K}^\mu)(\mathcal{U}_i) \right\}_{i\in I}$$

*form a global section $\mathrm{H}(v) \in H^0(S, \mathcal{D}_S^{\leq 2}(\mathcal{K}^\mu))$ with*

$$\sigma(\mathrm{H}(v)) = v, \quad \epsilon_2(\mathrm{H}(v)) = \frac{2}{1-2\mu}\delta_{\mathrm{H}}(v).$$

*Proof.* Let $\{d_S^i \in \mathcal{D}_S^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$ be such that $\sigma(d_S^i) = v|_{\mathcal{U}_i}$. Then

$$\{\mu(d_S^i - d_S^j) \in \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i \cap \mathcal{U}_j)\}$$

defines the class $b(\mathcal{K}^\mu, \delta_{\mathrm{H}}(v)) \in H^1(S, \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu))$, which is the obstruction for lifting $\delta_{\mathrm{H}}(v) \in H^0(S, S^2 T_{S/\mathcal{M}})$ to $H^0(S, S^2\mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu))$. Let

$$\{D_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 2}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$$

be local liftings of $\delta_{\mathrm{H}}(v)$. Then, by Proposition 1.7,

$$\{d_S^i - d_S^j\} = \frac{2}{2\mu - 1}\{D_{S/\mathcal{M}}^i - D_{S/\mathcal{M}}^j\}$$

as cohomology classes. Thus there are $\{d_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$ satisfying the requirements in the theorem. □

**Corollary 1.9.** *There exists uniquely a projective heat operator,*

$$\mathrm{H} : T_{\mathcal{M}} \to f_* \mathcal{W}(\Theta^k)/\mathcal{O}_{\mathcal{M}}$$

*such that $(f_*\epsilon_2) \cdot \mathrm{H} : T_{\mathcal{M}} \to f_* S^2 T_{S/\mathcal{M}}$ coincides with $f_*\delta_{\mathrm{H}}$.*

*Proof.* For any open set $U \subset \mathcal{M}$ and $v \in T_{\mathcal{M}}(U)$, by Theorem 1.8, we can construct a $\mathrm{H}(v) \in f_*\mathcal{W}(\Theta^k)(U)$. If $\mathrm{H}(v)'$ is another such operator, $\mathrm{H}(v) - \mathrm{H}(v)'$ must have symbol in $H^0(f^{-1}(U), T_{S/\mathcal{M}}) = 0$, so

$$\mathrm{H}(v) - \mathrm{H}(v)' \in H^0(f^{-1}(U), \mathcal{O}_S) = f_*\mathcal{O}_S(U) = \mathcal{O}_{\mathcal{M}}(U).$$

Hence a unique map $T_{\mathcal{M}} \to f_*\mathcal{W}(\Theta^k)/\mathcal{O}_{\mathcal{M}}$. □

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