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Delta 函數在 mean field 方程中的作用

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Let M be a compact Riemann surface without boundary, h be a C^2 function on M, $h > 0$, and $\rho \in \mathbf{R}$. We consider the following equation

$$\Delta u + \rho\left(\frac{he^u}{\int_M he^u} - 1\right) = 0 \quad \text{on} \quad M, \tag{0.1}$$

where Δ is the Betrami-Laplace operator on M. This is the second part of two papers concerning this equation. In the first paper, weobtained sharp estimates for blowup solutions. Based on these estimates, we will calculate the Leray-Schauder degree defined in a standard way for (0.1) in this paper.

When $\rho = 8\pi$ and M is a sphere, (0.1) is related to the problem of prescribed Gaussian curvature, called the Kazdan-Warner problem or the Nirenberg problem, see Kazdan-Warner [28]. In some Chern-Simons-Higgs model as discussed in Taubes [48]-[49], Hong, Kim and Pac [26], Jackiw and Weinberg [27], Spruck and Yang [44], Caffarelli and Yang [4], Tarantello [47], Struwe and Tarantello [45], Nolasco and Tarantello [39]-[41], Riciardi and Tarantello [42]-[43], Ding, Jost, Li and Wang [20]-[23] and the references therein, (0.1) play a role when M is a torus and $\rho = 4\pi N$, where u denotes a condensate solution and N is the vortex number. (0.1) on torus or on a bounded domain of $\mathbf{R}^2$ with Dirichlet boundary condition also arises from the mean field limit of point vortices of Euler flow as studied in Caglioti, Lions, Marchioro and Pulvirenti [5]-[6]and Kiessling [29].

The existence of a solution (0.1) for $\rho < 8\pi$ can be obtained by variational methods using the Moser-Trudinger inequality. For $\rho \geq 8\pi$, the existence problem is more difficult. When $\rho = 8\pi$ and M is the standard S^2, there have been a lot of related works. When $\rho = 8\pi$ and M is a general Riemann surface, Ding, Jost, Li and Wang in [20] gave sufficient conditions on h which implies the existence of (0.1). Using another approach, Nolasco and Tarantello in [39] obtained similar result for torus. When M is a flat torus with fundamental domain $[0,1] \times [0,1]$ and $h \equiv 1$, Struwe and Tarantello in [45] proved that (0.1) has a nontrivial solution for $8\pi < \rho < 4\pi^2$ by the mountain pass method. More recently, Ding, Jost, Li and Wang in [22] proved for $8\pi < \rho < 16\pi$ and $h > 0$, there exits a non-minimum solution of (0.1).

Another approach to study the existence problem of (0.1) is using the Leray-Schauder degree. Since $u + \mathit{constant}$ is still a solution for each solution u of (0.1), we can normalize u to satisfy $\int_M u = 0$. Then it can be shown by the Moser-Trudinger inequality known that when ρ lies in a compact subset of $(-\infty, 8\pi)$, all solutions of (0.1) stay bounded in $C^{2,\alpha}(M)$. When ρ is in a compact subset of $(8\pi m, 8\pi(m+1))$ with m a positive integer, the same conclusion were obtained by Brezis and Merle [2] and Li and Shafrir [32]. Let

$$X_\alpha = \left\{ v \in C^{2,\alpha}(M) : \int_M v = 0 \right\}$$

for $0 < \alpha < 1$,

$$B_R = \{ v \in X_\alpha : \|v\|_{C^{2,\alpha}(M)} < R \},$$

and $T(\rho) : X_\alpha \to X_\alpha$ be defined by

$$T(\rho) = \rho\Delta^{-1}\left(\frac{he^u}{\int_M he^u} - 1\right). \tag{0.2}$$

For large R, due to the results mentioned above, the Leray-Schauder degree

$$d_\rho \equiv deg(I + T(\rho), B_R, 0)$$

is well defined when $\rho \neq 8\pi m$. Due to the homotopy invariance, d_ρ for $\rho \neq 8\pi m$

The purpose of this paper is to find a complete formula of d_ρ for $\rho \neq 8\pi m$. Our main result is as follows. Let g denote the genus of M and $\chi(M)$ be the Euler characteristic of M, that is, $\chi(M) = 2 - 2g$. For two integers m_1 and m_2 with $m_2 \geq m_1 \geq 0$, let

$$\binom{m_2}{m_1} = \begin{cases} \dfrac{m_2(m_2-1)\cdots(m_2-m_1+1)}{m_1!} & \text{for } m_1 > 0 \\ 1 & \text{for } m_1 = 0 \end{cases}$$

Theorem 1.1. *Assume* $8\pi m < \rho < 8\pi(m+1)$ *with* m *a nonnegative integer. Then* $d_\rho = \binom{m-\chi(M)}{m}$, *that is,*

$$d_\rho = \begin{cases} \dfrac{1}{m!}(-\chi(M)+1)(-\chi(M)+2)\cdots(-\chi(M)+m) & \text{for } \ m > 0 \\ 1 & \text{for } \ m = 0. \end{cases}$$

As an application of Theorem 1.1, we have

Theorem 1.2. *For* $g \geq 1$, *there is* ρ_h *depending on* h *such that if* $\rho > \rho_h$, *then (0.1) has a solution. Moreover, we can choose*

$$\rho_h = \max_M (2K - \Delta \log h),$$

where K *is the Gaussian curvature of* M.

The main ideas of the proof of Theorem 1.1 is as follows. To find d_ρ for $\rho \neq 8\pi m$, it suffices to calculate the jump values at $8\pi m$. Instead of choosing $h = 1$ as in [33] to calculate the degree, we take a Morse like function h. When ρ approaches $8\pi m$, we will show the solutions of (0.1) separate into two groups: bounded solutions and blowup solutions. For solutions in the first group, there is a uniform constant $\bar{C}_m$ depending on m only such that the sup-norms of these solutions are bounded by $\bar{C}_m$. For solution in the second group, the sup-norms tend to infity as $\rho \to 8\pi m$. Since the Leray-Schauder degree of bounded solutions remains constant when ρ across $8\pi m$, the jump value of d_ρ at $8\pi m$ depends on the difference of the degree of blowup solutions for $\rho = 8\pi m - \epsilon$ and the degree of blowup solutions for $\rho = 8\pi m + \epsilon$, where $\epsilon > 0$ is a small number. Therefore, to determine the jump value, it is essential to study the blowup behavior of (0.1).

Let u_i be a solution of (0.1) with $\rho = \rho_i$ satisfying $\int_M u_i = 0$. Assume $\rho_i \to 8\pi m$ and $\sup u_i \to \infty$ as $i \to \infty$. By the results of Brezis and Merle in [2] and Li and Shafrir in [32], there are m local maximum points $p_{i,j}$ for $j = 1, ..., m$ of u_i such that $u_i(p_{i,j}) \to \infty$ and after passing to a subsequence, $p_{i,j} \to p_j$ for some p_j and

$$u_i(x) \to 8\pi \sum_j G(x, p_j) \tag{0.4}$$

in $C^2_{loc}(M \setminus \{p_1, ..., p_m\})$, where $G(x, p)$ is the Green function of $-\Delta$ on M satisfying

$$\begin{aligned} -\Delta G(x,p) &= \tilde{\delta}_p - 1, \\ \int_M G &= 0. \end{aligned} \tag{0.5}$$

On the other hand, let $\tilde{u}_i = u_i - \log \int_M he^{u_i}$ and

$$V_{i,j}(y) = \log \frac{e^{\tilde{u}_i(p_{i,j})}}{(1 + \frac{\rho_i h(p_{i,j})}{8} e^{\tilde{u}_i(p_{i,j})} |y|^2)^2}.$$

Then $\int_M he^{\tilde{u}_i} = 1$ and it is known for suitable $p_{i,j}$, there is $C > 0$ such that

$$|\tilde{u}_i - V_{i,j}(y)| \le C \tag{0.6}$$

on a small neighborhood of $p_{i,j}$, where y is an isothermal coordinates centered at $p_{i,j}$. For many applications, inequality (1.6) is not good enough since the neighborhood which it holds on may shrinks to a point as $i \to \infty$. Using the method of moving planes, Li proved in [30] that (1.6) holds on a fixed domain containing $p_{i,j}$ and moreover, $|\tilde{u}_i(p_{i,j}) - \tilde{u}_i(p_{i,l})| < C$ for $j \neq l$. This turns out to be very essential for further studying phenomena related to blowing up.

For the purpose to calculate the Leray-Schauder degree, more sharp estimates for blowup behavior of equation (1.1) are needed. This has been done by the authors in the previous paper [15]. Let

$$\lambda_{i,j} = \tilde{u}_i(p_{i,j})$$

and

$$\tilde{G}(x,p) = \frac{1}{2\pi} \log dist(x,p) + G(x,p), \tag{0.7}$$

where $dist(x, p)$ is the distant between x and p. Let

$$\eta_{i,j} = \tilde{u}_i - V_{i,j}(y) - 8\pi[\tilde{G}(x,p) - \tilde{G}(p,p)]$$

in a neighborhood of $p_{i,j}$. We improved (1.6) in Theorem 1.4 of [15] and obtained

$$|\eta_{i,j}| \le Ce^{-\lambda_{i,j}}\lambda_{i,j}^2. \tag{0.8}$$

Also a more sharp estimate for (1.4) are obtained as follows

$$|u_i(x) - 8\pi \sum_j G(x,p_j)| \le Ce^{-\lambda_{i,1}}\lambda_{i,1} \tag{0.9}$$

for x outside a fixed neighborhood of $\{p_1, ..., p_m\}$, see Lemma 5.3 in [15]. Based on these two estimates, further results concerning the location of $p_{i,j}$, the difference between $\lambda_{i,j}$ and $\lambda_{i,l}$, and the relation between $\rho_i - 8\pi m$, $\{\lambda_{i,j}\}$ and $\{p_{i,j}\}$ can be obtained as in Theorem 1.4 and Theorem 1.1 in [15]. That is, we have

$$\bigtriangledown \log h(x) + 8\pi[\bigtriangledown \tilde{G}(x,p_{i,j}) + \sum_{l\neq j} G(x,p_{i,l})]\Big|_{x=p_{i,j}} = O(e^{-\lambda_{i,j}}\lambda_{i,j}), \tag{0.10}$$

$$\begin{aligned} &\lambda_{i,j} + 8\pi\tilde{G}(p_{i,j},p_{i,j}) + 8\pi\sum_{l\neq j} G(p_{i,l},p_{i,l}) + 2\log\frac{\rho_i h(p_{i,j})}{8} \\ = \ &\lambda_{i,k} + 8\pi\tilde{G}(p_{i,k},p_{i,k}) + 8\pi\sum_{l\neq k} G(p_{i,k},p_{i,l}) + 2\log\frac{\rho_i h(p_{i,k})}{8} + O(e^{-\lambda_{i,j}}\lambda_{i,j}^2) \end{aligned} \tag{0.11}$$

$$\begin{aligned} \rho_i - 8\pi m \ = \ &\frac{2}{m}\sum_{j=1}^m \left(\Delta \log h(p_{i,j}) + 8\pi m - 2K(p_{i,j})\right) h(p_{i,j})^{-1}\frac{\lambda_{i,j}}{e^{\lambda_{i,j}}} \\ &+O(e^{-\lambda_{i,j}}), \end{aligned} \tag{0.12}$$

where K is the Gaussian curvature of M. These relations contain a lot of information of blowup behavior. For example, if the signs on both sides of (1.12) are not the same, then blowup can not happen. When $m = 1$, this gives a further explanation of the result by Ding, Jost, Li and Wang in [20] which states that the minimizers of the variational problem corresponding to (1.1) can not blow up when ρ tends to 8π from below if the sign in the right hand side of (1.12) is positive.

As mentioned above, to find the jump values of d_ρ, it suffices to calculate the degree of blowup solutions. To do this, we will first construct approximate blowup solutions which satisfy (1.10), (1.11) and (1.12). Then the neighborhoods of these approximate solutions contain all blowup solutions. The problem now is reduced to find all possible approximate blowup solutions and to calculat the action of the operator $T(\rho)$ defined in (1.2) on these neighborhoods. When $m > 1$, since there are more than one Green function in (1.10), it is hard to characterize all possible blowup points by using these relations. To overcome this difficulty, the key point is to explain the main term of (1.10) as the gradient of a function f_h defined on M^m, the m times product space of M. The function can be written as

$$f_h(x_1, x_2, ..., x_m) = \sum_j \left[\log h(x_j) + 4\pi\varphi(x_j) + \sum_{l \neq j} 8\pi G(x_j, x_l) \right],$$

where $(x_1, x_2, ..., x_m) \in M^m$ and $\varphi(x_j) = \tilde{G}(x_j, x_j)$ is the regular part of the Green function. Then it can be shown after ignoring the permutation of $(x_1, x_2, ..., x_m)$, the approximate solutions are one to one corresponding to critical points of f_h on M^m which also satisfy the compatibility conditions (1.11) and (1.12). When considering an equation (see (0.16) below) similar to equation (0.1) on a bounded domain in $\mathbf{R}^2$ with a constant h, the fucntion

$$f = \sum_j \left[4\pi\varphi(x_j) + \sum_{l \neq j} 8\pi G(x_j, x_l) \right]$$

has been used by Suzuki [46] and Nagasaki and Suzuki [38] to describe the locations of blowup points. When f has nondegenerate critical points, Baraket and Pacard [] constructed solutions with m bubbles.

Using a similar idea of Bahri and Coron in [1] (see also Li [31] for further application), the functions in a small neighborhood of an approximate solution can be represented in suitable terms orthogonal to each other. Then the dominant terms of $T(\rho)$ on the neighborhoods of the approximate solutions are found according to this representation. This is the main estimates of the paper and it will be stated in Lemma 3.2. When f_h is a Morse function and satisfies some additional nondegenerate condition at critical points, based on the calculation of the action of $T(\rho)$ near each approximate solution, the jump value of d_ρ can be found to equal plus or minus

$$\frac{1}{m!} \sum_{\nabla f_h(Q)=0} (-1)^{ind\, Q}, \tag{0.13}$$

where $Q \in M^m$ and *ind* Q is the Morse index of f_h at Q considered as a function on M^m. If we let

$$\Gamma = \{(x_1, x_2, ..., x_m) \in M^m : x_j = x_k \text{ for some } j \neq k\},$$

the value in (1.13) can be further shown to be a topological invariant and to equal the Euler characteristic of $M \setminus \Gamma$. To get a final formula of d_ρ for $\rho \neq 8\pi m$, we need some more effort to calculate this Euler characteristic. It turns out this characteristic can be express in a very simple formula and d_ρ can be written down in the explicit form as in Theorem 1.1. When f_h is a Morse function on $M \setminus \Gamma$, our argument also obtains a formula for d_ρ at $8\pi m$. In this case, d_ρ depends on the function h.

Some phenomena similar to the ones of (1.1) can be found for the conformal scalar curvature equation

$$\Delta u - \frac{n(n-2)}{4}u + \frac{n-2}{4(n-1)}R(x)u^{\frac{n+2}{n-2}} = 0, u > 0 \tag{0.14}$$

on S^n with $n \geq 3$. The Leray-Schauder degree for (1.14) is also close related to the critical points of $R(x)$. However, (1.1) is different from(1.14) in some aspects. The first is one needs to consider f_h on a product space M^m for (1.1), while one considers R, which is corresponding to the role play by f_h, on the space S^n itself for (1.14). This difference maybe is not essential since one can consider the function $\sum_{j=1}^m R(x_j)$ on the product space $(S^n)^m$ for equation (1.14) also. The second difference comes from the stronger interaction between the Green function $G(x,p)$ and the local data of a function near x for two dimension than for higher dimensions. For equation (1.1), we need to use some Green functions and h together to decide the locations of blowup points (that is, to decide the critical points of f_h and check the compatibility conditions (1.11) and (1.12)), while for equation (1.14), we only use R itself first to determine whether a point is a possible blowup point or not (that is, to find the critical points of R and the sign of ΔR on these points) and then only in a later step, the Green function is used to help determine the allowable combination of multiple blowing up. The effect of Green functions for (1.14) only comes in a more subtle or secondary way. For statements of theorems on (1.1), this fact may make the assumptions on h more complicated. To state the third different feature, we use (1.11) to write $\lambda_{i,j}$ in terms of $\lambda_{i,1}$ first and put them in to (1.12). Then it follows that

$$\rho - 8\pi m = l(Q)e^{-\lambda_{i,1}}\lambda_{i,1} + O(e^{-\lambda_{i,1}}), \tag{0.15}$$

for some $l(Q)$ which is given in (2.26) in Section 2 below, where Q is some critical point of f_h. For (1.14), both $\bigtriangledown R$ and ΔR are important for deciding the blowup points, but for (1.1), instead of using Δf_h, we use $\bigtriangledown f_h$ and $l(Q)$. It is $l(Q)$, not Δf_h, plays the role for (1.1) as ΔR for the higher dimensional problem. Usually, $l(Q)$ is not equal to $\Delta f_h(Q)$ at a critical point Q.

There is one more difference between (1.1) and (1.14). To construct an approximate solution, we need to glue bubbles near blowup points and Green functions away from blowup points together. As suggested by (1.10) and (1.12), the difference between a solution and the coresponding approximate solution should be within the order $e^{-\lambda_{i,j}}\lambda_{i,j}$. Unfortunately, in general, the error term defines in (0.8) is of order $e^{-\lambda_{i,j}}\lambda_{i,j}^2$ when h is not a constant. Therefore it is hard to obtain good approximate solutions. However, by Theorem 1.4 in [15],

$$\eta_{i,j}(x) = c_o e^{-\lambda_{i,j}}\lambda_{i,j}^2 + O(e^{-\lambda_{i,j}}\lambda_{i,j})$$

when x is away from blowup points, where c_o is a constant. Hence, we can add the constants $c_o e^{-\lambda_{i,j}}\lambda_{i,j}^2$ to the approximate solution around neighborhoods of blowup points and overcome the difficulty. For higher dimension problem (1.14), no such error term $\eta_{i,j}$ appears.

In [15], we showed similar sharp estimates for bubble solutions also hold for a similar equation

$$(0.16) \qquad \begin{cases} \Delta u + \rho \frac{he^u}{\int_\Omega he^u} = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a $C^{2,\beta}$ bounded domain in $\mathbf{R}^2$, $\rho \in \mathbf{R}$, and h is a positive $C^{2,\beta}$ function on the closure $\bar{\Omega}$ of Ω. When $\rho \neq 8\pi m$, since the sup-norm estimate holds, we can define the Leray-Schauder degree for (0.16)

$$d_{\Omega,\rho} \equiv deg(Iu + \rho\Delta^{-1}\left(\frac{he^u}{\int_\Omega he^u}\right), B_R, 0).$$

Since the solutions of (0.16) can not blowup at the boundary, the method to calculate the degree for (0.1) can also apply to (0.16). Let $\chi(\Omega)$ denote the Euler characteristic of $Omega$, that is, $\chi(\Omega) = 1 - g_\Omega$ with g_Ω the genus of Ω. We have a similar formula for $d_{\Omega,\rho}$ as the one in Theorem 1.1.

Theorem 1.3. *Assume $8\pi m < \rho < 8\pi(m+1)$ with m a nonnegative integer. Then $d_{\Omega,\rho} = \binom{m-\chi(\Omega)}{m}$, that is,*

$$d_{\Omega,\rho} = \begin{cases} \frac{1}{m!}(-\chi(\Omega)+1)(-\chi(\Omega)+2)\cdots(-\chi(\Omega)+m) & \text{for } m > 0 \\ 1 & \text{for } m = 0. \end{cases}$$

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