行政院國家科學委員會專題研究計畫　成果報告

交互作用粒子系統的流力極限 (8)

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行政院國家科學委員會專題研究計畫成果報告

交互作用粒子系統的流力極限 (8)

Hydrodynamic Limit of Interacting Particle Systems (8)

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一、中文摘要

本計劃中我們為一個固有值問題與 2 維格子點空間 Z^2 上對稱簡單互斥過程中一個位置被粒子佔據的時間的大離差估計建立起一個關聯。

關鍵詞：固有值、對稱簡單互斥過程、粒子佔據的時間、大離差估計

Abstract

In this project we develop a connection between an eigenvalue problem and the occupation time large deviations of two dimensional symmetric simple exclusion process established in [1].

Keywords: eigenvalue, symmetric simple exclusion process, occupation time, large deviations

二、報告內容

（報告正文以英文撰寫，於次頁起始）

In this report we briefly describe how to build a bridge between an eigenvalue problem and the occupation time large deviations established in [1]. We thank H.T. Yau for valuable discussions.

First we state the result of the occupation time large deviations of two dimensional symmetric simple exclusion process derived in [1]. Given $T > 0$, on the configuration space $\Omega = \{0,1\}^{\mathbb{Z}^2}$, consider the *speeded-up* symmetric simple exclusion process (SEP) generated by L_T given by

$$(L_T f)(\eta) \;=\; \frac{T}{2} \sum_{\substack{x,y\in\mathbb{Z}^2 \\ |x-y|=1}} [f(\sigma^{x,y}\eta) - f(\eta)]\;,$$

where the summation is carried over all nearest neighbor sites $x, y, |x-y| = 1$, of $\mathbb{Z}^2$. In this formula, f is a local function and $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x,\, y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x\;. \end{cases}$$

For each $0 \le \alpha \le 1$, denote by ν_α the Bernoulli product measure on Ω with marginals given by

$$\nu_\alpha\{\eta,\, \eta(x) = 1\} \;=\; \alpha$$

for $x \in \mathbb{Z}^2$. It is well known that $\{\nu_\alpha,\, 0 \le \alpha \le 1\}$ is a one-parameter family of reversible invariant measures. For $0 \le \alpha \le 1$, denote by $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ the probability on the path space $D(\mathbb{R}_+, \Omega)$ corresponding to SEP generated by L_T starting from ν_α. From now on we fix an $\alpha \in (0,1)$.

The large deviations principle of the occupation time of the origin:

$$O_T \;=\; \int_0^1 \eta_s(0)\, ds$$

under $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ as $T \to \infty$ has been established in [1]. It states that the decay rate is of order $T/\log T$, and the rate function $\Upsilon_\alpha : [0,1] \to \mathbb{R}_+$ is given by

$$\Upsilon_\alpha(\beta) \;=\; \frac{\pi}{2}\Big\{ \sin^{-1}(2\beta - 1) - \sin^{-1}(2\alpha - 1)\Big\}^2\;.$$

The corresponding Laplace-Varadhan Theorem [2] under the present setting has the following form.

Theorem 1 *Let $\mathcal{H} : [0,1] \to \mathbb{R}$ be a bounded continuous function on $[0,1]$. Then*

$$\lim_{T\to\infty} \frac{\log T}{T} \log \mathbb{E}_\alpha\Big[\exp\Big\{ \frac{T}{\log T}\mathcal{H}(O_T)\Big\}\Big] \;=\; \sup_{\beta\in[0,1]} \Big\{\mathcal{H}(\beta) - \Upsilon_\alpha(\beta)\Big\}. \tag{1}$$

For any configuration η, denote by $\mathbb{E}_\eta$ the conditional expectation of SEP $\mathbb{P}_\alpha$ given that the process starts from the specific configuration $\eta_0 = \eta$.

Let $V : \Omega = \{0,1\}^{\mathbb{Z}^2} \to \mathbb{R}$ be a local function on Ω satisfying $\nu_\alpha[\,V\,] \neq 0$ (in fact, more precisely, $\nu_\alpha[\,V \,|\, \bar\eta\,] \neq 0$). Denote the occupation time $O_T(V)$ associated with V by

$$O_T(V) = \int_0^1 V(\eta_s)\, ds\,, \quad \Big(\text{recall that } O_T = O_T(\eta(0)) = \int_0^1 \eta_s(0)\, ds\Big)\;.$$

Though no T appears on right hand side, the subscript T of $O_T(V)$ on left hand side indicates the dependency on T via the generator L_T of the process. By Feynman-Kac formula, the function $u(t,\eta) = u_T(t,\eta;\lambda), t \in [0,1]$, given by

$$u_T(t,\eta) = \mathbb{E}_\eta\Big[\exp\Big\{\int_0^t \frac{T}{\log T}\lambda V(\eta_s)\,ds\Big\}\Big] \tag{2}$$

solves the differential equation

$$\begin{cases} \partial_t u_T(t,\eta) = L_T u_T + \dfrac{T}{\log T}\,\lambda V(\eta) u_T,\ t \in [0,1]\,, \\ u_T(0,\eta) = 1\ . \end{cases} \tag{3}$$

We remark that the large deviations estimate of $O_T(V)$ is known when $V = \eta(0)$ as described above. Consequently, with the help of super-exponential estimate, since V is a local function, the large deviations estimate of $O_T(V)$ is fully understood.

Since u is positive, $\mathbb{E}[u] \le (\mathbb{E}[u^2])^{1/2} = \|u\|_{L^2(\nu_\alpha)}$. Furthermore

$$\begin{aligned} &\frac{1}{2}\partial_t \int u^2\,d\nu_\alpha = \int u\cdot\partial_t u\,d\nu_\alpha \\ = \ & \int u\cdot\Big\{L_T u + \frac{T}{\log T}\lambda V u\Big\}\ d\nu_\alpha = \|u\|^2_{L^2(\nu_\alpha)}\Big\{\int \bar{u}L_T\bar{u} + \frac{T}{\log T}\lambda V\bar{u}^2\,d\nu_\alpha\Big\} \\ = \ & \|u\|^2_{L^2(\nu_\alpha)}\Big\{\int \frac{T}{\log T}\lambda V\bar{u}^2\,d\nu_\alpha - D(\bar{u})\Big\} \\ \le \ & \|u\|^2_{L^2(\nu_\alpha)} \times \sup_{f,\ \|f\|_{L^2(\nu_\alpha)}=1}\Big\{\int \frac{T}{\log T}\lambda V f^2\,d\nu_\alpha - D(f)\Big\}\ , \end{aligned}$$

where $\bar{u} = u/\|u\|_{L^2(\nu_\alpha)}$ and $D(f) = \int f(-L_T)f\,d\nu_\alpha$. Denoting

$$\Gamma = \Gamma_T(\lambda) = \sup_{f,\ \|f\|_{L^2(\nu_\alpha)}=1}\Big\{\int \frac{T}{\log T}\lambda V f^2\,d\nu_\alpha - D(f)\Big\}\ ,$$

we obtain

$$\partial_t\|u\|^2_{L^2(\nu_\alpha)} \le 2\Gamma\|u\|^2_{L^2(\nu_\alpha)}\,.$$

By Gronwall's inequality we get

$$\|u(t)\|^2_{L^2(\nu_\alpha)} \le \|u(0)\|^2_{L^2(\nu_\alpha)}e^{2\Gamma t} = e^{2\Gamma t}$$

since $u(0,\eta) \equiv 1$. Now we evaluate $u(t)$ at $t = 1$, take logarithm, multiply $\log T/2T$ to get

$$\frac{\log T}{T}\Gamma_T(\lambda) \ge \frac{\log T}{T}\log\|u(1)\|_{L^2(\nu_\alpha)} \ge \frac{\log T}{T}\log\Big(\int u(1,\eta)\,d\nu_\alpha\Big)\ .$$

For simplicity, from now on we only consider the case $V(\eta) = \eta(0)$. Therefore

$$\int u(1,\eta;\lambda)\ d\nu_\alpha = \mathbb{E}_\alpha\Big[\exp\Big\{\int_0^1 \frac{T}{\log T}\lambda\eta_s(0)\,ds\Big\}\Big] = \mathbb{E}_\alpha\Big[\exp\Big\{\frac{T}{\log T}\lambda O_T\Big\}\Big].$$

Theorem 2 *For arbitrary* $\lambda \in \mathbb{R}$ *define*

$$\Gamma_T(\lambda) = \sup_{f,\, \|f\|_{L^2(\nu_\alpha)}=1} \left\{ \int \frac{T}{\log T} \lambda \eta(0) f^2 \, d\nu_\alpha - D(f) \right\} .$$

Then

$$\sup_{\beta \in [0,1]} \left\{ \lambda\beta - \Upsilon_\alpha(\beta) \right\} \le \lim_{T\to\infty} \frac{\log T}{T} \Gamma_T(\lambda) . \tag{4}$$

Proof: Since $\mathcal{H}(a) = \lambda a$ is a bounded continuous function on $[0, 1]$, we may apply Theorem 1 to conclude that

$$\sup_{\beta \in [0,1]} \left\{ \lambda\beta - \Upsilon_\alpha(\beta) \right\} = \lim_{T\to\infty} \frac{\log T}{T} \log \mathbb{E}_\alpha \left[\exp \left\{ \frac{T}{\log T} \mathcal{H}(O_T) \right\} \right] \le \lim_{T\to\infty} \frac{\log T}{T} \Gamma_T(\lambda)$$

by the discussion above. □

Naturally, the next step is to understand how close the two quantities appeared on the both sides of (4) are. This will be one of the subjects of the future project, and will appear in [3].

3 Self-evaluation

The eigenvalue problem is always an interesting subject in mathematics. In this project we develop an approach that connects the problem with the large deviation estimates. It is hoped that this method can give a different insight about the classical eigenvalue problem. Conceivably we believe that the progress made in this project is important.

References

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