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DERIVATIONS AND SKEW POLYNOMIAL RINGS

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Abstract. Let R be a prime ring and d a derivation of R. In the ring of additive endomorphisms of the abelian group $(R, +)$, let S be the subring generated by $a_L d^m$, where $a \in R$ and $m \geq 0$ and where $a_L : x \in R \mapsto ax \in R$ for $a \in R$. We compute the prime radical and minimal prime ideals of S via the skew polynomial ring $R[x; d]$ by the surjective ring homomorphism

$$\varphi : \sum_{i=0}^{n} a_i x^{n-i} \in R[x; d] \mapsto \sum_{i=0}^{n} (a_i)_L d^{n-i} \in S.$$

We compute explicitly the kernel $\mathcal{A}$ of φ, the prime radical $\mathcal{P}$ over $\mathcal{A}$ and minimal prime ideals over $\mathcal{A}$ (Theorem 2). We obtain a necessary and sufficient condition for S to be simple, prime or semiprime (Corollary 3). As an application, let d be nilpotent. We show that the d-extension of R defined in [6] is canonically isomorphic to the quotient ring of S modulo its prime radical (Corollary 14).

[1]

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1. Results

Throughout this paper, R is always a prime ring, *not* necessarily with 1, and d is a derivation of R. Additive endomorphisms of the abelian group $(R, +)$ form a ring End $(R, +)$ under the pointwise addition and the composition multiplication. Obviously, $d \in$ End $(R, +)$. For $a \in R$, let $a_L \in$ End $(R, +)$ be the left multiplication by a defined by $a_L : x \in R \mapsto ax \in R$. Let S be the subring generated by $a_L d^m$,

[1]

where $a \in R$ and $m \geq 0$ are arbitrary. If R possesses 1 then S is the ring generated by d and a_L, $a \in R$. We compute the prime radical and minimal prime ideals of S as follows: Let $R[x;d]$ be the skew polynomial ring with the multiplication rule: $xr = rx + d(r)$ for $r \in R$. Since $da_L = d(a)_L + a_L d$ for $a \in R$, the map

$$\varphi\colon a_0x^n + \cdots + a_{n-1}x + a_n \in R[x;d] \mapsto (a_0)_L d^n + \cdots + (a_{n-1})_L d + (a_n)_L \in S$$

defines a surjective ring homomorphism from $R[x;d]$ onto S. Let $\mathcal{A}$ denote the kernel of φ. The ring S is then isomorphic to the quotient ring $R[x;d]/\mathcal{A}$ and the prime radical of S corresponds to the ideal $\mathcal{P}$ of $R[x;d]$ such that $\mathcal{P} \supseteq \mathcal{A}$ and such that $\mathcal{P}/\mathcal{A}$ is the prime radical of $R[x;d]/\mathcal{A}$. We call $\mathcal{P}$ the prime radical of $R[x;d]$ over $\mathcal{A}$. We call an ideal $\mathcal{I}$ of $R[x;d]$ prime over $\mathcal{A}$ if $\mathcal{I} \supseteq \mathcal{A}$ and if $\mathcal{I}/\mathcal{A}$ is a prime ideal of $R[x;d]/\mathcal{A}$. Our aim is to describe explicitly the ideals $\mathcal{A}$, $\mathcal{P}$ and also minimal prime ideals over $\mathcal{A}$ in the ring $R[x,d]$. Notations introduced above will be retained throughout.

Let $R_{\mathcal{F}}$ and Q denote respectively the left Martindale quotient ring and the symmetric Martindale quotient ring of R. The center C of $R_{\mathcal{F}}$ coincides with the center of Q and is called the extended centroid of R. We refer these notions to [2] or [13] for details. It is well-known that d can be uniquely extended to a derivation of Q and also of $R_{\mathcal{F}}$, which we also denote by d. We thus form the skew polynomial ring $R_{\mathcal{F}}[x;d]$, which forms an overring of $R[x;d]$ in a natural way. Given $f(x)$ in the center of $R_{\mathcal{F}}[x;d]$, we define

$$\begin{aligned}\langle f(x)\rangle &\stackrel{\text{def.}}{=} R[x;d] \cap f(x)R_{\mathcal{F}}[x;d]\\ &= \{g(x) \in R[x;d] \mid g(x) \text{ is a multiple of } f(x) \text{ in } R_{\mathcal{F}}[x;d]\}.\end{aligned}$$

We will show that $\mathcal{A}, \mathcal{P}$ and all minimal prime ideals over $\mathcal{A}$ are of the form $\langle f(x)\rangle$ for some central elements $f(x) \in R_{\mathcal{F}}[x;d]$. We also want to compute these central elements $f(x)$ explicitly. For this purpose, we must investigate the center of $R_{\mathcal{F}}[x;d]$. Fortunately, this has been completely done in [14]. But we need some more notions given in the following to restate it.

Given $b \in R$, the map $\mathrm{ad}(b)\colon r \in R \mapsto [b,r] \stackrel{\text{def.}}{=} br - rb$ obviously defines a derivation, called the inner derivation defined by the element b. We call a derivation *outer* if it is *not* of this form. If a derivation of R extends to an inner derivation of $R_{\mathcal{F}}$, say $\mathrm{ad}(b)$, where $b \in R_{\mathcal{F}}$, we see easily that $b \in Q$. A derivation of R is called X-inner if its extension to $R_{\mathcal{F}}$ (or to Q) is inner, that is, it is of the form $r \in R \mapsto [b,r]$ for some $b \in Q$. We call a derivation X-outer if it is not X-inner. Given a subset S of $R_{\mathcal{F}}$, the set of constants of d on S, denoted by $S^{(d)}$, is defined by

$S^{(d)} \stackrel{\text{def.}}{=} \{r \in S \mid d(r) = 0\}$. Particularly, $C^{(d)} = \{\alpha \in C \mid d(\alpha) = 0\}$. We are ready to cite the following elegant result.

Theorem 1 (Matczuk [14]). *The center of $R_{\mathcal{F}}[x; d]$ is equal to the center of $Q[x; d]$ and can be described as follows:*

(1) Assume $\operatorname{char} R = 0$*. If $d + \operatorname{ad}(b) = 0$ for some $b \in R_{\mathcal{F}}$, then the center of $R_{\mathcal{F}}[x; d]$ is equal to $C^{(d)}[\zeta]$, where $\zeta \stackrel{\text{def.}}{=} x + b$. If there is no such $b \in R_{\mathcal{F}}$, then the center of $R_{\mathcal{F}}[x; d]$ is merely $C^{(d)}$.*

(2) Assume $\operatorname{char} R = p > 0$*. If there exists $b \in {R_{\mathcal{F}}}^{(d)}$ and $\alpha_1, \cdots, \alpha_s \in C^{(d)}$ such that*

$$d^{p^s} + \alpha_1 d^{p^{s-1}} + \cdots + \alpha_s d + \operatorname{ad}(b) = 0, \tag{1}$$

then we let (1) be the one with s as minimal as possible and set

$$\zeta \stackrel{\text{def.}}{=} x^{p^s} + \alpha_1 x^{p^{s-1}} + \cdots + \alpha_s x + b.$$

The center of $R_{\mathcal{F}}[x; d]$ is equal to $C^{(d)}[\zeta]$. If there is no such expression (1), then the center of $R_{\mathcal{F}}[x; d]$ is merely $C^{(d)}$.

We call the derivation d left algebraic over $R_{\mathcal{F}}$ or left $R_{\mathcal{F}}$-algebraic for short, if there exist $0 \neq b_0, b_1, \cdots, b_{n-1} \in R_{\mathcal{F}}$ such that

$$b_0 d^n(r) + b_1 d^{n-1}(r) + \cdots + b_{n-1} d(r) = 0$$

for all $r \in R$. If all $b_i \in R$ (or all $b_i \in C$ respectively), then we say that d is left R-algebraic (or C-algebraic respectively). It is easy to see that the left $R_{\mathcal{F}}$-algebraicity, the left R-algebraicity and the C-algebraicity of a derivation d are all equivalent. If d is *not* left R-algebraic, then $\mathcal{A} = 0$ and S is isomorphic to $R[x; d]$. Since the ring $R[x; d]$ is prime, we have $\mathcal{P} = 0$ and the only minimal prime ideal over $\mathcal{A}$ is $\{0\}$. There is nothing to prove in this case. We hence assume that our derivation d is left R-algebraic or, equivalently, left $R_{\mathcal{F}}$-algebraic. Our main theorem is as follows:

Theorem 2. *Let R be a prime ring and let d be a left R-algebraic derivation of R. Let ζ, b be as described in Theorem 1. Let $\mu(\lambda)$ be the minimal polynomial of b over $C^{(d)}$. Then the following hold:*

(1) $\mathcal{A} = \langle \mu(\zeta) \rangle$.

(2) We factorize $\mu(\lambda)$ into the product of monic irreducible factors in $C^{(d)}[\lambda]$: $\mu(\lambda) = \pi_1(\lambda)^{n_1} \pi_2(\lambda)^{n_2} \cdots \pi_k(\lambda)^{n_k}$. Then $\mathcal{P} = \langle \pi_1(\zeta)\pi_2(\zeta) \cdots \pi_k(\zeta) \rangle$ and minimal prime ideals of $R[x; d]$ over $\mathcal{A}$ are $\langle \pi_s(\zeta) \rangle$, $s = 1, \cdots, k$.

An ideal I of R is d-invariant if $d(I) \subseteq I$. Obviously, R has two d-invariant ideals R and $\{0\}$, which we call trivial d-invariant ideals. We have the following:

Corollary 3. *In the notations of Theorem 1, we have the following:*

(1) The ring S is semiprime if and only if the minimal polynomial of b over $C^{(d)}$ has no square factors.

(2) The ring S is prime if and only if the minimal polynomial of b over $C^{(d)}$ is irreducible.

(3) The ring S is simple if and only if R has no nontrivial d-invariant ideals and the minimal polynomial of b over $C^{(d)}$ is irreducible.

Before proceeding to the proof of Theorem 2, let us compute explicitly the ζ of Theorem 1 for a left R-algebraic derivation d: We apply Kharchenko's theorem [11, Corollaries 2 and 3]. If char $R = 0$, then $d + \mathrm{ad}(b) = 0$ for some $b \in R_{\mathcal{F}}$ and we set $\zeta \overset{\text{def.}}{=} x + b$. If char $R = p > 0$, then $d, d^p, d^{p^2}, \ldots$ are C-dependent modulo X-inner derivations. Let $s \geq 0$ be the minimal integer such that

$$d^{p^s}, d^{p^{s-1}}, \cdots, d^p, d$$

are C-dependent modulo X-inner derivations. By the minimality of s, there exist $\alpha_i \in C$ and $b \in Q$ such that

$$d^{p^s} + \alpha_1 d^{p^{s-1}} + \cdots + \alpha_s d + \mathrm{ad}(b) = 0. \tag{2}$$

By the minimality of s again, we see easily that $d(\alpha_i) = 0$ and $d(b) \in C$. We divide our discussion into two cases:

Case 1. $d(b) \in d(C)$: Say, $d(b) = d(\alpha)$, where $\alpha \in C$. Then $d(b - \alpha) = 0$. Since b and $b - \alpha$ define the same X-inner derivation, we may replace b by $b - \alpha$ and assume that $d(b) = 0$. So we have

$$\zeta = x^{p^s} + \alpha_1 x^{p^{s-1}} + \cdots + \alpha_s x + b.$$

Case 2. $d(b) \notin d(C)$: Since $d(\alpha_i) = 0$, all left multiplications $(\alpha_i)_L$ commutes with d. Since $d(b) \in C$, we have for $r \in R$,

$$d\big(\mathrm{ad}(b)(r)\big) = d([b, r]) = [d(b), r] + [b, d(r)] = [b, d(r)] = \mathrm{ad}(b)\big(d(r)\big).$$

So $\mathrm{ad}(b)$ also commutes with d. Using this commutativity and noting $\big(\mathrm{ad}(b)\big)^p = \mathrm{ad}(b^p)$, we raise both sides of (2) to the p-th power. This gives the equality:

$$d^{p^{s+1}} + \alpha_1^p d^{p^s} + \cdots + \alpha_s^p d^p + \mathrm{ad}(b^p) = 0.$$

Obviously, $d(b^p) = pb^{p-1}d(b) = 0$. So we have $\zeta = x^{p^{s+1}} + \alpha_1^p x^{p^s} + \cdots + \alpha_s^p x^p + b^p$.

The differential identity (2) of R also vanishes on $R_{\mathcal{F}}$ [12, Theorem 2]. In particular, the evaluation of (2) on C shows that the restriction of d to C is C-algebraic. In view of [1, Theorem 1], C is finite-dimensional over $C^{(d)}$. The left $R_{\mathcal{F}}$-algebraicity of d implies the left $R_{\mathcal{F}}$-algebraicity of $\mathrm{ad}(b)$ and the latter implies the C-algebraicity of b. In view of the finite-dimensionality of C over $C^{(d)}$, we see that b is $C^{(d)}$-algebraic. We summarize what we have shown in the following:

Lemma 4. *Let d be a left $R_{\mathcal{F}}$-algebraic derivation and let ζ be as described in Theorem 1. If $\operatorname{char} R = 0$, then $d + \mathrm{ad}(b) = 0$ for some $b \in Q$ and $\zeta = x + b$. If $\operatorname{char} R = p \geq 2$, then there exists the minimal integer $s \geq 0$ such that*

$$d^{p^s} + \alpha_1 d^{p^{s-1}} + \cdots + \alpha_s d + \mathrm{ad}(b) = 0$$

for some $\alpha_i \in C^{(d)}$ and $b \in Q$ with $d(b) \in C$. In the case of $d(b) \in d(C)$, we may choose $b \in Q^{(d)}$ and $\zeta = x^{p^s} + \alpha_1 x^{p^{s-1}} + \cdots + \alpha_s x + b$. In the case of $d(b) \notin d(C)$, we have $b^p \in Q^{(d)}$ and $\zeta = x^{p^{s+1}} + \alpha_1^p x^{p^s} + \cdots + \alpha_s^p x^p + b^p$. Moreover, b above is always $C^{(d)}$-algebraic.

To prove Theorem 2, we need another important result from [14], which, unfortunately, is not explicitly stated in [14]. For our purpose, we formulate it in the following form but refer its proof to [5, Lemma 1.3], [14] or [15, Theorem 3.3]. Although the ring R is assumed unital in [5, 14, 15], we can modify their proofs to our case that prime rings are not necessarily with an identity element.

Theorem 5 (Matczuk). *Given an ideal $\mathcal{I} \neq 0$ of $R[x; d]$, there exist an ideal $I \neq 0$ of R and a unique monic polynomial $f(x)$ in the center of $R_{\mathcal{F}}[x; d]$ such that $If(x) \subseteq \mathcal{I} \subseteq \langle f(x) \rangle$.*

Following [5], we call call $f(x)$ above the *canonical polynomial* of $\mathcal{I}$. Obviously, $f(x) = 1$ if and only if $\mathcal{I} \cap R \neq 0$. In this case, we can take the ideal I in Theorem 5 above to be $\mathcal{I} \cap R$. Following [4], we call an ideal of $R[x; d]$ *principal closed* if it is of the form $\langle f(x) \rangle$ for some $f(x)$ in the center of $R_{\mathcal{F}}[x; d]$. Let $\mathcal{I}, \mathcal{J}$ be ideals of $R[x; d]$. If $\mathcal{I} \subseteq \mathcal{J}$ then the canonical polynomial of $\mathcal{J}$ divides that of $\mathcal{I}$. This will be used frequently.

Following [9], we use the surjective ring homomorphism $\varphi \colon R[x; d] \to S$ to define an action $\rightharpoonup$ of $R[x; d]$ on R as follows: Given $g(x) \in R[x; d]$ and $r \in R$, we define $g(x) \rightharpoonup r$ to be $\varphi(g(x))(r)$. If $g(x) = a_0 x^n + \cdots + a_{n-1} x + a_n \in R[x; d]$, where $a_i \in R$, then $\varphi(g(x)) = (a_0)_L d^n + \cdots + (a_{n-1})_L d + (a_n)_L$ and hence

$$\begin{aligned} g(x) \rightharpoonup r &\overset{\text{def.}}{=} \big((a_0)_L d^n + \cdots + (a_{n-1})_L d + (a_n)_L\big)(r) \\ &= a_0 d^n(r) + \cdots + a_{n-1} d(r) + a_n r. \end{aligned}$$

It is well-known that R forms a left $R[x;d]$-module under the action $\rightharpoonup$. (See [9] for example.) Note that

$$\mathcal{A} = \{g(x) \in R[x;d] \mid g(x) \rightharpoonup R = 0\}.$$

We define similarly $g(x) \rightharpoonup r$ for $g(x) \in R_{\mathcal{F}}[x;d]$ and $r \in R_{\mathcal{F}}$. Obviously, the action $\rightharpoonup$ of $R_{\mathcal{F}}[x;d]$ on $R_{\mathcal{F}}$ extends the action $\rightharpoonup$ of $R[x;d]$ on R. We observe a simple but important property of central elements in $R_{\mathcal{F}}[x;d]$.

Lemma 6. *If $a_0x^n + \cdots + a_{n-1}x + a_n$ lies in the center of $R_{\mathcal{F}}[x;d]$, then*

$$(a_0x^n + \cdots + a_{n-1}x + a_n) \rightharpoonup r = ra_n \text{ for all } r \in R_{\mathcal{F}}.$$

Proof. We say that $g(x) \in R[x;d]$ has the constant term $c \in R$ if $g(x)$ can be written in the form $g(x) = c +$ terms ending in x. Set $f(x) = a_0x^n + \cdots + a_{n-1}x + a_n$. Let $r \in R_{\mathcal{F}}$. Note that $a_0d^n(r) + \cdots + a_{n-1}d(r) + a_nr$ is the constant term of $f(x)r$ and that ra_n is the constant term of $rf(x)$. Since $f(x)$ is central, the two skew polynomials $f(x)r$, $rf(x)$ are equal and hence so are their constant terms. Thus $a_0d^n(r) + \cdots + a_{n-1}d(r) + a_nr = ra_n$, proving the lemma.

An ideal $\mathcal{I}$ of $R[x;d]$ is called R-disjoint if $\mathcal{I} \cap R = \{0\}$. [5, Theorem 1.6] gives the following elegant characterization of R-disjoint prime ideals of $R[x;d]$.

Lemma 7. *An ideal of $R[x;d]$ is prime and R-disjoint if and only if it is of the form $\langle \pi(\zeta) \rangle$ for a monic irreducible $1 \neq \pi(\lambda) \in C^{(d)}[\lambda]$.*

We are now ready to give the

Proof of Theorem 2. By Theorem 5, there exist the canonical polynomial $f(x)$ of $\mathcal{A}$ and an ideal $I \neq 0$ of R such that $If(x) \subseteq \mathcal{A} \subseteq \langle f(x) \rangle$. Since $If(x) \subseteq \mathcal{A}$, we have $0 = If(x) \rightharpoonup R = I(f(x) \rightharpoonup R)$ and hence $f(x) \rightharpoonup R = 0$. This implies $\langle f(x) \rangle \rightharpoonup R = 0$ and hence $\mathcal{A} = \langle f(x) \rangle$.

By assumption, d is $R_{\mathcal{F}}$-algebraic. Let

$$\zeta = \begin{cases} x + b, & \text{if char } R = 0 \\ x^{p^s} + \alpha_1 x^{p^{s-1}} + \cdots + \alpha_s x + b, & \text{if char } R = p > 0 \end{cases}$$

be given as in Theorem 1. As in the proof of Lemma 6, we say that $g(x) \in R[x;d]$ has the constant term $c \in R$ if $g(x)$ can be written in the form $g(x) = c +$ terms ending in x.

Claim. ζ^k has the constant term b^k for $k \geq 0$. This is obvious if $k=0$. For $k \geq 1$, we have

$$\begin{aligned} \zeta^k &= \zeta^{k-1}(b + \text{terms ending in } x) \\ &= \zeta^{k-1}b + \text{terms ending in } x \\ &= b\zeta^{k-1} + \text{terms ending in } x \end{aligned}$$

If ζ^{k-1} has the constant term b^{k-1}, then ζ^k has the constant term b^k. The claim follows by induction on k.

Applying Theorem 1, we may write

$$f(x) = \zeta^n + \beta_1\zeta^{n-1} + \cdots + \beta_n, \tag{3}$$

where $\beta_i \in C^{(d)}$. We set

$$\widetilde{\mu}(\lambda) = \lambda^n + \beta_1\lambda^{n-1} + \cdots + \beta_n \in C^{(d)}[\lambda].$$

By Claim above, $f(x) = \widetilde{\mu}(\zeta)$ has the constant term $\widetilde{\mu}(b)$. By Lemma 6, we have $0 = f(x) \rightharpoonup R = R\,\widetilde{\mu}(b)$ and so $\widetilde{\mu}(b) = 0$. The minimal polynomial $\mu(\lambda)$ of b over $C^{(d)}$ thus divides $\widetilde{\mu}(\lambda)$ in $C^{(d)}[\lambda]$. On the other hand, $\mu(b)$ is the constant term of $\mu(\zeta)$ by Claim above. Since $\mu(\zeta)$ is central, we have $\mu(\zeta) \rightharpoonup R = R\mu(b) = 0$ by Lemma 6. Let J be a nonzero ideal of R such that $J\mu(\zeta) \subseteq R[x;d]$. Then $J\mu(\zeta) \subseteq \mathcal{A}$. It follows from the minimality of the degree of $f(x)$ in $\mathcal{A}$ that the degree of $\mu(\lambda)$ is equal to or greater than the degree of the canonical polynomial $f(x)$ of $\mathcal{A}$. So $\mu(\lambda) = \widetilde{\mu}(\lambda)$ and $f(x) = \mu(\zeta)$ follows.

We factorize $\mu(\lambda)$ into the product of monic irreducible factors in $C^{(d)}[\lambda]$: $\mu(\lambda) = \pi_1(\lambda)^{n_1}\pi_2(\lambda)^{n_2}\cdots\pi_k(\lambda)^{n_k}$, where each $n_i \geq 1$. The rest then follows by the more general Theorem 8 below.

Let $\mathcal{I}, \mathcal{J}$ be ideals of $R[x;d]$. We say that $\mathcal{J}$ is prime over $\mathcal{I}$ if $\mathcal{J} \supseteq \mathcal{I}$ and if $\mathcal{J}/\mathcal{I}$ is a prime ideal of $R[x;d]/\mathcal{I}$. We call $\mathcal{J}$ the prime radical over $\mathcal{I}$ if $\mathcal{J} \supseteq \mathcal{I}$ and if $\mathcal{J}/\mathcal{I}$ is the prime radical of $R[x;d]/\mathcal{I}$. The following theorem describes the prime radical and minimal prime ideals over a principal closed ideal.

Theorem 8. *Let $\mu(\lambda) \in C^{(d)}[\lambda]$ be monic. We factorize $\mu(\lambda)$ into the product of monic irreducible factors in $C^{(d)}[\lambda]$: $\mu(\lambda) = \pi_1(\lambda)^{n_1}\pi_2(\lambda)^{n_2}\cdots\pi_k(\lambda)^{n_k}$, where each $n_i \geq 1$. Then minimal prime ideals of $R[x;d]$ over $\langle\mu(\zeta)\rangle$ are $\langle\pi_s(\zeta)\rangle$, $s = 1, \cdots, k$, where ζ is given as in Theorem 1. Moreover, the prime radical of $R[x;d]/\langle\mu(\zeta)\rangle$ is equal to $\langle\pi_1(\lambda)\pi_2(\lambda)\cdots\pi_k(\lambda)\rangle/\langle\mu(\zeta)\rangle$.*

For the proof of Theorem 8, we need a lemma.

Lemma 9. *If $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are monic and relatively prime in $C^{(d)}[\lambda]$, then $\langle\mu_1(\zeta)\rangle \cap \langle\mu_2(\zeta)\rangle = \langle\mu_1(\zeta)\mu_2(\zeta)\rangle$, where ζ is given as in Theorem 1.*

Proof. The inclusion $\langle \mu_1(\zeta)\mu_2(\zeta)\rangle \subseteq \langle \mu_1(\zeta)\rangle \cap \langle \mu_2(\zeta)\rangle$ is obvious. For the reverse inclusion, let $f(x) \in \langle \mu_1(\zeta)\rangle \cap \langle \mu_2(\zeta)\rangle$. Write $f(x) = g_1(x)\mu_1(\zeta) = g_2(x)\mu_2(\zeta)$, where $g_i(x) \in R_{\mathcal{F}}[x; d]$. Since $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are relatively prime in $C^{(d)}[\lambda]$, there exist $A(\lambda), B(\lambda) \in C^{(d)}[\lambda]$ such that $A(\lambda)\mu_1(\lambda) + B(\lambda)\mu_2(\lambda) = 1$. So

$$A(\zeta)\mu_1(\zeta) + B(\zeta)\mu_2(\zeta) = 1.$$

Thus

$$\begin{aligned} g_1(x) =& A(\zeta)g_1(x)\mu_1(\zeta) + g_1(x)B(\zeta)\mu_2(\zeta) \\ =& A(\zeta)g_2(x)\mu_2(\zeta) + g_1(x)B(\zeta)\mu_2(\zeta) \\ =& \big(A(\zeta)g_2(x) + g_1(x)B(\zeta)\big)\mu_2(\zeta), \end{aligned}$$

implying that

$$f(x) = g_1(x)\mu_1(\zeta) = \big(A(\zeta)g_2(x) + g_1(x)B(\zeta)\big)\mu_1(\zeta)\mu_2(\zeta).$$

So $f(x) \in \langle \mu_1(\zeta)\mu_2(\zeta)\rangle$. Thus $\langle \mu_1(\zeta)\rangle \cap \langle \mu_2(\zeta)\rangle \subseteq \langle \mu_1(\zeta)\mu_2(\zeta)\rangle$, proving the lemma.

Proof of Theorem 8. Let $\mathcal{Q}$ be an ideal of $R[x; d]$, which is a minimal prime ideal over $\langle \mu(\zeta)\rangle$. Then

$$\mathcal{Q} \supseteq \langle \pi_1(\zeta)^{n_1} \cdots \pi_k(\zeta)^{n_k}\rangle \supseteq \langle \pi_1(\zeta)\rangle^{n_1} \cdots \langle \pi_k(\zeta)\rangle^{n_k}.$$

By the primeness of $\mathcal{Q}$, we see that $\mathcal{Q}$ includes $\langle \pi_i(\zeta)\rangle$ for some i. By Lemma 7, $\langle \pi_i(\zeta)\rangle$ is a prime ideal of $R[x; d]$. The minimality of $\mathcal{Q}$ implies that $\mathcal{Q} = \langle \pi_i(\zeta)\rangle$. This proves that all possible minimal prime ideals of $R[x; d]$ over $\langle \mu(\zeta)\rangle$ are $\langle \pi_s(\zeta)\rangle$, $s = 1, \cdots, k$. Conversely, we show each $\langle \pi_i(\zeta)\rangle$ is a minimal prime ideal over $\langle \mu(\zeta)\rangle$: Let $\mathcal{Q}_0$ be a prime ideal of $R[x; d]$ such that $\langle \pi_i(\zeta)\rangle \supseteq \mathcal{Q}_0 \supseteq \langle \mu(\zeta)\rangle$. Applying the same argument above yields $\mathcal{Q}_0 \supseteq \langle \pi_j(\zeta)\rangle$ for some j and so $\langle \pi_i(\zeta)\rangle \supseteq \langle \pi_j(\zeta)\rangle$. Then $\pi_i(\zeta)$ divides $\pi_j(\zeta)$. So $\pi_i(\zeta) = \pi_j(\zeta) = \mathcal{Q}_0$ follows as asserted. Let $\mathcal{H}$ be the prime radical over the ideal $\langle \mu(\zeta)\rangle$. Choose an integer $m \geq n_i$ for all i. Then

$$\langle \pi_1(\lambda)\pi_2(\lambda) \cdots \pi_k(\lambda)\rangle^m \subseteq \langle \mu(\zeta)\rangle \subseteq \mathcal{H}.$$

But $\mathcal{H}$ is a semiprime ideal of $R[x; d]$. So $\langle \pi_1(\lambda)\pi_2(\lambda) \cdots \pi_k(\lambda)\rangle \subseteq \mathcal{H}$ follows. On the other hand, by Lemma 7, each $\langle \pi_i(\zeta)\rangle$ is a prime ideal of $R[x; d]$ and so

$$\mathcal{H} \subseteq \langle \pi_1(\lambda)\rangle \cap \cdots \langle \pi_1(\lambda)\rangle \subseteq \langle \pi_1(\lambda)\pi_2(\lambda) \cdots \pi_k(\lambda)\rangle,$$

where the second inclusion is implied by Lemma 9. Thus $\mathcal{H} = \langle \pi_1(\lambda)\pi_2(\lambda) \cdots \pi_k(\lambda)\rangle$. The proof is now complete.

We conclude this section with

Proof of Corollary 3. (1) and (2) follows immediately from Theorem 2. For (3), let $\mu(\lambda) \in C^{(d)}[\lambda]$ denote the minimal polynomial of b over $C^{(d)}$. For the implication $\Leftarrow$, assume that R has no nontrivial d-invariant ideals and that $\mu(\lambda)$ is irreducible over $C^{(d)}$. Let $\mathcal{I}$ be an ideal of $R[x;d]$ properly larger than $\mathcal{A}$. By Theorem 2, $\mathcal{A} = \langle \mu(\zeta) \rangle$. Then the canonical polynomial of $\mathcal{I}$ is a proper divisor of $\mu(\zeta)$ and hence must be 1 by the irreducibility of $\mu(\zeta)$. So $\mathcal{I} \cap R \neq 0$. If $r \in \mathcal{I} \cap R$ then $d(r) = xr - rx \in \mathcal{I} \cap R$. So $\mathcal{I} \cap R$ is a d-invariant ideal of R. So $\mathcal{I} \cap R = R$, that is, $\mathcal{I} \supseteq R$. This implies $\mathcal{I} = R[x;d]$. The simplicity of $R[x;d]/\mathcal{A}$ follows as asserted.

For the implication $\Rightarrow$, we assume that S is simple. Then S is surely prime. By (2), the minimal polynomial $\mu(\lambda)$ of b over $C^{(d)}$ is irreducible. By Theorem 2, $\mathcal{A} = \langle \mu(\zeta) \rangle$. Let I be a nonzero d-invariant ideal of R. Set

$$I[x;d] \overset{\text{def.}}{=} \{a_0 + a_1 x + \cdots \in R[x;d] \mid a_0, a_1, \ldots \in I\}.$$

Then $I[x;d]$ forms an ideal of $R[x;d]$. Note that $R \cap I[x;d] = I$ but $R \cap \mathcal{A} = 0$. The ideal $\mathcal{A} + I[x;d]$ is thus properly larger than $\mathcal{A}$. But S is isomorphic to $R[x;d]/\mathcal{A}$. By the simplicity of S, $\mathcal{A} + I[x;d] = R[x;d]$. Given any $a \in R$, we may thus write

$$a = f(x) + a_n x^n + \cdots + a_0,$$

where $f(x) \in \mathcal{A}$ and where $a_n, \ldots, a_0 \in I$. Using the ring homomorphism $\varphi\colon R[x;d] \to S$ and noting that $\varphi(f(x)) = 0$, we have

$$a_L = (a_n)_L d^n + \cdots + (a_1)_L d + (a_0)_L.$$

That is, for all $y \in R$,

$$ay = a_n d^n(y) + \cdots + a_1 d(y) + a_0 y.$$

But this differential identity also holds for $y \in Q$ by [12, Theorem 2]. Setting $y = 1$, we have $a = a_0 \in I$. This is true for any given $a \in R$. It follows that $I = R$. So R has no d-invariant ideals other that R and 0, as asserted.

2. An Application to the Nilpotent Case

Firstly, we need an important notion discovered by Grzezczuk:

Definition ([6, 3]). Let R be a prime ring and let d be a nilpotent derivation of R. The least integer m such that $d^m(R)c = 0$ for some nonzero $c \in R$ is called the *annihilating* nilpotency of d and is denoted by $m_d(R)$.

Let d be a nilpotent derivation of R. We consider the prime subring $R + \mathbf{Z} \cdot 1$ of Q, where $\mathbf{Z}$ is the ring of integers. The derivation d extends to a nilpotent derivation of $R + \mathbf{Z} \cdot 1$ with the same nilpotencey and annihilating nilpotency. Replacing R by $R + \mathbf{Z} \cdot 1$, we always assume that R *has an identity element* 1 in this section. In the ring $R[x; d]$, we consider the two-sided ideal (x^m), where $m = m_d(R)$. For $r \in R$,

$$x^m r = rx^m + \binom{m}{1} d(r)x^{m-1} + \cdots + \binom{m}{m-1} d^{m-1}(r)x + d^m(r).$$

Therefore, if $a_0x^n + a_1x^{n-1} + \cdots + a_n \in (x^m)$, then $a_n \in Rd^m(R)$. This implies $Rd^m(R)$ includes $(x^m) \cap R$, which is an ideal of R. Since $Rd^m(R)$ has nonzero right annihilator, we have $(x^m) \cap R = 0$. We have now come to an interesting construction, which has been employed extensively and fruitfully in the literature [6]–[10]:

Definition ([8]). Let d be a nilpotent derivation of R. Write $m = m_d(R)$. Let $\mathcal{M}$ be an ideal of $R[x; d]$ which is maximal with respect to the property that $x^m \in \mathcal{M}$ and $\mathcal{M} \cap R = 0$. Obviously, $\mathcal{M}$ is a prime ideal of $R[x; d]$. The quotient ring $R[x; d]/\mathcal{M}$ is called the d-extension of R.

Although the $\mathcal{M}$ obtained above by Zorn's Lemma is not necessarily unique, our aim is to prove that $\mathcal{M}$ is equal to the ideal $\mathcal{P}$ described in Theorem 2. So it is unique. From now on we always fix such an $\mathcal{M}$. For this purpose we need a structure result of nilpotent derivations. The following is given in [3, Theorems 1–4].

Theorem 10. *Let d be a nilpotent derivation of a prime ring R.*

(1) In the case of $\operatorname{char} R = 0$*, there exists a nilpotent $b \in Q$ with the nilpotency l such that $d + \operatorname{ad}(b) = 0$ and $m_d(R) = l$.*

(2) In the case of $\operatorname{char} R = p \geq 2$*, let s be the least integer ≥ 1 such that d^{p^i}, $0 \leq i \leq s$, are C-dependent modulo X-inner derivations. Then there exists $b \in Q$ such that $d^{p^s} + \operatorname{ad}(b) = 0$ and such that the minimal polynomial of b over C assumes the form $(b^{p^t} - \alpha)^l = 0$, where $\alpha \in C^{(d)}$ and where l, t are integers ≥ 0 such that $(l, p) = 1$. Moreover, $m_d(R) = p^{s+t}l$.*

For a nilpotent derivation d, we have a detailed description of the ideal $\mathcal{M}$ defined above. We divide our statement into two cases according to whether the characteristic of R is 0 or not:

Theorem 11. *Let d be a nilpotent derivation of a prime ring R and let $\mathcal{A}$, $\mathcal{P}$ be as described in Theorem 2 and $\mathcal{M}$, the ideal described in the definition above. In the notation of Theorem 10, if* $\operatorname{char}\ R = 0$*, then $\mathcal{A} = \langle \zeta^l \rangle$ and $\mathcal{P} = \mathcal{M} = \langle \zeta \rangle$, where $\zeta = x + b$ and where l is the nilpotency of b.*

We need the following lemma. See [2, Theorem 2.3.3] for the proof.

Lemma 12. *Let $v_1, v_2, \cdots, v_n$ be C-independent elements in $R_{\mathcal{F}}$ and let I be a nonzero ideal of R. Then there exist finitely many $a_i, b_i \in I$ such that $\sum_i a_i v_j b_i = 0$ for $1 \le j \le n-1$ but $\sum_i a_i v_n b_i \ne 0$.*

Proof of Theorem 11. We retain the notation of Theorem 10 in the following. Then, by (1) of Theorem 10, $d + \mathrm{ad}(b) = 0$ for some nilpotent $b \in Q$ with the nilpotency l. That is, $b^l = 0$ but $b^{l-1} \ne 0$. By Theorem 1, the center of $R_{\mathcal{F}}[x; d]$ is equal to $C^{(d)}[\zeta]$, where $\zeta = x + b$. The minimal polynomial of b over $C^{(d)}$ is obviously the polynomial λ^l in $C^{(d)}[\lambda]$. It follows from Theorem 2 that $\mathcal{A} = \langle \zeta^l \rangle$ and $\mathcal{P} = \langle \zeta \rangle$. By Lemma 7, $\mathcal{P}$ is prime, as asserted. We compute the ideal $\mathcal{M}$: By Theorem 10, $m_d(R) = l$. Now, we look at the canonical polynomial of the ideal (x^l) of $R[x; d]$: Write $x = \zeta + b$. Noting that $b^l = 0$ and ζ is central, we have

$$x^l = (\zeta + b)^l = \zeta^l + \binom{l}{1}\zeta^{l-1} b + \cdots + \binom{l}{l-1}\zeta b^{l-1},$$

implying that $(x^l) \subseteq \langle \zeta \rangle$. Since $1, b, \ldots, b^{l-1}$ are C-independent, by Lemma 12 there exist finitely many $r_i, r_i' \in I$ such that $\sum_i r_i b^{l-1} r_i' \ne 0$ but such that $\sum_i r_i b^j r_i' = 0$ for $0 \le j < l-1$. We have

$$(x^l) \ni \sum_i r_i x^l r_i' = \binom{l}{l-1}\zeta\Big(\sum_i r_i b^{l-1} r_i'\Big) \ne 0.$$

This shows that ζ is the canonical polynomial of the ideal (x^l). But $\mathcal{M}$ extends (x^l). We see easily that the canonical polynomial of $\mathcal{M}$ divides the canonical polynomial ζ of (x^l). Since $\mathcal{M} \cap R = 0$, the canonical polynomial of $\mathcal{M}$ cannot be 1 and hence must be ζ. So $\langle \zeta \rangle \supseteq \mathcal{M}$. But $\langle \zeta \rangle$ also extends (x^l) and intersects R trivially. By the maximality of $\mathcal{M}$, it follows that $\mathcal{M} = \langle \zeta \rangle = \mathcal{P}$.

The case for char $R = p \ge 2$ is more complicate:

Theorem 13. *Let d be a nilpotent derivation of a prime ring R and let $\mathcal{A}$, $\mathcal{P}$ be as described in Theorem 2 and $\mathcal{M}$, the ideal described in the definition above. In the notation of Theorem 10, if* char $R = p \ge 2$, *then we have the following two cases:*

(1) Suppose that b is chosen such that $d(b) = 0$. Set $\zeta = x^{p^s} + b$. Then

$$\mathcal{A} = \langle (\zeta^{p^t} - \alpha)^l \rangle \quad \textit{and} \quad \mathcal{M} = \mathcal{P} = \langle \zeta^{p^{t-u}} - \alpha^{1/p^u} \rangle,$$

where u is the largest integer such that $0 \le u \le t$ and such that $\alpha^{1/p^u} \in C^{(d)}$.

(2) Suppose that $d(b) \notin d(C)$. Set $\zeta = x^{p^{s+1}} + b^p$. Then

$$\mathcal{A} = \langle (\zeta^{p^{t-1}} - \alpha)^l \rangle \quad \textit{and} \quad \mathcal{M} = \mathcal{P} = \langle \zeta^{p^{t-1-u}} - \alpha^{1/p^u} \rangle,$$

where u is the largest integer such that $0 \le u \le t-1$ and such that $\alpha^{1/p^u} \in C^{(d)}$.

Proof. Assume char $R = p \ge 2$. Let s be the least integer ≥ 1 such that d^{p^i}, $0 \le i \le s$, are C-dependent modulo X-inner derivations. By Theorem 10, there exists $b \in Q$ such that $d^{p^s} + \mathrm{ad}(b) = 0$ and such that the minimal polynomial of b over C assumes the form $(b^{p^t} - \alpha)^l = 0$, where $\alpha \in C$, $l, t \ge 0$ and $(l, p) = 1$. By Theorem 10 again, $m_d(R) = p^{s+t}l$.

Our next step is to find the canonical polynomial of the ideal (x^m), where $m = m_d(R) = p^{s+t}l$. For this purpose, we must first decide ζ described in Theorem 1. Analogous to Lemma 4, we divide our argument into two cases:

Case 1. $d(b) \in d(C)$: By Lemma 4, we may assume that $d(b) = 0$ and so $\zeta = x^{p^s} + b$. Applying d to $(b^{p^t} - \alpha)^l = 0$, we obtain

$$-l(b^{p^t} - \alpha)^{l-1} d(\alpha) = 0.$$

Since $l \not\equiv 0$ modulo p and $(b^{p^t} - \alpha)^{l-1} \ne 0$, it follows $d(\alpha) = 0$. So $\zeta - \alpha$ is also in the center of $R_{\mathcal{F}}[x; d]$. Using this and noting that $(b^{p^t} - \alpha)^l = 0$, we compute

$$\begin{aligned} x^m = x^{p^{s+t}l} &= (\zeta - b)^{p^t l} = (\zeta^{p^t} - b^{p^t})^l = \big((\zeta^{p^t} - \alpha) - (b^{p^t} - \alpha)\big)^l \\ &= \sum_{i=0}^{l} (-1)^i \binom{l}{i} (\zeta^{p^t} - \alpha)^{l-i} (b^{p^t} - \alpha)^i \\ &= \sum_{i=0}^{l-1} (-1)^i \binom{l}{i} (\zeta^{p^t} - \alpha)^{l-i} (b^{p^t} - \alpha)^i. \end{aligned}$$

So $x^m \in \langle \zeta^{p^t} - \alpha \rangle$ and hence $x^m \subseteq \langle \zeta^{p^t} - \alpha \rangle$. So $\zeta^{p^t} - \alpha$ divides the canonical polynomial of (x^m). On the other hand, since l is the nilpotency of $b^{p^t} - \alpha$, the elements $1, b^{p^t} - \alpha, \ldots, \big(b^{p^t} - \alpha\big)^{l-1}$ are C-independent. By Lemma 12, there exist $r_i, r_i' \in I$ such that $\sum_i r_i (b^{p^t} - \alpha)^{l-1} r_i' \ne 0$ but such that $\sum_i r_i (b^{p^t} - \alpha)^j r_i' = 0$ for $0 \le j < l-1$. Multiplying the above displayed expression of x^m by r_i, r_i' from the left and right respectively and then adding them up, we have

$$0 \ne \sum_i r_i x^m r_i' = l(\zeta^{p^t} - \alpha)\Big(\sum_i r_i \big(b^{p^t} - \alpha\big)^{l-1} r_i'\Big) \in (x^m).$$

The canonical polynomial of the ideal (x^m) thus has ζ-degree $\le p^t$ and hence must be equal to $\zeta^{p^t} - \alpha$.

Let u be the largest integer such that $0 \le u \le t$ and such that $\alpha^{1/p^u} \in C^{(d)}$. Set $\beta = \alpha^{1/p^u}$. Then $\zeta^{p^t} - \alpha = (\zeta^{p^{t-u}} - \beta)^{p^u}$. Note that $\lambda^{p^{t-u}} - \beta \in C^{(d)}[\lambda]$ is irreducible. Since $\mathcal{M} \supseteq (x^m)$, the canonical polynomial of $\mathcal{M}$ is a divisor of $(\lambda^{p^{t-u}} - \beta)^{p^u}$. Say,

$(\lambda^{p^{t-u}} - \beta)^v$, where $0 \le v \le p^u$, is the canonical polynomial of $\mathcal{M}$. Since $\mathcal{M} \cap R = 0$, v must be > 0. We have

$$\mathcal{M} \subseteq \langle (\zeta^{p^{t-u}} - \beta)^v \rangle \subseteq \langle \zeta^{p^{t-u}} - \beta \rangle.$$

Since $\langle \zeta^{p^{t-u}} - \beta \rangle \cap R = 0$, it follows that $\mathcal{M} = \langle \zeta^{p^{t-u}} - \beta \rangle$ by the maximality of $\mathcal{M}$.

We now compute the ideals $\mathcal{A}$ and $\mathcal{P}$ by Theorem 2: Since the minimal polynomial of b over $C^{(d)}$ is $(\lambda^{p^t} - \alpha)^l = (\lambda^{p^{t-u}} - \beta)^{p^u l} \in C^{(d)}[\lambda]$, the ideal $\mathcal{A}$ is thus equal to

$$\langle (\zeta^{p^{t-u}} - \beta)^{p^u l} \rangle.$$

The only irreducible factor of $(\lambda^{p^{t-u}} - \beta)^{p^u l}$ is $\lambda^{p^{t-u}} - \beta$. So the ideal $\mathcal{P}$ is given by $\langle \zeta^{p^{t-u}} - \beta \rangle$ and is hence equal to $\mathcal{M}$, as asserted.

Case 2. $d(b) \notin d(C)$: We have $\zeta = x^{p^{s+1}} + b^p$ by Lemma 4. We claim $t > 0$: Assume otherwise $t = 0$. That is, $(b - \alpha)^l = 0$. Applying d and noting $d(b) \in C$, we obtain

$$l(b - \alpha)^{l-1}(d(b) - d(\alpha)) = 0.$$

Since $d(b) \notin d(C)$, $d(b) - d(\alpha) \neq 0$. But $l \not\equiv 0$ modulo p by our assumption and $(b^{p^t} - \alpha)^{l-1} \neq 0$ by the minimality of l. This contradiction shows $t > 0$ as claimed. Applying d to $(b^{p^t} - \alpha)^l = 0$ and using $d(b) \in C$ and $t > 0$, we obtain $l(b^{p^t} - \alpha)^{l-1} d(\alpha) = 0$. It follows that $d(\alpha) = 0$. So α is also in the center of $R_{\mathcal{F}}[x; d]$. Using this and noting that $(b^{p^t} - \alpha)^l = 0$, we compute analogously

$$\begin{aligned} x^m = x^{p^{s+t} l} &= (\zeta - b^p)^{p^{t-1} l} = (\zeta^{p^{t-1}} - b^{p^t})^l = \big((\zeta^{p^{t-1}} - \alpha) - (b^{p^t} - \alpha) \big)^l \\ &= \sum_{i=0}^{l} (-1)^i \binom{l}{i} (\zeta^{p^{t-1}} - \alpha)^{l-i} (b^{p^t} - \alpha)^i \\ &= \sum_{i=0}^{l-1} (-1)^i \binom{l}{i} (\zeta^{p^{t-1}} - \alpha)^{l-i} (b^{p^t} - \alpha)^i. \end{aligned}$$

As in Case 1, the canonical polynomial of (x^m) is $\zeta^{p^{t-1}} - \alpha$. We let u be the largest integer such that $0 \le u \le t - 1$ and such that $\alpha^{1/p^u} \in C^{(d)}$. Set $\beta = \alpha^{1/p^u}$. Then $\zeta^{p^{t-1}} - \alpha = (\zeta^{p^{t-1-u}} - \beta)^{p^u}$. Arguing as in Case 1, we have

$$\mathcal{M} = \langle \zeta^{p^{t-1-u}} - \beta \rangle.$$

The minimal polynomial of b^p over $C^{(d)}$ is $(\lambda^{p^{t-1}} - \alpha)^l = (\lambda^{p^{t-1-u}} - \beta)^{p^u l} \in C^{(d)}[\lambda]$. By Theorem 2, $\mathcal{A}$ is equal to

$$\langle (\zeta^{p^{t-1-u}} - \beta)^{p^u l} \rangle.$$

Since the only irreducible factor of $(\lambda^{p^{t-1-u}} - \beta)^{p^u l}$ is $\lambda^{p^{t-1-u}} - \beta$, the ideal $\mathcal{P}$ is given by $\langle \zeta^{p^{t-1-u}} - \beta \rangle$ and is also equal to $\mathcal{M}$, as asserted.

We conclude our paper with the following immediate

Corollary 14. *Let d be a nilpotent derivation of a prime ring R. The d-extension of R is isomorphic to S modulo its prime radical via the ring homomorphism $\varphi\colon R[x;d] \to S$.*

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