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Ranking of Interval-Valued Fuzzy Sets
Based on Signed Distance

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Abstract:

In this article, we use the signed distance to consider the ranking problem of interval-valued fuzzy sets. This ranking can be used in decision-making problem based on interval-valued fuzzy sets, or in the fuzzification or defuzzification of two sides of an equation in the fuzzy sense. We can learn that how far it is between two interval-valued fuzzy sets by the ranking of the signed distance.

Keywords: Ranking, interval-valued fuzzy sets, signed distance.

§1. Introduction

In Yao and Wu [12], we considered the family F_N of the fuzzy numbers in R which are convex and normal, and also defined the signed distance and ranking of the fuzzy numbers on F_N. In section 3 of [12], we compared the ranking using the signed distance in [12] with the ranking of [2, 3, 4, 6, 7, 11]'s. All the ranking's in papers [2, 3, 4, 5, 6, 7, 8, 10, 11,] are dealing with the fuzzy numbers which are different from the ranking interval-valued fuzzy sets in this paper. It ought to use interval-valued fuzzy set to handle the practical situation in order to fit the reality.

In the inventory problem for some material discussed in Yao and Su [13], to find out the exact value r_0 of the total demand during the period of time T is rather difficult, but it would be easier to know the total demand " around r_0 " (which is a fuzzy language). Since the membership grade of r_0 during the time period T may not equal to 1, therefore assume it belongs to the interal $[\lambda, 1]$, $0 \leq \lambda \leq 1$. Thus they could express it in the fuzzy language " around r_0 " by using an interval-valued fuzzy set. As we mentioned in Abstract, an interval-valued fuzzy sets is needed in the discussion of decision-making problem. These, bring us to discuss the problem of how to rank the interval-valued fuzzy sets.

In section 2, we use the method of signed distance similar to the one in [12] to define the ranking of interval-valued fuzzy sets. Then we have some results closed to the properties of signed distance ranking over R.

In section 3, we compare this article with the one by Liou and Wan's [10], and with [6, 7, 12]'s works as well as to the Yager's [11].

§2. Ranking of Interval-valued Fuzzy Sets.

In this paper, we consider the problem for ranking of interval-valued fuzzy sets as follows:

Definition 1. An interval-valued fuzzy set (i-v fuzzy set for short) A on R is given by

$$A \triangleq \{x, [\mu_{A^L}(x), \mu_{A^U}(x)]\}, \quad x \in R \text{ or } A = [A^L, A^U] \tag{1}$$

$$\text{where } 0 \le \mu_{A^L}(x) \le \mu_{A^U}(x) \le 1 \qquad \forall x \in R$$

Let $\overline{\mu}_A(x) = [\mu_{A^L}(x), \mu_{A^U}(x)]$, $x \in R$, then the grade of membership of the i-v fuzzy set A at x belongs to the interval $[\mu_{A^L}(x), \mu_{A^U}(x)]$.

Insert Fig. 1 here

Let us assume the i-v fuzzy set A satisfies the following conditions:

(1°) $\mu_{A^L}(x;\lambda)$, $\mu_{A^U}(x;\rho)$ are continuous functions of x in R. (see Fig.1)

(2°) $max_x \mu_{A^L}(x;\lambda) = \lambda$, $max_x \mu_{A^U}(x;\rho) = \rho$, $0 < \lambda \le \rho \le 1$ and $0 \le \mu_{A^L}(x;\lambda) \le \mu_{A^U}(x;\rho) \le 1$ $\forall x \in R$

(3°) For $0 \le \alpha \le \lambda$, let $A_l^L(\alpha)$, $A_r^L(\alpha)$ be the left and right end points of the α-cut of A^L, respectively. For $0 \le \alpha \le \rho$, let $A_l^U(\alpha)$, $A_r^U(\alpha)$ be the left and right end points of the α-cut of A^U, respectively. Also $A_l^L(\alpha)$, $A_r^L(\alpha)$ are continuous in α for $0 \le \alpha \le \lambda$; and $A_l^U(\alpha)$, $A_r^U(\alpha)$ are continuous in α for $0 \le \alpha \le \rho$.

Let F_{iv} be the family of all the i-v fuzzy sets on R which satisfy the conditions

$(1°) - (3°)$. For $A \in F_{iv}$, Definition 1 is replaced by

$$A \triangleq (\, x, [\mu_{_{A^L}}(x;\lambda),\ \mu_{_{A^U}}(x;\rho)]\,), \quad x \in R$$

or

$$\overline{\mu}_A(x) = [\mu_{_{A^L}}(x;\lambda), \mu_{_{A^U}}(x;\rho)], \quad x \in R \quad \text{or} \quad A = [A^L, A^U]$$

$\{(x, \overline{\mu}_A(x)) | x \in R\}$, and $\overline{\mu}_A(x)$ is an interval such that it is not degenerated to one point except points p and q. This area, shaded in Fig.1, is denoted by $R(A)$.

We will consider the ranking of i-v fuzzy sets in F_{iv} as the following:

Definition 2. For $b \in R$, we define the signed distance from the origin O to b by $d(b) = b$.

Remark 1. If $b > 0$, then $d(b) = b$ means b has distance b to the right of O. If $b < 0$, then $d(b) = b$ means b has distance $-b$ to the left of O. Therefore we say this distance $d(b)$, a signed distance.

For $A \in F_{iv}$ and each $0 \le \alpha < \lambda$, the α-cut of A in $R(A)$ is $[A_l^U(\alpha),\ A_l^L(\alpha)] \cup [A_r^L(\alpha),\ A_r^U(\alpha)]$, see (Fig.1). The signed distance from the origin O to $[A_l^U(\alpha), A_l^L(\alpha)]$, $[A_r^L(\alpha), A_r^U(\alpha)]$ are defined as

$$d([A_l^U(\alpha),\ A_l^L(\alpha)]) = \frac{1}{2}(A_l^U(\alpha) + A_l^L(\alpha)) \quad \text{and}$$

$$d([A_r^L(\alpha),\ A_r^U(\alpha)]) = \frac{1}{2}(A_r^L(\alpha) + A_r^U(\alpha)) \quad \text{respectively .}$$

Since $[A_l^U(\alpha),\ A_l^L(\alpha)] \cap [A_r^L(\alpha),\ A_r^U(\alpha)] = \emptyset$, therefore the signed distance from the origin O to $[A_l^U(\alpha)\ A_l^L(\alpha)] \cup [A_r^L(\alpha),\ A_r^U(\alpha)]$ is defined by

$$d([A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)])$$

$$= \frac{1}{2}(d([A_l^U(\alpha),\, A_l^L(\alpha)]) + d([A_r^L(\alpha),\, A_r^U(\alpha)]))$$
$$= \frac{1}{4}(A_l^U(\alpha) + A_l^L(\alpha) + A_r^L(\alpha) + A_r^U(\alpha)) \tag{2}$$

The d in (2) is a continuous function in α for $0 \leq \alpha \leq \lambda$.

Hence by the definition of the mean of an integral, we have

$$\begin{aligned} &\frac{1}{\lambda - 0}\int_0^\lambda d([A_l^U(\alpha),\, A_l^L(\alpha)] \cup [A_r^L(\alpha),\, A_r^U(\alpha)])d\alpha \\ &= \frac{1}{4\lambda}\int_0^\lambda (A_l^U(\alpha) + A_l^L(\alpha) + A_r^L(\alpha) + A_r^U(\alpha))d\alpha \end{aligned} \tag{3}$$

When $\lambda \leq \alpha \leq \rho$, the α-cut of A in $R(A)$ is $[A_l^U(\alpha),\, A_r^U(\alpha)]$ (see Fig.1). So the distance from O to $[A_l^U(\alpha),\, A_r^U(\alpha)]$ is defined as

$$d([A_l^U(\alpha),\, A_r^U(\alpha)]) = \frac{1}{2}(A_l^U(\alpha) + A_r^U(\alpha)) \tag{4}$$

The d in (4) is a continuous function in α for $\lambda \leq \alpha \leq \rho$. Hence by the definition of the mean of an integral, we have

$$\frac{1}{\rho - \lambda}\int_\lambda^\rho d([A_l^U(\alpha),\, A_r^U(\alpha)])d\alpha = \frac{1}{2(\rho - \lambda)}\int_\lambda^\rho (A_l^U(\alpha) + A_r^U(\alpha))d\alpha \tag{5}$$

From (3) - (5) and the Decomposition theory

$$A = \bigcup_{0 \leq \alpha \leq \lambda} \alpha([A_l^U(\alpha),\, A_r^L(\alpha)]) \cup ([A_r^L(\alpha),\, A_r^U L(\alpha)]) \cup \bigcup_{\lambda \leq \alpha \leq \rho} \alpha([A_l^U(\alpha), A_r^U(\alpha)])$$

(by Fig.1). The signed distance from y - axis to $A \in F_{iv}$, may be defined as follows:

Definition 3 For $A \in F_{iv}$, $\overline{\mu}_A(x) = [\mu_{_{A^L}}(x; \lambda),\, \mu_{_{A^U}}(x; \rho)]$, (1°) If $0 < \lambda < \rho \leq 1$, the signed distance from y - axis to A is

$$d^*(A;\lambda,\rho) = \frac{1}{4\lambda}\int_0^{\lambda}(A_l^U(\alpha) + A_l^L(\alpha) + A_r^L(\alpha) + A_r^U(\alpha)]d\alpha \\ + \frac{1}{2(\rho-\lambda)}\int_{\lambda}^{\rho}(A_l^U(\alpha) + A_r^U(\alpha))d\alpha$$

(2°) If $0 < \lambda = \rho \leq 1$, the signed distance from y - axis to A is

$$d^*(A;\lambda,\lambda) = \frac{1}{4\lambda}\int_0^{\lambda}(A_l^U(\alpha) + A_l^L(\alpha) + A_r^L(\alpha) + A_r^U(\alpha))d\alpha$$

Remark 2.

(1) The d^* defined on F_{iv} is different from the d defined on R.

(2) In Fig.1, A^U is a fuzzy set on R with α-cut ($0 \leq \alpha \leq \rho$) $[A_l^U(\alpha),\ A_r^U(\alpha)]$. But the α-cut of the i-v fuzzy set A $(0 \leq \alpha \leq \lambda(< \rho))$ is $[A_l^U(\alpha),\ A_l^L(\alpha)] \cup [A_r^L(\alpha),\ A_r^U(\alpha)]$ Hence these two α -cuts do not have the same meaning.

We now consider the ranking on F_{iv} as the following:

Definition 4. For $A \triangleq (\, x, [\mu_{A^L}(x;\lambda),\ \mu_{A^U}(x;\rho)]\,)$,

$$B \triangleq (\, x, [\mu_{B^L}(x;\eta),\ \mu_{B^U}(x;\zeta)]\,),\ x \in R;\ 0 < \lambda \leq \rho \leq 1,\ 0 < \eta \leq \zeta \leq 1,$$

we say

$$A \succ B \text{ iff } d^*(A;\lambda,\rho) > d^*(B;\eta,\zeta)$$

$$A \approx B \text{ iff } d^*(A;\lambda,\rho) = d^*(B;\eta,\zeta)$$

Example 1. If two i-v fuzzy sets A, B are defined as the following

$$\mu_{A^L}(x;0.5) = \begin{cases} \frac{0.5(x-20)}{15}, & 20 \leq x \leq 35 \\ \frac{0.5(50-x)}{15}, & 35 \leq x \leq 50 \\ 0, & \text{elsewhere} \end{cases}$$

$$\mu_{A^U}(x;0.9) = \begin{cases} \frac{0.9(x-10)}{20}, & 10 \le x \le 30 \\ \frac{0.9(60-x)}{30}, & 30 \le x \le 60 \\ 0, & \text{elsewhere} \end{cases}$$

i.e. $A \triangleq (\, x, [\mu_{A^L}(x;0.5), \mu_{A^U}(x;0.9)]\,),\ x \in R$

$$\mu_{B^L}(x;0.4) = \begin{cases} \frac{0.4(x-10)}{20}, & 10 \le x \le 30 \\ \frac{0.4(50-x)}{20}, & 30 \le x \le 50 \\ 0, & \text{elsewhere} \end{cases}$$

$$\mu_{B^U}(x;0.8) = \begin{cases} \frac{0.8(x-10)}{25}, & 10 \le x \le 35 \\ \frac{0.8(50-x)}{15}, & 35 \le x \le 50 \\ 0, & \text{elsewhere} \end{cases}$$

i.e. $B \triangleq (\, x, [\mu_{B^L}(x;0.4), \mu_{B^U}(x;0.8)]\,),\ x \in R$ When $0 \le \alpha \le 0.5$, from the α-cut of A^L, A^U, we have

$$A_l^L(\alpha) = 20 + 30\alpha, \quad A_r^L(\alpha) = 50 - 30\alpha$$
$$A_l^U(\alpha) = 10 + \frac{200}{9}\alpha, \quad A_r^U(\alpha) = 60 - \frac{100}{3}\alpha$$

When $0.5 \le \alpha \le 0.9$, from the α-cut of A^U, we have

$$A_l^U(\alpha) = 10 + \frac{200}{9}\alpha, \quad A_r^U(\alpha) = 60 - \frac{100}{3}\alpha$$

When $0 \le \alpha \le 0.4$, from the α-cut of B^L, B^U, we have

$$B_l^L(\alpha) = 10 + 50\alpha, \quad B_r^L(\alpha) = 50 - 50\alpha$$

$$B_l^U(\alpha) = 10 + 31.25\alpha, \quad B_r^U(\alpha) = 50 - 18.75\alpha$$

When $0.4 \le \alpha \le 0.8$, from the α-cut of B^U, we have

$$B_l^U(\alpha) = 10 + 31.25\alpha, \quad B_r^U(\alpha) = 50 - 18.75\alpha$$

By Definition 3, 4, $\lambda = 0.5$, $\rho = 0.9$

$$\begin{aligned} d^*(A; 0.5, 0.9) =& \frac{1}{4 \times 0.5} \int_0^{0.5} [30 + (30 + \frac{200}{9})\alpha + 110 - (30 + \frac{100}{3})\alpha] d\alpha \\ & + \frac{1}{2 \times 0.4} \int_{0.5}^{0.9} [70 + (\frac{200}{9} - \frac{100}{3})\alpha] d\alpha = 65.4167 \\ d^*(B; 0.4, 0.8) =& \frac{1}{4 \times 0.4} \int_0^{0.4} [60 + 60 + 12.5\alpha] d\alpha + \frac{1}{2 \times 0.4} \int_{0.4}^{0.8} [60 + 12.5\alpha] d\alpha \\ =& 64.375 \end{aligned}$$

$$d^*(A; 0.5, 0.9) > d^*(B; 0.4, 0.8)$$

Therefore $A \succ B$.

We have the following property.

Property 1. If

$$A \triangleq (\, x, [\mu_{A^L}(x; \lambda), \mu_{A^U}(x; \rho)]\,),$$
$$B \triangleq (\, x, [\mu_{B^L}(x; \eta), \mu_{B^U}(x; \zeta)]\,),$$
$$\text{and} \quad C \triangleq (\, x, [\mu_{C^L}(x; \delta), \mu_{C^U}(x; \gamma)]\,), x \in R \quad \text{are in } F_{iv}$$

then

(1°) $(F_{iv}, \approx, \succ)$ satisfy the law of trichotomy. i.e. one of $A \succ B$ or $B \succ A$ or $A \approx B$ holds.

(2°) $A \approx A$

(3°) If $A \succapprox B$, $B \succapprox A$, then $A \approx B$

(4°) If $A \succapprox B$, $B \succapprox C$, then $A \succapprox C$

Proof. Follows from Definitions 3, 4.

Definition 5. For $A, B \in F_{iv}$, $A = \{x, [\mu_{A^L}(x; \lambda), \mu_{A^U}(x; \rho)]\}$, $B = \{x, [\mu_{B^L}(x; \lambda), \mu_{B^U}(x; \rho)]\}$, $x \in R$, $0 < \lambda \leq \rho \leq 1$, we define the i-v fuzzy sets $A \oplus^* B$ and kA by $\overline{\mu}_{A\oplus^* B}(x) = [\mu_{A^L \oplus B^L}(x;\lambda),\, \mu_{A^U \oplus B^U}(x;\rho)]$ and $\overline{\mu}_{kA}(x) = [\mu_{kA^L}(x;\lambda),\, \mu_{kA^U}(x;\rho)]$, $x \in R$ respectively, where

$$\mu_{A^L \oplus B^L}(x;\lambda) = \sup_{x=y+z} \mu_{A^L}(y,\lambda) \wedge \mu_{B^L}(z;\lambda)$$

$$\mu_{kA^L}(x;\lambda) = \sup_{x=ky} \mu_{A^L}(y;\lambda)$$

Similarly, for $\mu_{A^U \oplus B^U}$ and μ_{kA^U}.

Property 2. If $A = \{x, [\mu_{A^L}(x;\lambda),\, \mu_{A^U}(x;\rho)]\}$, $B = \{x, [\mu_{B^L}(x;\lambda),\, \mu_{B^U}(x;\rho)]\}$, $x \in R$, in $F_{iv}, k \in R$, then

(1°) $d^*(A \oplus^* B;\lambda,\rho) = d^*(A;\lambda,\rho) + d^*(B;\lambda,\rho)$

(2°) $d^*(kA;\lambda,\rho) = kd^*(A;\lambda,\rho)$

Proof. By Definition 5, for $0 \le \alpha \le \lambda$, the α - cut of $A^L \oplus B^L$ is

$$[A_l^L(\alpha) + B_l^L(\alpha),\, A_r^L(\alpha) + B_r^L(\alpha)]$$

and for $0 \le \alpha \le \rho$, the α - cut of $A^U \oplus B^U$ is

$$[A_l^U(\alpha) + B_l^U(\alpha),\, A_r^U(\alpha) + B_r^U(\alpha)]$$

and by Definition 3, we have (1°).

Similarly, for (2°).

Property 3. For $A_j = \{x,\, [\mu_{A_j^L}(x;\lambda),\, \mu_{A_j^U}(x;\rho)]\}, j = 1,2,3,4$ in F_{iv}.

(1°) If $A_1 \succ A_2$, then $A_1 \oplus A_3 \succ A_2 \oplus A_3$.

(2°) If $A_1 \succ A_2$, and $A_3 \succ A_4$, then $A_1 \oplus A_3 \succ A_2 \oplus A_4$.

Proof.

(1°) Since $A_1 \succ A_2$ by Definition 4, $d^*(A_1;\lambda,\rho) > d^*(A_2;\lambda,\rho)$, therefore $d^*(A_1;\lambda,\rho) + d^*(A_3;\lambda,\rho) \geq d^*(A_2,\lambda,\rho) + d^*(A_3,\lambda,\rho)$. By Property 2, (1°), $d^*(A_1 \oplus A_3;\lambda,\rho) > d^*(A_2 \oplus A_3;\lambda,\rho)$, then follows by Definition 4, we have (1°).

(2°) Use the same way in the proof of (1°), we can prove (2°).

Property 4. For $A_j = \{x, [\mu_{A_j^L}(x;\lambda), \mu_{A_j^U}(x;\rho)], \}, j = 1,2;\ k \in R$. If $A_1 \succ A_2$, then $kA_1 \succ kA_2$ for $k > 0$ and $kA_2 \succ kA_1$ for $k < 0$.

Proof. Follows by Definition 4 and Property 2 (2°).

Property 5. For $A = \{x, [\mu_{A^L}(x;\lambda), \mu_{A^U}(x;\rho)]\}$, $B = \{x, [\mu_{B^L}(x;\eta), \mu_{B^U}(x;\zeta)]\}$ $\in F_{iv}, 0 < a,\ 0 < b$, we have

(1°) If $a < b, d^*(A;\ \lambda,\rho) > 0$, then $bA \succ aA$

(2°) If $a < b, d^*(A;\ \lambda,\rho) < 0$, then $aA \succ bA$

(3°) If $0 < a < b$ and $A \succ B, d^*(B;\ \eta,\zeta) > 0$, then $bA \succ aB$

(4°) If $0 < a < b$ and $A \succ B, d^*(A;\ \lambda,\rho) < 0$, then $aA \succ bB$

Proof.

(1°) By Property 2 (2°), $d^*(bA;\lambda,\rho) = bd^*(A;\lambda,\rho) > ad^*(A;\lambda,\rho) = d^*(aA;\lambda,\rho)$. Then by Definition 4, we have (1°).

(2°) Similar to (1°).

(3°) Since $A \succ B$, by Definition 4, $d^*(A:\lambda,\rho) > d^*(B;\eta,\zeta)$, Hence $bd^*(A;\lambda,\rho) > ad^*(B;\eta,\zeta)$. Therefore by Property 2 and Definition 4, we have (3°).

(4°) Similar to (3°).

Property 6. For $A_j = \{x, [\mu_{A_j^L}(x;\lambda), [\mu_{A_j^U}(x;\rho)]\}; j = 1,2,3,4 \in F_{iv}$, if $A_1 \succ A_2$ and $A_3 \succ A_4$, $a > 0, b > 0$, then $(aA_1) \oplus (bA_3) \succ (aA_2) \oplus (bA_4)$

Proof. It follows from Property 3(2°) and Property 4.

Insert Fig.2 here

In Fig.1, let $\rho = \lambda = 1$, and $\{x \,|\, max\mu_{A^L}(x;1) = 1\} = [b,c]$, $\{x \,|\, max\mu_{A^U}(x;1) = 1\} = [a,e]$. Then we obtained the i-v fuzzy set in Fig.2. Let F_{iv}^* be the family of these fuzzy sets, then for each $0 \le \alpha \le 1$, the α-cut of $A(\in F_{iv}^*)$ in $R^*(A)$ (the shaded area in Fig.2) is $[A_l^U(\alpha),\, A_l^L(\alpha)] \cup [A_r^L(\alpha),\, A_r^U(\alpha)]$ (see Fig.2). Together with Definition 3 (2°), we have

$$d^*(A;1,1) = \frac{1}{4}\int_0^1 [A_l^L(\alpha) + A_r^L(\alpha) + A_l^U(\alpha) + A_r^U(\alpha)]\, d\alpha \tag{6}$$

§3 Discussion.

[A]. Compare with Liou and Wan [10], and [6, 7, 12].

Look at the graph of Fig.1, when $\rho = 1$, and $\{x \,| max\mu_{A^U}(x;1\} = [b,c]$, $\mu_{A^U}(x;1)$ is strictly increasing when $p \le x \le b$; and $\mu_{A^U}(x;1)$ is strictly decreasing when $c \le x \le q$. Fuzzy number A^U is the fuzzy number discussed in [10]. For $A = [A^L, A^U]$, by Definition 3(1°) and L'Hopital's rule , we have

$$\begin{aligned}
&\lim_{\lambda\to 0} \frac{1}{4\lambda}\int_0^\lambda [A_l^U(\alpha) + A_l^L(\alpha) + A_r^L(\alpha) + A_r^U(\alpha)]\, d\alpha \\
&= \lim_{\lambda\to 0} \frac{1}{4}[A_l^U(\lambda) + A_l^L(\lambda) + A_r^L(\lambda) + A_r^U(\lambda)] \\
&= \frac{1}{4}[A_l^U(0) + A_l^L(0) + A_r^L(0) + A_r^U(0)] \equiv I \quad \text{(say)}
\end{aligned}$$

Thus we can define $d^*(A;0,1)$ by the following:

$$\begin{aligned} d^*(A:0,1) &\equiv \lim_{\lambda\to 0} d^*(A;\,\lambda,1) \\ &= I + \frac{1}{2}\int_0^1 [A_l^U(\alpha)+A_r^U(\alpha)]\,d\alpha \end{aligned} \tag{7}$$

The index of optimism $\alpha=\frac{1}{2}$ in the total integral value defined in [10] is

$$I_T^{\frac{1}{2}}(A^U) = \frac{1}{2}\int_0^1 [A_l^U(\alpha)+A_r^U(\alpha)]\,d\alpha \tag{8}$$

From (7) and (8), the fuzzy numbers in [10] has the following relation with the one we discussed in this paper.

$$d^*(A;0,1) = I + I_T^{\frac{1}{2}}(A^U) \tag{9}$$

Insert Fig.3 here

For example, let

$$\mu_{A^L}(x) = \begin{cases} \frac{\lambda(x-p)}{q-p}, & p \le x \le q \\ \frac{\lambda(r-x)}{r-q}, & q \le x \le r \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{A^U}(x) = \begin{cases} \frac{(x-a)}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \frac{(d-x)}{d-c}, & c \le x \le d \\ 0, & \text{otherwise} \end{cases}$$

where $a<p<b<q<c<r<d, 0<\lambda<1$ (see Fig.3). Then we have the i-v fuzzy set $A=[A^L,A^U]$. For each $0\le\alpha\le\lambda$, the left and right points of the α -cut of A^L are $A_l^L(\alpha)=p+(q-p)\frac{\alpha}{\lambda}$, $A_r^L(\alpha)=r-(r-q)\frac{\alpha}{\lambda}$. For each $0\le\alpha\le 1$, the left and right points of the α -cut of A^U are $A_l^U(\alpha) = a + (b-a)\alpha$, $A_l^U(\alpha) = d - (d-c)\alpha$. Therefore

$$I = \frac{1}{4}(p + r + a + d)$$

$$I_T^{\frac{1}{2}}(A^U) = \frac{1}{2}\int_0^1 [a + d + (b-a)\alpha - (d-c)\alpha]\, d\alpha = \frac{1}{4}[a + b + c + d],$$

$$d^*(A; 0.1) = \frac{1}{4}(p + r + b + c + 2a + 2d)$$

Look at the graph of $\mu_{A^U}(x;\rho)$ in Fig.1. When $\rho = \sup_x \mu_{A^U}(x;1) = 1$, the fuzzy number A^U is the fuzzy number discussed in [12]. The ranking fuzzy number discussed in section 3 of [12] has the same result as those discussed in [6], [7], and the measure, $\frac{1}{2}\int_0^1 [A_l^U(\alpha) + A_r^U(\alpha)]\, d\alpha$, for the fuzzy set A^U in [6], [7], are the same as the signed distance in [12]. Therefore the fuzzy number in this paper is linked with the one in [6, 7, 12] by (9).

[B] Compare with Yager [11]

In [11], p161, he defined the measure of the fuzzy number B by

$$F(B) = \int_0^{\alpha max} M(B_\alpha)\, d\alpha \tag{10}$$

where $M(B_\alpha)$ is the mean of the B_α level sets of B. Here $A = [A^L, A^U]$ is the i-v fuzzy set in Fig.1. If we only consider the graph of $\mu_{A^U}(x;\rho)$, we see A^U is the fuzzy number in [11]. As discussed in [A], by Definition 3(1°), we can define $d^*(A;0,\rho)$ by the following:

$$d^*(A;0,\rho) = \lim_{\lambda \to 0} d^*(A;\lambda,\rho) = I + \frac{1}{2\rho}\int_0^\rho [A_l^U(\alpha) + A_r^U(\alpha)]\, d\alpha \tag{11}$$

From (10), (11), then we have

$$d^*(A;0,\rho) = I + \frac{1}{\rho}F(A^U)$$

[C]. The signed distance of the level $(\lambda, 1)$ interval-valued fuzzy number in F_{iv} (see section 2).

In section 2, suppose $A = [A^L, A^U]$ is an interval-valued fuzzy set. Here A^L is not a normal triangular fuzzy number, denoted as $A^L = (a, b, c; \lambda)$ and A^U is a triangular fuzzy number, denoted as $A^U = (p, b, q; 1)$. The membership function of A^L is

$$\mu_{A^L}(x) = \begin{cases} \frac{\lambda(x-a)}{b-a}, & a \le x \le b \\ \frac{\lambda(c-x)}{c-b}, & b \le x \le c \\ 0, & \text{otherwise} \end{cases}$$

Then we have $A = [(a, b, c; \lambda), (p, b, q; 1)]$ is called a level $(\lambda, 1)$ interval-valued fuzzy number. The α-cut $(0 < \alpha < 1)$ of A^L is $A^L_l(\alpha) = a + (b-a)\frac{\alpha}{\lambda}$, $A^L_r(\alpha) = c - (c-b)\frac{\alpha}{\lambda}$. The α-cut of A^U is $A^U_l(\alpha) = p + (b-p)\alpha$, $A^U_r(\alpha) = q - (q-b)\alpha$. From Definition 3 ($1^\circ$), we have

$$d^*(A; \lambda, 1) = \frac{1}{8}[6b + a + c + 4p + 4q + 3\lambda(2b - p - q)] \tag{12}$$

For example, in the inventory problem [13] mentioned in tne Introduction, the total demand during the whole period T is around r_0. They were using the following interval-valued fuzzy set to express it.

$$\mu_{r^L}(x) = \begin{cases} \frac{\lambda(x-r_0+\Delta_3)}{\Delta_3}, & r_0 - \Delta_3 \le x \le r_0 \\ \frac{\lambda(r_0+\Delta_4-x)}{\Delta_4}, & r_0 \le x \le r_0 + \Delta_4 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{r^U}(x) = \begin{cases} \frac{(x-r_0+\Delta_1)}{\Delta_1}, & r_0 - \Delta_1 \le x \le r_0 \\ \frac{(r_0+\Delta_2-x)}{\Delta_2}, & r_0 \le x \le r_0 + \Delta_2 \\ 0, & \text{otherwise} \end{cases}$$

where $0 < \Delta_3 < \Delta_1 < r_0$, $0 < \Delta_4 < \Delta_2$, $0 < \lambda \leq 1$ (see Fig. 4).

Insert Fig. 4 here

Let $r = [r^L, r^U]$ be this interval-valued fuzzy set. From (12), we have

$$\begin{aligned} d^*(r; \lambda, 1) &= 2r_0 + \frac{1}{8}[\Delta_4 - \Delta_3 + (4 - 3\lambda)(\Delta_2 - \Delta_1)] \\ &= \frac{11 + 3\lambda}{8} r_0 + \frac{1}{8}\Delta_4 + \frac{4 - 3\lambda}{8}\Delta_2 + \frac{1}{8}(r_0 - \Delta_3) + \frac{4 - 3\lambda}{8}(r_0 - \Delta_1) \end{aligned}$$

Since $4 - 3\lambda > 0, r_0 - \Delta_1 > 0, r_0 - \Delta_3 > 0$, we have $d^*(r; \lambda, 1) > 0$. Let

$$r^* = \frac{1}{2} d^*(r; \lambda, 1) = r_0 + \frac{1}{16}[\Delta_4 - \Delta_3 + (4 - 3\lambda)(\Delta_2 - \Delta_1)]$$

This is the estimation of the total demand during the period T in the fuzzy sense. When $\Delta_1 = \Delta_2, \Delta_3 = \Delta_4$, then the two triangulars in Fig.4 are isosceles and $r^* = r_0$. That is, the estimation r^* in the fuzzy sense equals to the crisp value r_0. When $\Delta_1 < \Delta_2, \Delta_3 < \Delta_4$, then both the two triangulars in Fig.4 are skew to the right and $r^* > r_0$. When $\Delta_1 > \Delta_2, \Delta_3 > \Delta_4$, then both the two triangulars in Fig.4 are skew to the left and $r^* < r_0$. This is a reasonable result as we can see from Fig.4.

[D]. The signed distance in this paper is an extension of the signed distance on R.

For any real number $x \in R$, we can fuzzify x and get a triangular fuzzy number $\widetilde{x} = (x - \Delta_1, x, x + \Delta_2; \lambda), 0 < \Delta_j, j = 1, 2$ and $\widetilde{x}$ is not normal. Also we can have a level (λ, ρ) interval-valued fuzzy number $\widetilde{x}^* = [(x - \Delta_1, x, x + \Delta_2; \lambda), (x - \Delta_3, x, x + \Delta_4; \rho)], 0 < \Delta_1 < \Delta_3, 0 < \Delta_2 < \Delta_4; 0 < \lambda \leq \rho \leq 1$. All these are extension of R.

If $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0$ and $\rho = \lambda$ then we have $\widetilde{x} = \widetilde{x}^* = x_\lambda$ (level λ fuzzy point. i.e. if $t = x, \mu_{x_\lambda}(t) = \lambda$, and if $t \neq x, \mu_{x_\lambda}(t) = 0$.) which corresponds to $x \in R$. i.e. the level (λ, λ) interval-valued fuzzy numbers $\widetilde{x}^*$ are all extension points of R. Hence we extend this concept of ranking on R to the ranking of the fuzzy numbers and the ranking of the interval-valued fuzzy sets. Meanwhile we also maintain the properties of the order relation on R.

The comparison of the ranking of fuzzy numbers in [2, 3, 4, 6, 7] with the signed distance ranking of fuzzy numbers in [12] is stated in section 3 of [12]. Also we learn the relation among them through the discussions in [A], [B] in this section.

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1.0

ρ

$\overline{\mu}_A(x)$

λ

$\mu_{_{A^U}}(x;\rho)$ $\mu_{_{A^L}}(x;\lambda)$ $\mu_{_{A^U}}(x;\rho)$

α

0 p $A_l^U(\alpha)$ $A_l^L(\alpha)$ x $A_r^L(\alpha)$ $A_r^U(\alpha)$ q

Fig. 1. $i-v$ fuzzy set A. Case 1.

1.0 $\overline{\mu}_A(x)$

$\mu_{_{A^U}}(x)$ $\mu_{_{A^L}}(x)$ $\mu_{_{A^L}}(x)$ $\mu_{_{A^U}}(x)$

α $P^{(1)}$ $P^{(2)}$ $Q^{(1)}$ $Q^{(2)}$

0 a' $A_l^U(\alpha)$ b' $A_l^L(\alpha)$ a b c e x $A_r^L(\alpha)$ c' $A_r^U(\alpha)$ e'

Fig. 2. $i-v$ fuzzy set A. Case 2.

1.0

λ

0 a p b q c r d

Fig. 3. $i-v$ fuzzy set $D=[D^L,\, D^U]$

1.0

λ

0 $r_0 - \Delta_1$ $r_0 - \Delta_3$ r_0 $r_0 + \Delta_4$ $r_0 + \Delta_2$

Fig. 4. $i-v$ fuzzy set $[(r_0-\Delta_3, r_0, r_0+\Delta_4; \lambda),\ (r_0-\Delta_1, r_0, r_0+\Delta_2; 1)]$