行政院國家科學委員會專題研究計畫 成果報告

在模糊拓樸空間內水準(λ, ρ)的區間值模糊集合的收斂問題及其應用

研究成果報告(精簡版)

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Limit Theorem on a Family of Level (λ, ρ) Interval-Valued Fuzzy Sets

with

Fuzzy Topological Space and Its Application

1. **Limit theorem on a family of level (λ, ρ) interval-valued fuzzy sets with fuzzy topological space**.

1.1 A family of level (λ, ρ) interval-valued fuzzy sets.

Def.1. For $a, b \in R, a < b,\ 0 \le \alpha \le 1$, we called the fuzzy set $[a, b; \alpha]$ is a level α closed fuzzy interval if its membership function is

$$\mu_{[a,b;\alpha]}(x) = \begin{cases} \alpha, & a \le x \le b \\ 0, & otherwise \end{cases} \tag{1.1}$$

Def.2. For $a, b \in R, a < b,\ 0 \le \alpha \le 1$ if

$$\mu_{(a,b;\alpha)}(x) = \begin{cases} \alpha, & a < x < b \\ 0, & otherwise \end{cases} \tag{1.2}$$

Then we called the fuzzy set , a level α open fuzzy interval.

And if $$\mu_{[a,b;\alpha)}(x) = \begin{cases} \alpha, & a \le x < b \\ 0, & otherwise \end{cases} \tag{1.3}$$

We called the fuzzy set , a level α right hand open fuzzy interval.

Again, if

$$\mu_{(a,b;\alpha]}(x) = \begin{cases} \alpha, & a < x \le b \\ 0, & otherwise \end{cases} \tag{1.4}$$

Then we called the fuzzy set , a level α left hand open fuzzy interval.

All the fuzzy sets defined above in these forms, $[a, b; \alpha],\ (a, b; \alpha), [a, b; \alpha), (a, b; \alpha],$ are called level α fuzzy interval.

Def.3. We say, a level α triangular fuzzy number, if its membership function is

$$\mu_{\tilde{D}}(x) = \begin{cases} \dfrac{\lambda(x-p)}{q-p}, & p \le x \le q \\ \dfrac{\lambda(r-x)}{r-q}, & q \le x \le r \\ 0, & otherwise \end{cases}$$

And denoted by $\tilde{D} = (p, q, r; \lambda)$. When $\lambda = 1$, we say $\tilde{D}$ is a triangular fuzzy number, and denoted by (p, q, r) or . $(p, q, r; 1)$.

Let $F_N = \{(a, b, c; 1) \mid \forall\, a, b, c, \in R,\ a < b < c\}$ denote the family of all triangular fuzzy numbers.

Def.4. (Gorzalezang [4]) We call $\tilde{E}$, an interval-valued fuzzy set, if the membership grade at $x\,(\in R)$ of the fuzzy set $\tilde{E}$, on R belongs to the interval $[\mu_{\tilde{E}^L}(x), \mu_{\tilde{E}^U}(x)]$, $0 \le \mu_{\tilde{E}^L}(x) \le \mu_{\tilde{E}^U}(x) \le 1$. And denoted by $\mu_{\tilde{E}}(x) = [\mu_{\tilde{E}^L}(x), \mu_{\tilde{E}^U}(x)]$ or $\tilde{E} = [\tilde{E}^L, \tilde{E}^U]$.

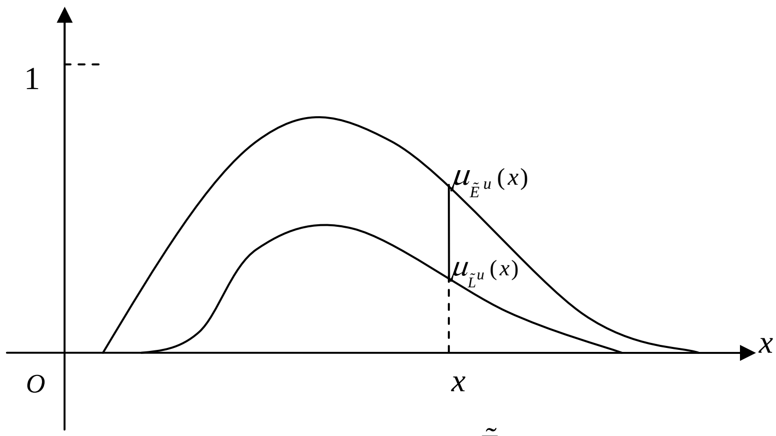

Fig. 1 Interval-valued fuzzy set $\tilde{E}$

When $\tilde{E}^L = (a,b,c;\lambda)$, $\tilde{E}^U = (p,b,q;\rho)$; $0 < \lambda \le \rho \le 1$, and $p < a < b < c < q$. $\tilde{E} = [\tilde{E}^L, \tilde{E}^U]$ is called a level (λ,ρ) interval-valued fuzzy number.

Let $F_{IVFN}(\lambda,\rho) = \{[(a,b,c;\lambda),(p,b,q;\rho)] \,|\, a,b,c,p,q \in R, p < a < b < c < q\}$ be the family of all interval-valued fuzzy number.

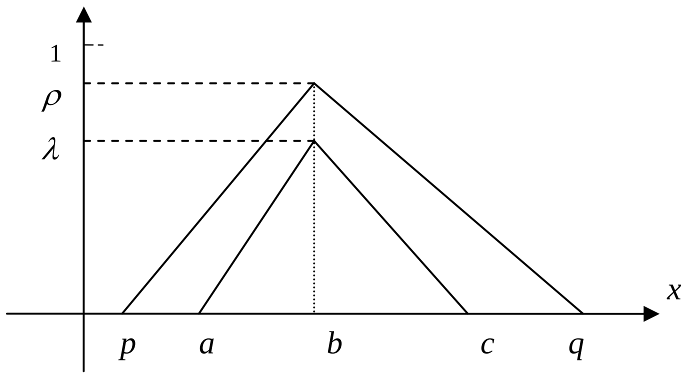

Fig. 2 Level (λ,ρ) interval-valued fuzzy number

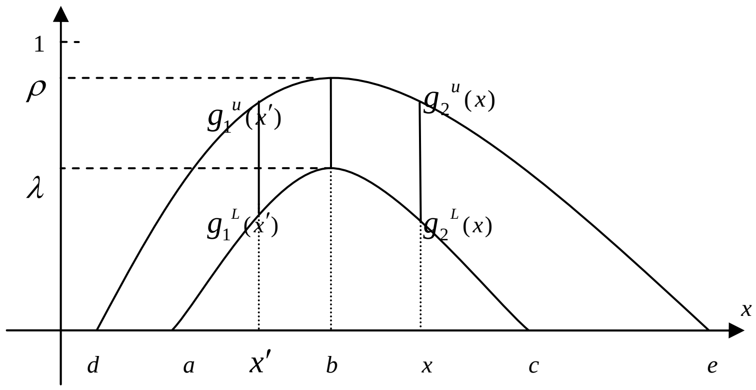

Fig 3 level (λ,ρ) interval-valued fuzzy set

Def.5. For $d < a < b < c < e \quad \in R$ and functions $g_j^L(x), g_j^U(x),\ j = 1,2$. If $g_1^L(x), g_1^U(x)$ are strictly increasing functions for $x \in [a,b], [d,b]$ respectively, and $g_1^{\ L}(a) = 0,\ g_1^{\ U}(d) = 0$; $g_1^{\ L}(b) = \lambda,\ g_1^{\ U}(b) = \rho$. Also $g_2^{\ L}\ (x), g_2^{\ U}(x)$ are strictly decreasing functions for $x \in [b,c], [b,e]$ respectively, and $g_2^L(c) = g_2^U(e) = 0$, $g_2^L(b) = \lambda,\ g_2^U(b) = \rho$. Besides, for each $j = 1,2,\ 0 \le g_j^L(x) \le g_j^U(x) \quad \forall\, x\,(d,e)$.

And $g_1^L(x) = 0, \quad x \le a; \quad g_2^L(x) = 0, \quad x \ge c. \qquad g_1^U(x) = 0, \quad x \le d; \quad g_2^U(x) = 0, \quad x \ge e$

Then we call $\tilde{G} = [\tilde{G}^L, \tilde{G}^U]$, a level (λ, ρ) interval-valued fuzzy set; where the fuzzy sets $\tilde{G}^L, \tilde{G}^U$ each has respectively the following membership function:

$$\mu_{\tilde{G}}(x) = \begin{cases} g_1^L(x), & a \le x \le b \\ g_2^L(x), & b \le x \le c \\ 0, & otherwise \end{cases} \tag{1}$$

$$\mu_{\tilde{G}}(x) = \begin{cases} g_1^U(x), & d \le x \le b \\ g_2^U(x), & b \le x \le e \\ 0, & otherwise \end{cases} \tag{2}$$

And hence the level (λ, ρ) interval-valued fuzzy set is $\tilde{G} = [\tilde{G}^L, \tilde{G}^U]$.

Let $F_{IV}(\lambda, \rho)$ be the family of all such level (λ, ρ), $0 < \lambda \le \rho \le 1$ interval-valued fuzzy sets $\tilde{G} = [\tilde{G}^L, \tilde{G}^U]$ (3)

When $g_1^L(x) = \dfrac{\lambda(x-a)}{b-a}, \qquad g_2^L(x) = \dfrac{\lambda(c-x)}{c-b}; \qquad g_1^U(x) = \dfrac{\rho(x-d)}{b-d}, \qquad g_2^U(x) = \dfrac{\rho(e-x)}{e-b}$

then $\tilde{G} = [\tilde{G}^L, \tilde{G}^U] \in F_{IVFN}(\lambda, \rho) \subset F_{IV}(\lambda, \rho)$.

From Kaufmann and Gupta [5], we can have the following operation rules:

For $a,b,c,d,e \in R$,

$$[a,b] + [c,d] = [a+c, b+d] \tag{4}$$

$$e[a,b] = \begin{cases} [ea, eb], & if\ e > 0 \\ [eb, ea], & if\ e < 0 \end{cases} \tag{5}$$

If $0 \le a < b$ and $0 \le c < d$, then $[[a,b] \times [c,d] = [ac, bd]$ (6)

If $a < b \le 0$ and $c < d \le 0$, then $[[a,b] \times [c,d] = [ad, bc]$ (7)

In general,

$$[[a,b]\times[c,d]=[MIN,MAX], \text{ where } MIN=\min(ac,ad,)\ MAX=\max(bc,bd) \quad (8)$$

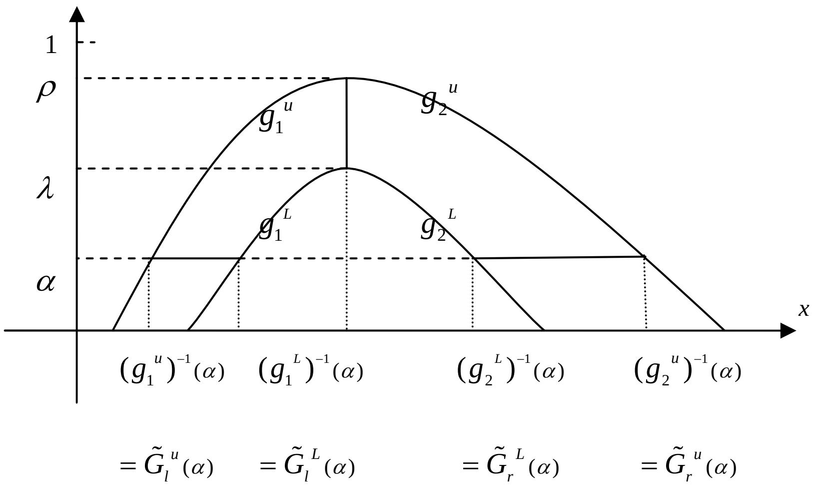

Fig.4 α -cut of level (λ,ρ) interval-valued fuzzy set

The α -cut of $\tilde{G}=[\tilde{G}^L,\tilde{G}^U]\in F_{IV}(\lambda,\rho)$ is defined as follows:

From Fig. 4, $g_1^U(x)=\alpha\to x=(g_1^U)^{-1}(\alpha)$, $g_1^L(x)=\alpha\to x=(g_1^L)^{-1}(\alpha)$

$g_2^L(x)=\alpha\to x=(g_2^L)^{-1}(\alpha)$, $g_2^U(x)=\alpha\to x=(g_2^U)^{-1}(\alpha)$

Let $\tilde{G}_l^U(\alpha)=(g_1^U)^{-1}(\alpha)$, $\tilde{G}_l^L(\alpha)=(g_1^L)^{-1}(\alpha)$; $\tilde{G}_r^L(\alpha)=(g_2^L)^{-1}(\alpha)$, $\tilde{G}_r^U(\alpha)=(g_2^U)^{-1}(\alpha)$ (9)

When $0\le\alpha<\lambda$, define the α - level set of $\tilde{G}$ be

$$\{x\mid\mu_{\tilde{G}^U}(x)\ge\alpha\}-\{x\mid\mu_{\tilde{G}^L}(x)\ge\alpha\} = [(g_1^U)^{-1}(\alpha),(g_1^L)^{-1}(\alpha)]\cup[(g_2^L)^{-1}(\alpha),(g_2^U)^{-1}(\alpha)]$$

$$= [\tilde{G}_l^U(\alpha),\tilde{G}_l^L(\alpha)]\cup[\tilde{G}_r^L(\alpha),\tilde{G}_r^U(\alpha)] \quad \text{(see Fig.4)} \quad (10)$$

When $\lambda\le\alpha\le\rho$, the α - level set of $\tilde{G}$ is

$$\{x\mid\mu_{\tilde{G}^U}(x)\ge\alpha\}=[(g_1^U)^{-1}(\alpha),(g_2^U)^{-1}(\alpha)]=[\tilde{G}_l^U(\alpha),\tilde{G}_r^U(\alpha)] \quad \text{(see Fig.4)} \quad (11)$$

For each $\alpha\in[0,1]$, the function maps the real interval $[a,b]$ to the level α fuzzy interval $[a,b;\alpha]$ is one-to-one onto. (12)

By Def.1, (9) ~ (12), Fig.4 and the Decomposition theorem, we may express the level (λ,ρ) interval-valued fuzzy set $\tilde{G}=[\tilde{G}^L,\tilde{G}^U]$ as

$$\tilde{G} = \bigcup_{0\le\alpha<\lambda}([\tilde{G}_l^U(\alpha),\tilde{G}_l^L(\alpha);\alpha]\cup[\tilde{G}_r^L(\alpha),\tilde{G}_r^U(\alpha);\alpha])\cup(\bigcup_{\lambda\le\alpha\le\rho}[\tilde{G}_l^U(\alpha),\tilde{G}_r^U(\alpha);\alpha]) \quad (13)$$

Def.6. For $\tilde{F}=[\tilde{F}^L,\tilde{F}^U],\tilde{G}=[\tilde{G}^L,\tilde{G}^U]\in F_{IV}(\lambda,\rho), 0<\lambda\le\rho\le 1$ and $e\in R$, define the following operations:

$$\tilde{F}(+)\tilde{G}=[\tilde{F}^L(+)\tilde{G}^L,\ \tilde{F}^U(+)\tilde{G}^U]$$
$$e\tilde{F}=[e\tilde{F}^L,e\tilde{F}^U]$$
$$\tilde{F}(\times)\tilde{G}=[\tilde{F}^L(\times)\tilde{G}^L,\ \tilde{F}^U(\times)\tilde{G}^U]$$

From (4) ~ (12), and Def. 6, we have

(1°) The α-cut $(0\le\alpha\le 1)$ of $\tilde{F}^L(+)\tilde{G}^L$ is

$$[\tilde{F}_l^L(\alpha),\tilde{F}_r^L(\alpha)]+[\tilde{G}_l^L(\alpha),\tilde{G}_r^L(\alpha)]=[\tilde{F}_l^L(\alpha)+\tilde{G}_l^L(\alpha),\ \tilde{F}_r^L(\alpha)+\tilde{G}_r^L(\alpha)] \quad \text{by (4)}$$

Thus $(\tilde{F}^L(+)\tilde{G}^L)_l(\alpha)=\tilde{F}_l^L(\alpha)+\tilde{G}_l^L(\alpha)$ and $(\tilde{F}^L(+)\tilde{G}^L)_r(\alpha)=\tilde{F}_r^L(\alpha)+\tilde{G}_r^L(\alpha)$; $0\le\alpha\le\lambda$

Similarly,

$$(\tilde{F}^U(+)\tilde{G}^U)_l(\alpha)=\tilde{F}_l^U(\alpha)+\tilde{G}_l^U(\alpha) \text{ and } (\tilde{F}^U(+)\tilde{G}^U)_r(\alpha)=\tilde{F}_r^U(\alpha)+\tilde{G}_r^U(\alpha);\ 0\le\alpha\le\rho \quad (14)$$

(2°) The α-cut $(0\le\alpha\le 1)$ of $e\tilde{F}^P$ is $[e\tilde{F}_l^P(\alpha), e\tilde{F}_r^P(\alpha)]$, if $e>0$; and is $[e\tilde{F}_r^P(\alpha), e\tilde{F}_l^P(\alpha)]$, if $e<0$; by (5), where $P=L,U$

Then, if $e>0$, $(e\tilde{F}^L)_l(\alpha)=e\tilde{F}_l^L(\alpha)$, $(e\tilde{F}^L)_r(\alpha)=e\tilde{F}_r^L(\alpha)$; $0\le\alpha\le\lambda$

$$(e\tilde{F}^U)_l(\alpha)=e\tilde{F}_l^U(\alpha),\ (e\tilde{F}^U)_r(\alpha)=e\tilde{F}_r^U(\alpha);\quad 0\le\alpha\le\rho$$

And if $e<0$, $(e\tilde{F}^L)_l(\alpha)=e\tilde{F}_r^L(\alpha)$, $(e\tilde{F}^L)_r(\alpha)=e\tilde{F}_l^L(\alpha)$; $0\le\alpha\le\lambda$

$$(e\tilde{F}^U)_l(\alpha)=e\tilde{F}_r^U(\alpha),\ (e\tilde{F}^U)_r(\alpha)=e\tilde{F}_l^U(\alpha);\quad 0\le\alpha\le\rho \quad (15)$$

(3°) The α-cut $(0\le\alpha\le 1)$ $\tilde{F}^L(\times)\tilde{G}^L$ is $[\tilde{F}_l^L(\alpha),\tilde{F}_r^L(\alpha)]\times[\tilde{G}_l^L(\alpha),\tilde{G}_r^L(\alpha)]$.

From (6), (7), we have the following

(3°.1) When $0\le\tilde{F}_l^L(\alpha)<\tilde{F}_r^L(\alpha)$, and $0\le\tilde{G}_l^L(\alpha)<\tilde{G}_r^L(\alpha)$ $\forall\alpha\in[0,\lambda]$

$$(\tilde{F}^L(\times)\tilde{G}^L)_l(\alpha)=\tilde{F}_l^L(\alpha)\tilde{G}_l^L(\alpha),\ (\tilde{F}^L(\times)\tilde{G}^L)_r(\alpha)=\tilde{F}_r^L(\alpha)\tilde{G}_r^L(\alpha);\ 0\le\alpha\le\lambda$$

Similarly, when $0\le\tilde{F}_l^U(\alpha)<\tilde{F}_r^U(\alpha)$, and $0\le\tilde{G}_l^U(\alpha)<\tilde{G}_r^U(\alpha)$ $\forall\alpha\in[0,\rho]$

$$(\tilde{F}^U(\times)\tilde{G}^U)_l(\alpha)=\tilde{F}_l^U(\alpha)\tilde{G}_l^U(\alpha),\ (\tilde{F}^U(\times)\tilde{G}^U)_r(\alpha)=\tilde{F}_r^U(\alpha)\tilde{G}_r^U(\alpha);0\le\alpha\le\rho \quad (16)$$

(3°.2) When $\tilde{F}_l^L(\alpha) < \tilde{F}_r^L(\alpha) \le 0$, and $\tilde{G}_l^L(\alpha) < \tilde{G}_r^L(\alpha) \le 0\ \forall \alpha \in [0,\lambda]$

$$(\tilde{F}^L(\times)\tilde{G}^L)_l(\alpha) = \tilde{F}_r^L(\alpha)\tilde{G}_r^L(\alpha), \quad (\tilde{F}^L(\times)\tilde{G}^L)_r(\alpha) = \tilde{F}_l^L(\alpha)\tilde{G}_l^L(\alpha); \quad 0 \le \alpha \le \lambda$$

When $\tilde{F}_l^U(\alpha) < \tilde{F}_r^U(\alpha) \le 0$ and $G_l^U(\alpha) < \tilde{G}_r^U(\alpha) \le 0\ \forall \alpha \in [0,\rho]$

$$(\tilde{F}^U(\times)\tilde{G}^U)_l(\alpha) = \tilde{F}_r^U(\alpha)\tilde{G}_r^U(\alpha), \quad (\tilde{F}^U(\times)\tilde{G}^U)_r(\alpha) = \tilde{F}_l^U(\alpha)\tilde{G}_l^U(\alpha); \quad 0 \le \alpha \le \rho \qquad (17)$$

1.2 Fuzzy topological space for level (λ,ρ) interval-valued fuzzy sets

Def.7 (Chang [3] Def. 2.2) A fuzzy topology T on fuzzy sets in X satisfying the following conditions:

(a) $\Phi, X \in T$

(b) If $\tilde{A},\tilde{B} \in T$, then $\tilde{A} \cap \tilde{B} \in T$

(©) If $\tilde{A}_i \in T, \forall\, i \in I$, where I is any index set, then $\bigcup_{i\in I} \tilde{A}_i \in T$

T is called a fuzzy topology for X and the pair (X,T) is a fuzzy topological space (FTS)

Def. 8. (Chang [3], Def. 2.3) A fuzzy set $\tilde{U}$ in a $FTS(X,T)$ is a neighborhood of a fuzzy set $\tilde{A}$ iff there exists a fuzzy set $\tilde{O} \in T$ such that $\tilde{A} \subset \tilde{O} \subset \tilde{U}$.

Def. 9. (Chang [3], Def.3.1) A sequence of fuzzy sets, say $\{\tilde{A}_n, n = 1,2,\cdots\}$ is eventually contained in a fuzzy set $\tilde{A}$ iff there exists a natural number m such that whenever $n \ge m$, $\tilde{A}_n \subset \tilde{A}$. If $\{\tilde{A}_n, n = 1,2,\cdots\}$ is a sequence in $FTS(X,T)$, then we say that this sequence converges to a fuzzy set $\tilde{A}$ iff it is eventually contained in each neighborhood of $\tilde{A}$. (i.e. if $\tilde{B}$ is any neighborhood of $\tilde{A}$, there is a positive integer m such that whenever $n \ge m$, $\tilde{A}_n \subset \tilde{B}$).

Let $O_F = \{(a,b;\alpha) \mid \forall a,b \in R,\ a < b$ and $\alpha \in [0,1]\}$, then O_F is the family of all level α $(\alpha \in [0,1])$ open fuzzy intervals (fuzzy sets in R, see Def.2).

Let T_F be the family of all sets which is any union of elements in O_F and contain Φ, R such that for any element $\tilde{O} \in O_F$. Define $\tilde{O} \cap \Phi = \Phi, \tilde{O} \cup \Phi = \tilde{O}, \tilde{O} \cap R = \tilde{O}, \tilde{O} \cup R = R$ and $\mu_\Phi(x) = 0,\ \mu_R(x) = 1\ \forall x \in R$. Thus, we have

$$\mu_{(a,b;\alpha)\cap(c,d;\beta)}(x) = \mu_{(a,b;\alpha)}(x) \wedge \mu_{(c,d;\beta)}(x)$$

Case 1: $c < a < b < d$; $\alpha \le \beta$

Case 2: $a < c < d < b$; $\alpha \le \beta$

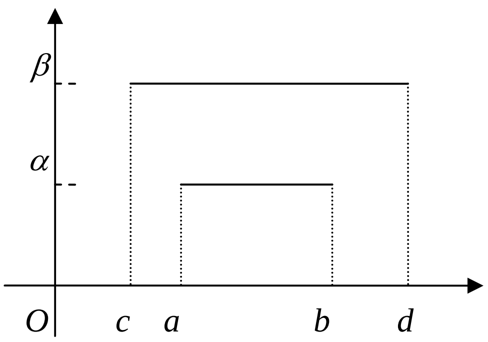

Fig. 5 $c < a < b < d$ (case 1)

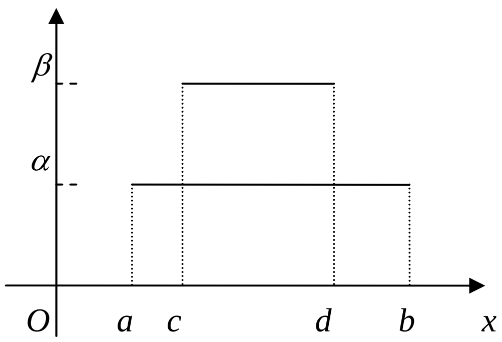

Fig. 6 $a < c < d < b$ (case 2)

Case 3: $c < a < d < b$; $\alpha \le \beta$

Case 4: $a < c < d < b$; $\alpha \le \beta$

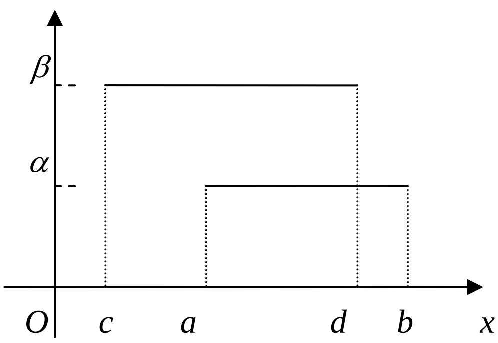

Fig. 7 $c < a < d < b$ (case 3)

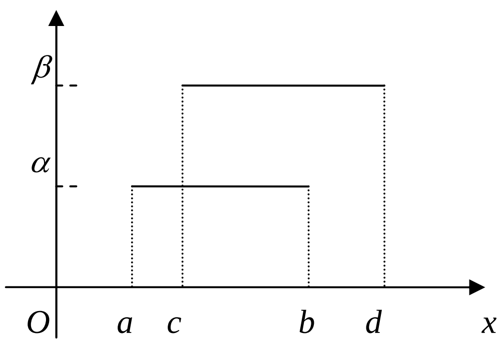

Fig. 8 $a < c < b < d$ (case 4)

Case 5: $c < d \le a < b$ or $a < b \le c < d$

We conclude the above results and have the following Table 1.

Table 1

case	1	2	3	4	5
$(a,b;\alpha)\cap(c,d;\beta)$	$(a,b;\alpha)$	$(c,d;\alpha)$	$(a,d;\alpha)$	$(c,b;\alpha)$	ϕ

Proposition 1. Let $T = T_F, X = R$ in Def. 7. Then T_F is a fuzzy topology for R, and (R, T_F) is a FTS.

Proof. (a) By the definition of T_F, $\Phi, R \in T_F$. Definition 7(a) is satisfied.

(b) For any element $\tilde{B}_i$ in T_F, it can be written as $\tilde{B}_i = \bigcup_{i \in I} \tilde{O}_i$, $I = \{i\}$. Hence the element of T_F has the form $\Phi, R,$ or $\tilde{B}_i = \bigcup_{i \in I} \tilde{O}_i$ with $\tilde{O}_i \in O_F$ $\forall i \in I$, an index set. Therefore, the intersection of any two elements in T_F, they are $\Phi \cap R = \Phi \in T_F$ or $\Phi \cap (\bigcup_{i \in I} \tilde{O}_i) = \Phi \in T_F$ or $R \cap (\bigcup_{i \in I} \tilde{O}_i) = \bigcup_{i \in I} \tilde{O}_i \in T_F$, or for $Q_j \in O_F$ $\forall j \in J$, $(\bigcup_{i \in I} \tilde{O}_i) \cap (\bigcup_{j \in J} \tilde{Q}_j) = \bigcup_{i \in I} \bigcup_{j \in J} (\tilde{O}_i \cap \tilde{Q}_j)$, where $\tilde{O}_i \cap \tilde{Q}_j \in O_F$ or equals to Φ. (see Table 1) i.e. $(\bigcup_{i \in I} \tilde{O}_i) \cap (\bigcup_{i \in j} \tilde{Q}_j) \in T_F$. 7(b) is satisfied.

(c) As stated in (b), any element of T_F, is $\Phi, R,$ or $\tilde{B}_i = \bigcup_{j \in J_i} \tilde{O}_{ij}$, $i \in I$, $\tilde{O}_{ij} \in O_F$ $\forall i \in I, j \in J$, Then $\bigcup_{i \in I} \tilde{B}_i = \bigcup_{i \in I} \bigcup_{j \in J_i} \tilde{O}_{ij} \in T_F$, $\Phi \cup (\bigcup_{i \in I} \tilde{B}_i) = \bigcup_{i \in I} \tilde{B}_i \in T_F$, $R \cup (\bigcup_{i \in I} \tilde{B}_i) = R \in T_F$. 7(c) is satisfied.

By Def. 7, T_F is a topology on R, and (R, T_F) is a FTS.

1.3 Limit theorem on $F_{IV}(\lambda, \rho)$

Def.10. A level α fuzzy point on R at b, denoted by $Fp\alpha$ of b if its membership function is

$$\mu_{Fp\alpha\ of\ b}(x) = \begin{cases} \alpha, & x = b \\ 0, & otherwise \end{cases}$$

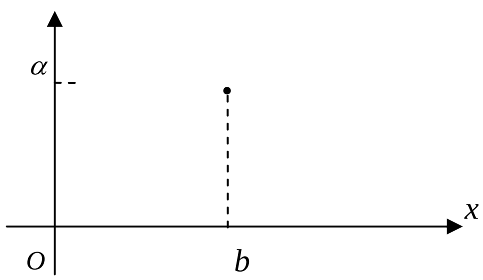

Fig. 9 Level α fuzzy point $Fp\alpha$ of b

From (13)

$$\tilde{A}_n = [\tilde{A}_n^L, \quad \tilde{A}_n^U]$$

$$= \bigcup_{0\le\alpha<\lambda} \{[(\tilde{A}_n^U)_l(\alpha), \quad (\tilde{A}_n^L)_l(\alpha);\ \alpha] \cup [(\tilde{A}_n^L)_r(\alpha), \quad (\tilde{A}_n^U)_r(\alpha);\ \alpha])$$

$$\cup \bigcup_{\lambda\le\alpha\le\rho} [(\tilde{A}_n^U)_l(\alpha), \quad (\tilde{A}_n^U)_r(\alpha);\ \alpha]) \in F_{IV}(\lambda, \rho), \qquad n = 1, 2, \cdots \quad 0 < \lambda \le \rho \le 1 \quad (18)$$

Similarly, as in Fig.(4), we can have the α-cut of $\tilde{A}_n$ as follows:

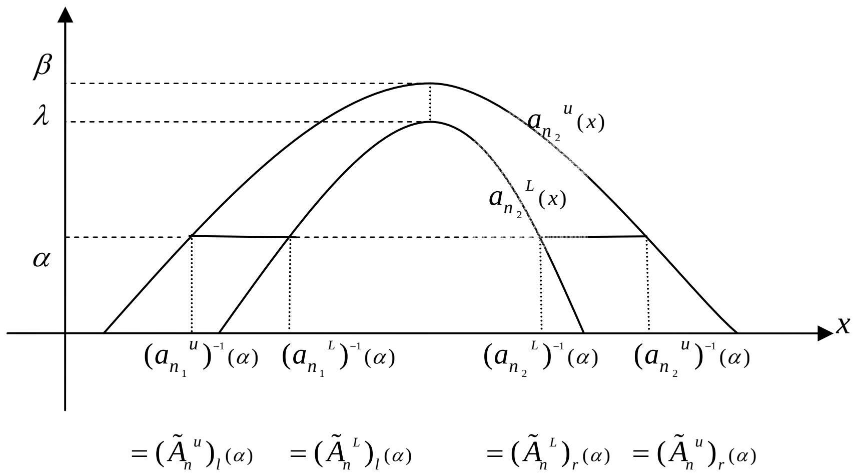

Fig. 10 α-cut of $\tilde{A}_n = [\tilde{A}_n{}^L, \tilde{A}_n{}^u]$

From Fig.10, when $0 \le \alpha < \lambda$, we have the end points of the α-cut are $(\tilde{A}_n^U)_l(\alpha)$, $(\tilde{A}_n^L)_l(\alpha)$, $(\tilde{A}_n^L)_r(\alpha)$, $(\tilde{A}_n^U)_r(\alpha)$ $\in R$. So we have the level α fuzzy points $Fp\alpha$ of $(\tilde{A}_n^U)_l(\alpha)$, and $Fp\alpha$ of $(\tilde{A}_n^L)_l(\alpha)$, $Fp\alpha$ of $(\tilde{A}_n^L)_r(\alpha)$, $Fp\alpha$ of $(\tilde{A}_n^U)_r(\alpha)$. And when $\lambda \le \alpha \le \rho$, we have the level α fuzzy points $Fp\alpha$ of $(\tilde{A}_n^U)_l(\alpha)$, $Fp\alpha$ of $(\tilde{A}_n^U)_r(\alpha)$.

Def.11 For $\tilde{A}_n = [\tilde{A}_n^L, \quad \tilde{A}_n^U]$ (in (18)),

$$\tilde{A} = [\tilde{A}_n^L, \quad \tilde{A}_n^U]$$

$$= \bigcup_{0\le\alpha<\lambda} [\tilde{A}_l^U(\alpha), \quad \tilde{A}_l^L(\alpha);\alpha] \cup [\tilde{A}_r^L(\alpha), \quad \tilde{A}_r^U(\alpha);\alpha] \bigcup_{\lambda\le\alpha\le\rho} [\tilde{A}_l^U(\alpha), \quad \tilde{A}_r^U(\alpha);\alpha] \in F_{IV}(\lambda, \rho)$$

For any arbitrary $\varepsilon > 0$, there exists a positive integer N which is independent of $\alpha(\in [0, \rho])$ such that whenever $n \ge N$, the following hold:

For each $\alpha(\in[0,\lambda])$, $Fp\alpha$ of $(\tilde{A}_n^U)_l(\alpha)\subset(\tilde{A}_l^U(\alpha)-\varepsilon,\tilde{A}_l^U(\alpha)+\varepsilon;\alpha)$ $(\in T_F)$,

$Fp\alpha$ of $(\tilde{A}_n^L)_l(\alpha)\subset(\tilde{A}_l^L(\alpha)-\varepsilon,\tilde{A}_l^L(\alpha)+\varepsilon;\alpha)$ $(\in T_F)$,

$Fp\alpha$ of $(\tilde{A}_n^L)_r(\alpha)\subset(\tilde{A}_r^L(\alpha)-\varepsilon,\tilde{A}_r^L(\alpha)+\varepsilon;\alpha)$ $(\in T_F)$,

$Fp\alpha$ of $(\tilde{A}_n^U)_r(\alpha)\subset(\tilde{A}_r^U(\alpha)-\varepsilon,\tilde{A}_r^U(\alpha)+\varepsilon;\alpha)$ $(\in T_F)$.

And for each $\alpha(\in[\lambda,\rho])$, $Fp\alpha$ of $(\tilde{A}_n^U)_l(\alpha)\subset(\tilde{A}_l^U(\alpha)-\varepsilon,\tilde{A}_l^U(\alpha)+\varepsilon;\alpha)$ $(\in T_F)$,

$Fp\alpha$ of $(\tilde{A}_n^U)_r(\alpha)\subset(\tilde{A}_r^U(\alpha)-\varepsilon,\tilde{A}_r^U(\alpha)+\varepsilon;\alpha)$ $(\in T_F)$.

Then define $\lim_{n\to\infty}\tilde{A}_n=\tilde{A}$.

NB.1 The meaning of Definition 11 is:

For each $\alpha\in[0,1]$, the following hold.

$Fp\alpha$ of $b\leftrightarrow$ $b(\in R)$ is an one-to-one onto mapping (19). Corresponds to the conditions of Definition 11 on R, we can have

For each $\alpha\in[0,\lambda)$, $(\tilde{A}_n^U)_l(\alpha)\in(\tilde{A}_l^U(\alpha)-\varepsilon,\ \tilde{A}_l^U(\alpha)+\varepsilon)$,

$$(\tilde{A}_n^L)_l(\alpha)\in(\tilde{A}_l^L(\alpha)-\varepsilon,\ \tilde{A}_l^L(\alpha)+\varepsilon),$$

$$(\tilde{A}_n^L)_r(\alpha)\in(\tilde{A}_r^L(\alpha)-\varepsilon,\ \tilde{A}_r^L(\alpha)+\varepsilon),$$

$$(\tilde{A}_n^U)_r(\alpha)\in(\tilde{A}_r^U(\alpha)-\varepsilon,\ \tilde{A}_r^U(\alpha)+\varepsilon). \quad (20)$$

For each $\alpha\in[\lambda,\rho]$, $(\tilde{A}_n^U)_l(\alpha)\in(\tilde{A}_l^U(\alpha)-\varepsilon,\ \tilde{A}_l^U(\alpha)+\varepsilon)$,

$$(\tilde{A}_n^U)_r(\alpha)\in(\tilde{A}_r^U(\alpha)-\varepsilon,\ \tilde{A}_r^U(\alpha)+\varepsilon). \quad (21)$$

From Definition 11, (20), and (21) we have. For each $\varepsilon>0$, there exists a positive integer N, independent of $\alpha,(\in[0,\rho])$ such that whenever $n\ge N$, the following hold:

For each $\alpha\in[0,\lambda)$, $[(\tilde{A}_n^U)_l(\alpha),(\tilde{A}_n^L)_l(\alpha)]\subset(\tilde{A}_l^U(\alpha)-\varepsilon,\ \tilde{A}_l^L(\alpha)+\varepsilon)$,

$$[(\tilde{A}_n^L)_r(\alpha),(\tilde{A}_n^U)_r(\alpha)]\subset(\tilde{A}_r^L(\alpha)-\varepsilon,\ \tilde{A}_r^U(\alpha)+\varepsilon).$$

And $\lim_{n\to\infty}(\tilde{A}_n^U)_l=\tilde{A}_l^U(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_n^L)_l=\tilde{A}_l^L(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_n^L)_r=\tilde{A}_r^L(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_n^U)_r=\tilde{A}_r^U(\alpha)$.

For each $\alpha \in [\lambda, \rho]$, $[(\tilde{A}_n^U)_l(\alpha), (\tilde{A}_n^U)_r(\alpha)] \subset (\tilde{A}_l^U(\alpha) - \varepsilon, \tilde{A}_r^U(\alpha) + \varepsilon)$.

And $\lim_{n\to\infty}(\tilde{A}_n^U)_l = \tilde{A}_l^U(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_n^U)_r = \tilde{A}_r^U(\alpha)$.

Thus, when $n \to \infty$,

if $\alpha \in [0, \lambda)$, $[(\tilde{A}_n^U)_l(\alpha), (\tilde{A}_n^L)_l(\alpha)] \to (\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)]$,

and $[(\tilde{A}_n^L)_r(\alpha), (\tilde{A}_n^U)_r(\alpha)] \to [\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)]$.

if $\alpha \in [\lambda, \rho]$, $[(\tilde{A}_n^U)_l(\alpha), (\tilde{A}_n^U)_r(\alpha)] \to (\tilde{A}_l^U(\alpha), \tilde{A}_r^U(\alpha)]$.

Hence by (19),

if $\alpha \in [0, \lambda)$, $[(\tilde{A}_n^U)_l(\alpha), (\tilde{A}_n^L)_l(\alpha); \alpha] \to (\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha); \alpha]$,

and $[(\tilde{A}_n^L)_r(\alpha), (\tilde{A}_n^U)_r(\alpha); \alpha] \to [\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha); \alpha]$.

if $\alpha \in [\lambda, \rho]$, $[(\tilde{A}_n^U)_l(\alpha), (\tilde{A}_n^U)_r(\alpha); \alpha] \to (\tilde{A}_l^U(\alpha), \tilde{A}_r^U(\alpha); \alpha]$.

1.3 Limit theorem on a family of interval valued fuzzy sets

Thm 1. $\tilde{A}_n = [\tilde{A}_n^L, \tilde{A}_n^U]$, $\tilde{A} = [\tilde{A}^L, \tilde{A}^U] \in F_{IV}(\lambda, \rho)$, $(0 < \lambda \le \rho \le 1)$. If $\lim_{n\to\infty}(\tilde{A}_n^L)_l(\alpha) = \tilde{A}_l^L(\alpha)$,

and $\lim_{n\to\infty}(\tilde{A}_n^L)_r(\alpha) = \tilde{A}_r^L(\alpha)$ are uniformly convergent for $0 \le \alpha < \lambda$. And if $\lim_{n\to\infty}(\tilde{A}_n^U)_l(\alpha) = \tilde{A}_l^U(\alpha)$,

and $\lim_{n\to\infty}(\tilde{A}_n^U)_r(\alpha) = \tilde{A}_r^U(\alpha)$ are uniformly convergent for $0 \le \alpha \le \rho$,

Then $\lim_{n\to\infty} \tilde{A}_n = \tilde{A}$.

Proof. By the definition of uniformly convergent, we have, for any $\varepsilon > 0$, there exists a natural number N, which is independent of $\alpha (\in [0, \rho])$ such that for all $n \ge N$, the following hold.

For each $\alpha \in [0, \lambda]$, $\tilde{A}_l^L(\alpha) - \varepsilon < (\tilde{A}_n^L)_l(\alpha) < \tilde{A}_l^L(\alpha) + \varepsilon$,

and $\tilde{A}_r^L(\alpha) - \varepsilon < (\tilde{A}_n^L)_r(\alpha) < \tilde{A}_r^L(\alpha) + \varepsilon$.

For each $\alpha \in [0, \rho]$, $\tilde{A}_l^U(\alpha) - \varepsilon < (\tilde{A}_n^U)_l(\alpha) < \tilde{A}_l^U(\alpha) + \varepsilon$.

and $$\tilde{A}_r^U(\alpha)-\varepsilon<(\tilde{A}_n^U)_r(\alpha)<\tilde{A}_r^U(\alpha)+\varepsilon \quad . \tag{22}$$

By Definition 10, for each $\alpha\in[0,\lambda]$, the membership function of $Fp\alpha$ of $(\tilde{A}_n^U)_l(\alpha)$ is

$$\mu_{Fp\alpha\ of\ (\tilde{A}_n^U)_l(\alpha)}(x)=\begin{cases}\alpha, & if \quad x=(\tilde{A}_n^U)_l(\alpha)\\ 0, & otherwise\end{cases} \tag{23}$$

Bt Definition 2, for each $\alpha\in[0,\lambda]$, the membership function of $(\tilde{A}_l^U(\alpha)-\varepsilon,\tilde{A}_l^U(\alpha)+\varepsilon;\alpha)$ is

$$\mu_{(\tilde{A}_l^U(\alpha)-\varepsilon,\tilde{A}_l^U(\alpha)+\varepsilon;\alpha)}(x)=\begin{cases}\alpha, & if \quad \tilde{A}_l^U(\alpha)-\varepsilon<x<\tilde{A}_l^U(\alpha)+\varepsilon\\ 0, & otherwise\end{cases} \tag{24}$$

Since $\lambda\le\rho$, and the third formula in (22), by (23), (24) we have

$\mu_{Fp\alpha\ of\ (\tilde{A}_n^U)_l(\alpha)}(x)\le\mu_{(\tilde{A}_l^U(\alpha)-\varepsilon,\tilde{A}_l^U(\alpha)+\varepsilon;\alpha)}(x) \quad \forall x\in R$ and $\alpha\in[0,\lambda]$. Hence

$Fp\alpha$ of $(\tilde{A}_n^U)_l(\alpha)\subset(\tilde{A}_l^U(\alpha)-\varepsilon,\tilde{A}_l^U(\alpha)+\varepsilon;\alpha) \quad \forall\alpha\in[0,\lambda]$.

Similarly, $Fp\alpha$ of $(\tilde{A}_n^L)_l(\alpha)\subset(\tilde{A}_l^L(\alpha)-\varepsilon,\tilde{A}_l^L(\alpha)+\varepsilon;\alpha)$,

$Fp\alpha$ of $(\tilde{A}_n^L)_r(\alpha) \quad \subset(\tilde{A}_r^L(\alpha)-\varepsilon,\tilde{A}_r^L(\alpha)+\varepsilon;\alpha)$,

$Fp\alpha$ of $(\tilde{A}_n^U)_r(\alpha)\subset(\tilde{A}_r^U(\alpha)-\varepsilon,\tilde{A}_r^U(\alpha)+\varepsilon;\alpha)$.

Also, for each $\alpha\in[\lambda,\rho]$, $Fp\alpha$ of $(\tilde{A}_n^U)_l(\alpha)\subset(\tilde{A}_l^U(\alpha)-\varepsilon,\tilde{A}_l^U(\alpha)+\varepsilon;\alpha)$,

$Fp\alpha$ of $(\tilde{A}_n^U)_r(\alpha)\subset(\tilde{A}_r^U(\alpha)-\varepsilon,\tilde{A}_r^U(\alpha)+\varepsilon;\alpha)$.

Thus the conditions of Definition 11 are fulfilled, and hence by Definition 11, $\lim_{n\to\infty}\tilde{A}_n=\tilde{A}$.

Thm 2. $\tilde{A}_{jn}=[\tilde{A}_{jn}^L,\tilde{A}_{jn}^U], \quad \tilde{A}_j=[\tilde{A}_j^L,\tilde{A}_j^U]\in F_{IV}(\lambda,\rho),\ j=1,2;\ n=1,2,\cdots$

For each $j=1,2;$ if $\lim_{n\to\infty}(\tilde{A}_{jn}^L)_l(\alpha)=(\tilde{A}_j^L)_l(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_{jn}^L)_r(\alpha)=(\tilde{A}_j^L)_r(\alpha)$ are uniformly convergent for $0\le\alpha\le\lambda$; and if $\lim_{n\to\infty}(\tilde{A}_{jn}^U)_l(\alpha)=(\tilde{A}_j^U)_l(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_{jn}^U)_r(\alpha)=(\tilde{A}_j^U)_r(\alpha)$ are uniformly convergent for $0\le\alpha\le\rho$, Then

$\lim_{n\to\infty}(\tilde{A}_{1n}(+)\tilde{A}_{2n})=\tilde{A}_1(+)\tilde{A}_2$.

Proof. From Def. 6, we have $\tilde{A}_{1n}(+)\tilde{A}_{2n}=[\tilde{A}_{1n}^L(+)\tilde{A}_{2n}^L,\ \tilde{A}_{1n}^U(+)\tilde{A}_{2n}^U]$, and

$\tilde{A}_1(+)\tilde{A}_2=[\tilde{A}_1^L(+)\tilde{A}_2^L,\ \tilde{A}_1^U(+)\tilde{A}_2^U]$. By the assumption of Thm. 2, and the theorem of uniform convergence (unions) and (14), we have

$\lim_{n\to\infty}(\tilde{A}_{1n}^L(+)\tilde{A}_{2n}^L)_l(\alpha)=\lim_{n\to\infty}(\tilde{A}_{1n}^L)_l(\alpha)+\lim_{n\to\infty}(\tilde{A}_{2n}^L)_l(\alpha)=(\tilde{A}_1^L)_l(\alpha)+(\tilde{A}_2^L)_l(\alpha)=(\tilde{A}_1^L(+)\tilde{A}_{2l}^L)_l(\alpha)$ is uniformly convergent for $0\le\alpha\le\lambda$.

Similarly, $\lim_{n\to\infty}(\tilde{A}_{1n}^L(+)\tilde{A}_{2n}^L)_r(\alpha)=(\tilde{A}_1^L(+)\tilde{A}_{2l}^L)_r(\alpha)$ is uniformly convergent for $0\le\alpha\le\lambda$.

Also, $\lim_{n\to\infty}(\tilde{A}_{1n}^U(+)\tilde{A}_{2n}^U)_l(\alpha)=(\tilde{A}_1^U(+)\tilde{A}_2^U)_l(\alpha)$, and $\lim_{n\to\infty}(\tilde{A}_{1n}^U(+)\tilde{A}_{2n}^U)_r(\alpha)=(\tilde{A}_1^U(+)\tilde{A}_2^U)_r(\alpha)$ are all uniformly convergent for $0\le\alpha\le\rho$.

Nor replace $\tilde{A}_n$ by $\tilde{A}_{1n}(+)\tilde{A}_{2n}$, $\tilde{A}$ by $\tilde{A}_1(+)\tilde{A}_2$ in Thm 1, we thus have

$\lim_{n\to\infty}(\tilde{A}_{1n}(+)\tilde{A}_{2n})=\tilde{A}_1(+)\tilde{A}_2$.

Thm. 3. $\tilde{A}_n=[\tilde{A}_n^L,\tilde{A}_n^U]$, $\tilde{A}=[\tilde{A}^L,\tilde{A}^U]\in F_{IV}(\lambda,\rho)$, $n=1,2,\cdots; e\in R$.

If $\lim_{n\to\infty}(\tilde{A}_n^L)_l(\alpha)=\tilde{A}_l^L(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_n^L)_r(\alpha)=\tilde{A}_r^L(\alpha)$, are uniformly convergent for $0\le\alpha\le\lambda$. And

If $\lim_{n\to\infty}(\tilde{A}_n^U)_l(\alpha)=\tilde{A}_l^U(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_n^U)_r(\alpha)=\tilde{A}_r^U(\alpha)$ are uniformly convergent for $0\le\alpha\le\rho$,

Then $\lim_{n\to\infty}e\tilde{A}_n=e\tilde{A}$.

Proof. By Def. 6, $e\tilde{A}_n=[e\tilde{A}_n^L,e\tilde{A}_n^U]$, . Similar to the proof of Thm. 2, by the assumptions of Thm. 3, the theorem of uniform convergence (scalar product) and (15), we have ,

for $e>0$, $\lim_{n\to\infty}(e\tilde{A}_n^L)_l(\alpha)=\lim_{n\to\infty}e(\tilde{A}_n^L)_l(\alpha)=e\tilde{A}_l^L(\alpha)=(e\tilde{A}^L)_l(\alpha)$, $\lim_{n\to\infty}(e\tilde{A}_n^L)_r(\alpha)=$ $(e\tilde{A}^L)_r(\alpha)$ are uniformly convergent for $0\le\alpha\le\lambda$.

And $\lim_{n\to\infty}(e\tilde{A}_n^U)_l(\alpha)=(e\tilde{A}^U)_l(\alpha)$, $\lim_{n\to\infty}(e\tilde{A}_n^U)_r(\alpha)=(e\tilde{A}^U)_r(\alpha)$ are uniformly convergent for $0\le\alpha\le\rho$.

Now, replace $\tilde{A}_n$ by $e\tilde{A}_n$, $\tilde{A}$ by $e\tilde{A}$ in Thm. 1, we have $\lim_{n\to\infty}e\tilde{A}_n=e\tilde{A}$.

Same argument for the case $e<0$.

Thm. 4. $\tilde{A}_{jn} = [\tilde{A}_{jn}^L, \tilde{A}_{jn}^U]$, $\tilde{A}_j = [\tilde{A}_j^L, \tilde{A}_j^U] \in F_{IV}(\lambda, \rho)$, $n = 1, 2, \cdots$ satisfy the two conditions in Thm. 2, and also satisfy either case 1 or case 2 of the followings:

Case 1: $0 \le (\tilde{A}_{jn}^L)_l(\alpha) < (\tilde{A}_{jn}^L)_r(\alpha) \quad \forall \alpha \in [0, \lambda],\ j = 1, 2$ and

$$0 \le (\tilde{A}_{jn}^U)_l(\alpha) < (\tilde{A}_{jn}^U)_r(\alpha) \quad \forall \alpha \in [0, \rho],\ j = 1, 2.$$

Case 2: $(\tilde{A}_{jn}^L)_l(\alpha) < (\tilde{A}_{jn}^L)_r(\alpha) \le 0, \quad \forall \alpha \in [0, \lambda],\ j = 1, 2$ and

$$(\tilde{A}_{jn}^U)_l(\alpha) < (\tilde{A}_{jn}^U)_r(\alpha) \le 0, \quad \forall \alpha \in [0, \rho],\ j = 1, 2,$$

Then, $\lim_{n \to \infty} (\tilde{A}_{1n}(\times)\tilde{A}_{2n}) = \tilde{A}_1(\times)\tilde{A}_2$.

Proof. (a) $\tilde{A}_{jn}, \tilde{A}_j$ satisfy the conditions of Thm. 2 and Case 1 above, by Def. 6,

$$(\tilde{A}_{1n}(\times)\tilde{A}_{2n}) = [\tilde{A}_{1n}^L(\times)\tilde{A}_{2n}^L, \quad \tilde{A}_{1n}^U(\times)\tilde{A}_{2n}^U],\ (\tilde{A}_1(\times)\tilde{A}_2) = [\tilde{A}_1^L(\times)\tilde{A}_2^L, \quad \tilde{A}_1^U(\times)\tilde{A}_2^U].$$

Then, $0 \le (\tilde{A}_j^L)_l(\alpha) < (\tilde{A}_j^L)_r(\alpha) \quad \forall \alpha \in [0, \lambda],\ j = 1, 2$ and

$$0 \le (\tilde{A}_j^U)_l(\alpha) < (\tilde{A}_j^U)_r(\alpha) \quad \forall \alpha \in [0, \rho],\ j = 1, 2.$$

Hence, by the condition of (a), theorem of uniform convergence (product), and (16), we have $\lim_{n \to \infty} (\tilde{A}_{1n}^L(\times)\tilde{A}_{2n}^L)_l(\alpha) = \lim_{n \to \infty} (\tilde{A}_{1n}^L)_l(\alpha)(\tilde{A}_{2n}^L)_l(\alpha) = (\tilde{A}_1^L)_l(\alpha)(\tilde{A}_2^L)_l(\alpha) = (\tilde{A}_1^L(\times)\tilde{A}_2^L)_l(\alpha)$ is uniformly convergent for $0 \le \alpha \le \lambda$.

Similarly, $\lim_{n \to \infty} (\tilde{A}_{1n}^L(\times)\tilde{A}_{2n}^L)_r(\alpha) = (\tilde{A}_1^L(\times)\tilde{A}_2^L)_r(\alpha)$ is uniformly convergent for $0 \le \alpha \le \lambda$.

Also $\lim_{n \to \infty} (\tilde{A}_{1n}^U(\times)\tilde{A}_{2n}^U)_l(\alpha) = (\tilde{A}_1^U(\times)\tilde{A}_2^U)_l(\alpha)$, $\lim_{n \to \infty} (\tilde{A}_{1n}^U(\times)\tilde{A}_{2n}^U)_r(\alpha) = (\tilde{A}_1^U(\times)\tilde{A}_2^U)_r(\alpha)$ for $0 \le \alpha \le \rho$.

Now, replace $\tilde{A}_n$ by $(\tilde{A}_{1n}(\times)\tilde{A}_{2n})$, $\tilde{A}$ by $\tilde{A}_1(\times)\tilde{A}_2$ in Thm. 1, we have $\lim_{n \to \infty} (\tilde{A}_{1n}(\times)\tilde{A}_{2n}) = \tilde{A}_1(\times)\tilde{A}_2$.

(b) $\tilde{A}_{jn}, \tilde{A}_j$ satisfy the conditions of Thm. 2 and Case 2 above, we have

$$(\tilde{A}_j^L)_l(\alpha) < (\tilde{A}_j^L)_r(\alpha) \le 0 \quad \forall \alpha \in [0, \lambda],\ j = 1, 2 \text{ and}$$

$$(\tilde{A}_j^U)_l(\alpha) < (\tilde{A}_j^U)_r(\alpha) \le 0 \quad \forall \alpha \in [0, \rho],\ j = 1, 2.$$

By (17), we have $\lim_{n\to\infty}(\tilde{A}_{1n}(\times)\tilde{A}_{2n}) = \tilde{A}_1(\times)\tilde{A}_2$.

Now we consider the set $\tilde{G} = [\tilde{G}^L, \tilde{G}^U]$ (in (1), (2)) $\in F_{IV}(\lambda,\rho)$, $V(x)$ is a real function satisfies either Case 1 or Case 2 of the following. Then find out the level (λ,ρ) interval-valued fuzzy set $V(\tilde{F})$ and both left, right point of its α -cut.

Case 1: $V(x)$ is a strictly decreasing continuous function on R. From (1), (2) and Extension Principle, let $z = V(x)$, we have

$$\mu_{V(\tilde{G}^L)}(z) = \sup_{z=V(x)} \mu_{\tilde{G}^L}(V^{-1}(z)) = \begin{cases} g_2^L(V^{-1}(z)), & V(c) \le z \le V(b) \\ g_1^L(V^{-1}(z)), & V(b) \le z \le V(a) \\ 0, & elsewhere \end{cases} \quad (25)$$

$$\mu_{V(\tilde{G}^U)}(z) = \begin{cases} g_2^U(V^{-1}(z)), & V(e) \le z \le V(b) \\ g_1^U(V^{-1}(z)), & V(b) \le z \le V(d) \\ 0, & elsewhere \end{cases} \quad (26)$$

$\mu_{V(\tilde{G}^L)}V(c) = g_2^L(c) = 0$, $\mu_{V(\tilde{G}^L)}V(b) = g_2^L(b) = \lambda$, $\mu_{V(\tilde{G}^L)}V(a) = 0$;

$\mu_{V(\tilde{G}^U)}V(e) = \mu_{V(\tilde{G}^U)}V(d) = 0$, $\mu_{V(\tilde{G}^L)}V(b) = \rho$, $0 \le \mu_{V(\tilde{G}^L)}(z) \le \mu_{V(\tilde{G}^U)} \le 1 \quad \forall z$

By (25), (26), we have the level (λ,ρ) interval-valued fuzzy set $V(\tilde{G}) = [V(\tilde{G}^L), V(\tilde{G}^U)]$ (27)

From (25), $g_2^L(V^{-1}(z)) = \alpha \to z = ((g_2^L)^{-1}(\alpha))$ the left point $(V(\tilde{G}^L))_l(\alpha)$ of the α -cut of $(V(\tilde{G}^L))$, the right point $(V(\tilde{G}^L))_r(\alpha)$ of the α -cut of $V(\tilde{G}^L)$ respectively are

$$(V(\tilde{G}^L)_l)(\alpha) = V((g_2^L)^{-1}(\alpha)) = V(\tilde{G}_r^L(\alpha)), \ (V(\tilde{G}^L)_r)(\alpha) = V((g_1^L)^{-1}(\alpha)) = V(\tilde{G}_l^L(\alpha)) \ ; \ 0 \le \alpha \le \lambda \quad (28)$$

by second formula of (1). And the α -cut of $V(\tilde{G}^U)$ are

$$(V(\tilde{G}^U)_l)(\alpha) = V((g_2^U)^{-1}(\alpha)) = V(\tilde{G}_r^U(\alpha)), \quad (V(\tilde{G}^U)_r)(\alpha) = V((g_1^U)^{-1}(\alpha)) = V(\tilde{G}_l^U(\alpha)); \ 0 \le \alpha \le \rho \quad (29)$$

by second formula of (2).

Case 2: $V(x)$ is a strictly increasing continuous function on R similar to Case 1, we have

$$\mu_{V(\tilde{G}^L)}(z)=\begin{cases} g_1^L(V^{-1}(z)), & V(a)\le z\le V(b) \\ g_2^L(V^{-1}(z)), & V(b)\le z\le V(c) \\ 0, & elsewhere \end{cases} \tag{30}$$

$$\mu_{V(\tilde{G}^U)}(z)=\begin{cases} g_1^U(V^{-1}(z)), & V(d)\le z\le V(b) \\ g_2^U(V^{-1}(z)), & V(b)\le z\le V(e) \\ 0, & elsewhere \end{cases} \tag{31}$$

From (30), (31), we have the level (λ,ρ) interval-valued fuzzy set $V(\tilde{G})=[V(\tilde{G}^L),V(\tilde{G}^U)]$ (32)
Similar to (28), (29), we can get

$$(V(\tilde{G}^L)_l)(\alpha)=V((g_1^L)^{-1}(\alpha))=V(\tilde{G}_l^L(\alpha)),(V(\tilde{G}^L)_r)(\alpha)=V((g_1^L)^{-1}(\alpha))=V(\tilde{G}_l^L(\alpha));\ \ 0\le\alpha\le\lambda \tag{33}$$

$$(V(\tilde{G}^U)_l)(\alpha)=V((g_2^U)^{-1}(\alpha))=V(\tilde{G}_r^U(\alpha)),(V(\tilde{G}^U)_r)(\alpha)=V((g_1^U)^{-1}(\alpha))=V(\tilde{G}_l^U(\alpha));\ \ 0\le\alpha\le\rho \tag{34}$$

Thm. 5. $\tilde{G}_n=[\tilde{G}_n^L,\tilde{G}_n^U]$, $\tilde{G}=[\tilde{G}^L,\tilde{G}^U]$ (in (1), (2)) $\in F_{IV}(\lambda,\rho)$, $n=1,2,\cdots$. $V(x)$ defined on R (or $R^+=(0,\infty)$, if $d>0$ in Fig. 3) is a strictly decreasing continuous function or a strictly increasing continuous function.

If $\lim_{n\to\infty}(\tilde{G}_n^L)_l(\alpha)=\tilde{G}_l^L(\alpha)$, $\lim_{n\to\infty}(\tilde{G}_n^L)_r(\alpha)=\tilde{G}_r^L(\alpha)$ are uniformly convergent in $0\le\alpha\le\lambda$

And $\lim_{n\to\infty}(\tilde{G}_n^U)_l(\alpha)=\tilde{G}_l^U(\alpha)$, $\lim_{n\to\infty}(\tilde{G}_n^U)_r(\alpha)=\tilde{G}_r^U(\alpha)$ are uniformly convergent in $0\le\alpha\le\rho$.

Then $\lim_{n\to\infty}V(\tilde{G}_n)=V(\tilde{G})$, where for each $n=1,2,\cdots$

$$\mu_{\tilde{G}_n^L}(x)=\begin{cases} g_{1n}^L(x), & a\le x\le b \\ g_{2n}^L(x), & b\le x\le c \\ 0, & elsewhere \end{cases}\quad,\quad \mu_{\tilde{G}_n^U}(x)=\begin{cases} g_{1n}^U(x), & a\le x\le b \\ g_{2n}^U(x), & b\le x\le e \\ 0, & elsewhere \end{cases}$$

$g_{1n}^L(x), g_{1n}^U(x)$ are strictly increasing continuous function in $[a,b],[d,b]$ respectively.

$g_{2n}^L(x), g_{2n}^U(x)$ are strictly decreasing continuous function in $[b,c],[b,e]$ respectively.

And $g_{1n}^L(a)=g_{2n}^L(c)=0$, $g_{1n}^L(b)=g_{2n}^L(b)=\lambda$; $g_{1n}^U(d)=g_{2n}^U(e)=0$, $g_{1n}^U(b)=g_{2n}^U(b)=\rho$

And $0 \le \mu_{\tilde{G}_n^L}(x) < \mu_{\tilde{G}_n^U}(x) \le 1 \quad \forall x, n$. $V(\tilde{G}_n) = [V(\tilde{G}_n^L), V(\tilde{G}_n^U)]$ and $V(\tilde{G}) = [V(\tilde{G}), V(\tilde{G})]$

$\in F_{IV}(\lambda, \rho)$.

Proof. (a) If $V(x)$ defined on R is a strictly decreasing continuous function, by the assumption of Thm.4, theorem of uniform convergent, and (28), (29), we have

$(V(\tilde{G}_n^L))_l(\alpha) = V(\tilde{G}_n^L))_r(\alpha)$, $(V(\tilde{G}^L))_l(\alpha) = V(\tilde{G}^L))_r(\alpha)$.

By the assumption of Thm.5, theorem of uniform convergent, we have

$\lim_{n\to\infty}(V(\tilde{G}_n^L))_l(\alpha) = \lim_{n\to\infty} V((\tilde{G}_n^L)_r(\alpha)) = V((\tilde{G}^L)_r(\alpha)) = (V(\tilde{G}^L))_l(\alpha)$ is uniformly convergent in

$0 \le \alpha \le \lambda$.

Similarly, $\lim_{n\to\infty}(V(\tilde{G}_n^L))_r(\alpha) = (V(\tilde{G}^L))_r(\alpha)$ is uniformly convergent in $0 \le \alpha \le \lambda$.

By (29), we have $\lim_{n\to\infty}(V(\tilde{G}_n^U))_l(\alpha) = (V(\tilde{G}^U))_l(\alpha)$, $\lim_{n\to\infty}(V(\tilde{G}_n^U))_r(\alpha) = (V(\tilde{G}^U))_r(\alpha)$ are uniformly convergent in $0 \le \alpha \le \rho$.

Replace $\tilde{A}_n$ by $V(\tilde{G}_n)$, $\tilde{A}$ by $V(\tilde{G})$, by Thm.1, $\lim_{n\to\infty} V(\tilde{G}_n) = V(\tilde{G})$.

(b) $V(x)$ defined on R, is a strictly increasing continuous function, by (33), (34) and the same argument above, we have $\lim_{n\to\infty} V(\tilde{G}_n) = V(\tilde{G})$.

1.4 Limit theorem on a family of level ζ fuzzy sets.

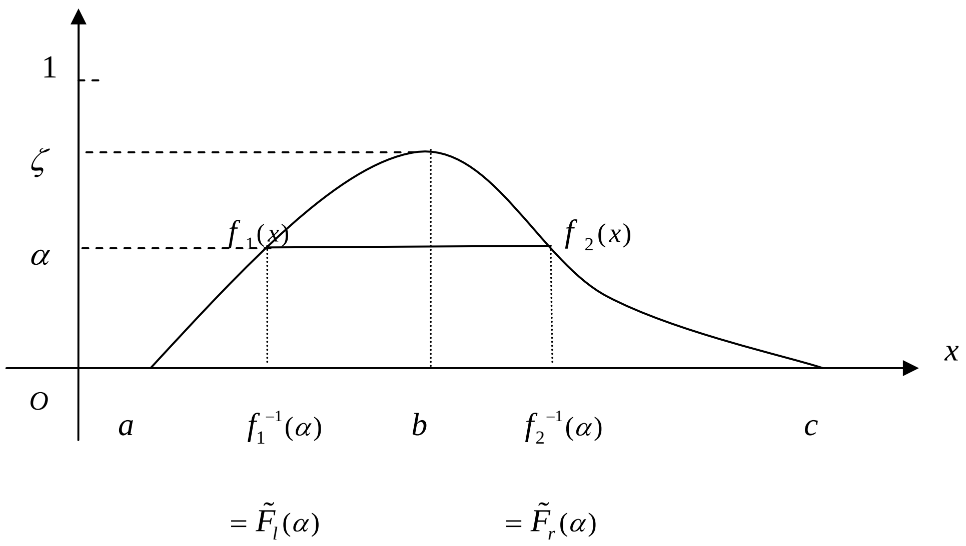

Fig. 11 level ζ fuzzy set

There exist $a < b < c \in R$ and functions $f_1(x)$, $f_2(x)$ satisfy the following set is called a level ζ fuzzy set $\tilde{F}$, $0 < \zeta \le 1$, where $f_1(x)$ is a strictly increasing continuous function and $f_1(x) = 0 \quad \forall\, x \le a$. $f_2(x)$ is a strictly decreasing continuous function and $f_2(x) = 0 \quad \forall\, x \ge c$ and $f_1(b) = f_2(b) = \zeta$.

Let the membership function of the level ζ fuzzy set $\tilde{F}$ be

$$\mu_{\tilde{F}}(x) = \begin{cases} f_1(x), & a \le x \le b \\ f_2(x), & b \le x \le c \\ 0, & elsewhere \end{cases}$$

$\tilde{F}_l(x) = f_1^{-1}(\alpha)$, $\tilde{F}_r(x) = f_2^{-1}(\alpha)$ (see Fig.11) are the left, right end points of the α-cut of $\tilde{F}$ respectively.

Let $F_S(\zeta)$ be the family of all such level ζ fuzzy sets. In Fig. 3, let $a = d, c = e$ and $\lambda = 0$. Then the level (λ, ρ) interval-valued fuzzy set $\tilde{G} = [\tilde{G}^L, \tilde{G}^U]$ will turn to be a level ρ fuzzy set $\tilde{G}^U \in F_S(\rho)$. It is trivial that $\tilde{G}^U \in F_S(\lambda)$ and $\tilde{F}_n = \bigcup_{0 \le \alpha \le \zeta} [(\tilde{F}_n)_l(\alpha), (\tilde{F}_n)_r(\alpha); \alpha] \in F_S(\zeta)$, $n = 1, 2, \cdots$.

Then Def. 10 becomes

Def. $10'$. For $\tilde{F}_n, F \in F_S(\zeta)$, $n = 1, 2, \cdots$, $0 < \zeta \le 1$. For each $\varepsilon > 0$, there exists N, a natural number which is independent of α, $(\in [0, \zeta])$ such that $\forall\, n \ge N$, the following holds:

For each $\alpha \in [0, \zeta]$, $[(\tilde{F}_n)_l(\alpha), (\tilde{F}_n)_r(\alpha); \alpha) \subset (\tilde{F}_l(\alpha) - \varepsilon, (\tilde{F}_n)_l(\alpha) + \varepsilon; \alpha)$ $(\in T_F)$, Then define $\lim_{n\to\infty} \tilde{F}_n = \tilde{F}$. Thus Thm. 1 ~ Thm. 5 will become Thm. 6 ~ Thm 8 as follows:

Thm. 6. For $\tilde{F}_n$, $\tilde{F} \in F_S(\zeta)$, $n = 1, 2, \cdots;$ $0 < \zeta \le 1$.

If $\lim_{n\to\infty} (\tilde{F}_n)_l(\alpha) = \tilde{F}_l(\alpha)$, $\lim_{n\to\infty} (\tilde{F}_n)_r(\alpha) = \tilde{F}_r(\alpha)$ are uniformly convergent in $0 \le \alpha \le \zeta$,

Then (a) $\lim_{n\to\infty} \tilde{F}_n = \tilde{F}$, (b) $\lim_{n\to\infty} e\tilde{F}_n = e\tilde{F}$, (c) $\lim_{n\to\infty} V(\tilde{F}_n) = V(\tilde{F})$.

Thm 7. For $\tilde{F}_{jn}, \tilde{F}_j \in F_S(\zeta)$, $j = 1, 2$ and $n = 1, 2, \cdots$ $(0 < \zeta \le 1)$; if

$\lim_{n\to\infty} (\tilde{F}_{jn})_l(\alpha) = (\tilde{F}_j)_l(\alpha)$, $\lim_{n\to\infty} (\tilde{F}_{jn})_r(\alpha) = (\tilde{F}_j)_r(\alpha)$ are uniformly convergent in $0 \le \alpha \le \zeta$, Then $\lim_{n\to\infty}(\tilde{F}_{1n}(+)\tilde{F}_{2n})=\tilde{F}_1(+)\tilde{F}_2$

Thm 8. If $\tilde{F}_{jn},\tilde{F}_j \in F_S(\zeta)$, satisfy either the following condition of Case 1 or Case 2 beside the condition of Thm. 7,

Case 1: $0<(\tilde{F}_{jn})_l(\alpha)<(\tilde{F}_{jn})_r(\alpha) \quad \forall\ j=1,2$ and $n=1,2\cdots$

Case 2: $(\tilde{F}_{jn})_l(\alpha)<(\tilde{F}_{jn})_r(\alpha)\le 0 \quad \forall\ j=1,2$ and $n=1,2\cdots$

Then $\lim_{n\to\infty}(\tilde{F}_{1n}(\times)\tilde{F}_{2n})=\tilde{F}_1(\times)\tilde{F}_2$

Thm. 9 $\tilde{A}_n=[A_n^L,A_n^U]$, $\tilde{A}=[A^L,A^U]$, $\in F_{IV}(\lambda,\rho)$. If $\lim_{n\to\infty}(\tilde{A}_n^L)_l(\alpha)=\tilde{A}_l^L(\alpha)$ and $\lim_{n\to\infty}(\tilde{A}_n^L)_r(\alpha)=\tilde{A}_r^L(\alpha)$ are uniformly convergent for $0\le\alpha\le\lambda$; and $\lim_{n\to\infty}(\tilde{A}_n^U)_l(\alpha)=\tilde{A}_l^U(\alpha)$, $\lim_{n\to\infty}(\tilde{A}_n^U)_r(\alpha)=\tilde{A}_r^U(\alpha)$ for $\lambda\le\alpha\le\rho$, then $\lim_{n\to\infty}\tilde{A}_n=\tilde{A}$, $\lim_{n\to\infty}\tilde{A}_n^L=\tilde{A}^L$, $\lim_{n\to\infty}\tilde{A}_n^U=\tilde{A}^U$.

Proof. By Thm. 1, we have $\lim_{n\to\infty}\tilde{A}_n=\tilde{A}$. Since $\tilde{A}_n^L,\tilde{A}^L \in F_S(\lambda)$, $n=1,2,\cdots$, by the first condition of Thm. 9 and Thm. 6, $\lim_{n\to\infty}\tilde{A}_n^L=\tilde{A}^L$. Similarly, since $\tilde{A}_n^U,\tilde{A}^U \in F_S(\rho)$, $n=1,2,\cdots$, by the second condition of Thm.9 and Thm. 6, $\lim_{n\to\infty}\tilde{A}_n^U=\tilde{A}^U$.

Now, since $[a,b;\alpha)=\bigcap_{n=1}^{\infty}(a-\frac{1}{n},b)$, and for each $\alpha\in[0,1]$, $[a,b)\leftrightarrow[a,b;\alpha)$, $(a-\frac{1}{n},b)\leftrightarrow(a-\frac{1}{n},b;\alpha)$ are one-to-one mapping, therefore $[a,b;\alpha)=\bigcap_{n=1}^{\infty}(a-\frac{1}{n},b;\alpha)\in T_F$ for each $\alpha\in[0,1]$. It is obvious that for any arbitrary unions of all these fuzzy intervals $[a,b;\alpha)$ ($\forall\ a<b$ and $\alpha\in[0,1]$) $\in T_F$. Similarly $(a,b;\alpha]=\bigcap_{n=1}^{\infty}(a,b+\frac{1}{n};\alpha)\in T_F$ for each $\alpha\in[0,1]$. And any arbitrary unions of these fuzzy interval $(a,b;\alpha]$ ($\forall\ a<b$ and $\alpha\in[0,1]$) $\in T_F$. Also, $[a,b;\alpha]=\bigcap_{n=1}^{\infty}(a-\frac{1}{n},b+\frac{1}{n};\alpha)\in T_F$ for each $\alpha\in[0,1]$. Hence. Any arbitrary unions of these fuzzy interval $[a,b;\alpha),(a,b;\alpha],[a,b;\alpha]$ ($\forall\ a<b$ and $\alpha\in[0,1]$) and any arbitrary unions of elements in O_F are all $\in T_F$.

For any $\tilde{G}=[\tilde{G}^L,\tilde{G}^U]\in F_{IV}(\lambda,\rho)$. By (13), for each $\alpha\in[0,\lambda)$, $[\tilde{G}_l^U(\alpha),\tilde{G}_l^L(\alpha);\alpha]$,

$[\tilde{G}_r^L(\alpha),\tilde{G}_r^U(\alpha);\alpha]\in T_F$ and for each $\alpha\in[\lambda,\rho]$, $[\tilde{G}_l^U(\alpha),\tilde{G}_r^U(\alpha);\alpha]\in T_F$. Hence

$$\tilde{G}=\bigcup_{0\le\alpha<\lambda}([\tilde{G}_l^U(\alpha),\tilde{G}_l^L(\alpha);\alpha]\cup[\tilde{G}_r^L(\alpha),\tilde{G}_r^U(\alpha);\alpha]\bigcup_{\lambda\le\alpha\le\rho}[\tilde{G}_l^U(\alpha),\tilde{G}_r^U(\alpha);\alpha]\in T_F.$$

i.e. $F_{IV}(\lambda,\rho)\subset T_F \quad \forall\,\lambda,\rho;\quad 0<\lambda\le\rho\le 1$. Similarly, level ρ fuzzy set $\tilde{F}=\bigcup_{0\le\alpha\le\rho}[\tilde{F}_l(\alpha),\tilde{F}_r(\alpha);\alpha]$ in (35) $\in T_F$. Hence $F_S(\rho)\subset T_F \quad \forall\, 0<\rho\le 1$.

Def. 12. $\tilde{G}_n=[\tilde{G}_n^L,\tilde{G}_n^U]$, $\tilde{G}=[\tilde{G}^L,\tilde{G}^U]$, $\tilde{E}=[\tilde{E}^L,\tilde{E}^U]$ $\in F_{IV}(\lambda,\rho)$, $n=1,2,\cdots$

(a) $\lim_{n\to\infty}\mu_{\tilde{G}_n^L}(x)=\mu_{\tilde{G}^L}(x)$ and $\lim_{n\to\infty}\mu_{\tilde{G}_n^U}(x)=\mu_{\tilde{G}^U}(x) \quad \forall\, x\in R \Leftrightarrow \lim_{n\to\infty}\mu_{\tilde{G}^L}(x)=\mu_{\tilde{G}}(x) \quad \forall\, x\in R$.

(b) $\tilde{G}^L=\bigcup_{n=1}^{\infty}\tilde{G}_n^L$ and $\tilde{G}^U=\bigcup_{n=1}^{\infty}\tilde{G}_n^U \Leftrightarrow \tilde{G}=\bigcup_{n=1}^{\infty}\tilde{G}_n$.

(c) $\tilde{E}^L\subset\tilde{G}^L$ and $\tilde{E}^U\subset\tilde{G}^U \Leftrightarrow \tilde{E}\subset\tilde{G}$.

Note 3. By Def. 4, $\mu_{\tilde{G}}(x)=[\mu_{\tilde{G}_L}(x),\mu_{\tilde{G}_U}(x)], \quad x\in R$

Thm. 10 $\tilde{G}_n=[\tilde{G}_n^L,\tilde{G}_n^U]$, $\tilde{G}=[\tilde{G}^L,\tilde{G}^U]\in F_{IV}(\lambda,\rho)$ $(\subset T_F)$, $n=1,2,\cdots$

If $\tilde{G}_1\subset\tilde{G}_2\subset\cdots\cdots\subset\tilde{G}_n\subset\cdots\cdots\tilde{G}$ and $\lim_{n\to\infty}\mu_{\tilde{G}_n}(x)=\mu_{\tilde{G}}(x)\ \forall\, x\in R$, then $\tilde{G}=\bigcup_{n=1}^{\infty}\tilde{G}_n$

Case 1: If $\tilde{G}_1^L\subset\tilde{G}_2^L\subset\cdots\cdots\subset\tilde{G}_n^L\subset\cdots\cdots\tilde{G}^L$ (in $F_S(\lambda)\subset T_F$) and $\lim_{n\to\infty}\mu_{\tilde{G}_n^L}(x)=\mu_{\tilde{G}^L}(x)\ \forall\, x\in R$,

then $\tilde{G}^L=\bigcup_{n=1}^{\infty}\tilde{G}_n^L$.

Case 2: If $\tilde{G}_1^U\subset\tilde{G}_2^U\subset\cdots\cdots\subset\tilde{G}_n^U\subset\cdots\cdots\tilde{G}^U$ (in $F_S(\rho)\subset T_F$) $\forall\, x\in R$, then $\tilde{G}^U=\bigcup_{n=1}^{\infty}\tilde{G}_n^U$.

By Def.12, we shall prove for each case 1, 2, theorem holds.

Proof. Case 1: Let $\tilde{A}\,(\in F_S(\lambda))$ be any neighborhood of $\tilde{G}_L$. By Def. 7, there exists a fuzzy set $\tilde{O}\,(\in T_F)$ such that $\tilde{G}^L\subset\tilde{O}\subset\tilde{A}$. By condition of Case 1, we have $\tilde{G}_n^L\subset\tilde{G}^L\subset\tilde{A}$, $\forall\, n=1,2,\cdots$.

Hence from Def. 8, we have , the sequence $\{\tilde{G}_n^L;n=1,2,\cdots\}$ converges to $\tilde{G}^L$. And also since $\tilde{G}_k^L\subset\tilde{G}_n^L \quad \forall\, k=1,2,\cdots,n$; we have $\bigvee_{k=1}^{n}\mu_{\tilde{G}_k^L}(x)=\mu_{\tilde{G}_n^L}(x)$. By the assumption of Case 1 of the theorem, $\mu_{\tilde{G}^L}(x) = \lim_{n\to\infty} \mu_{\tilde{G}_n^L}(x) = \lim_{n\to\infty} \bigvee_{k=1}^{n} \mu_{\tilde{G}_k^L}(x) = \bigvee_{k=1}^{\infty} \mu_{\tilde{G}_k^L}(x) = \mu_{\bigcup_{k=1}^{\infty} \tilde{G}_k^L}(x)$. Thus $\tilde{G}^L = \bigcup_{k=1}^{\infty} \tilde{G}_k^L$.

By the same argument as in Case 1, Case 2 holds too. Hence Thm 10 holds.

2. **Example**

In Brigham [2] and Shappe etc. [7], there was a formula for present value of expected future dividend at time 0 with zero growth:

$$P_0^* = \frac{D}{1+k} + \frac{D}{(1+k)^2} + \cdots\cdots = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{D}{(1+k)^j} = \frac{D}{k} \tag{36}$$

Where D: dividends, the stockholder expects to receive at the end of every year.

k: required rate of return at every year.

P_0^*: the intrinsic, expected present value at time 0 of an infinite stream of dividends.

If the preferred red stock (preferred shares) can earn the dividends D. By (35), he can get P_0^*. But it is difficult to earn the fixed dividend D for the common stock. If the management of the company is stable, we can consider the dividend each year will be in the interval

$$[D-\Delta_1, D+\Delta_2],\quad 0<\Delta_1<D, \quad 0<\Delta_2 \tag{37}$$

Since (37) is an interval in stead of a single value, hence the decision maker needs to evaluate a certain value as the estimates dividend for each year. Should the decision maker choose a value D, Just exact the same as the original D, as it has zero error. Through the point of view in fuzzy concept, by using confidence level to express this. The zero error means it has the maximum confidence level 1. that is, the farther the chosen value from D, the larger of the error in $[D-\Delta_1, D)$ or $(D, D+\Delta_2]$ will be. If the chosen value is $D-\Delta_1$ or $D+\Delta_2$, then the confidence level reaches its least value 0. Therefore, relative to the interval (37), let the triangular fuzzy number $\tilde{D}^U$ be

$$\tilde{D}^U = (D-\Delta_1, D, D+\Delta_2; 1) \tag{38}$$

The decision maker should choose Δ_1, Δ_2 suitably so that they satisfy $0<\Delta_1<D, \quad 0<\Delta_2$ (39)

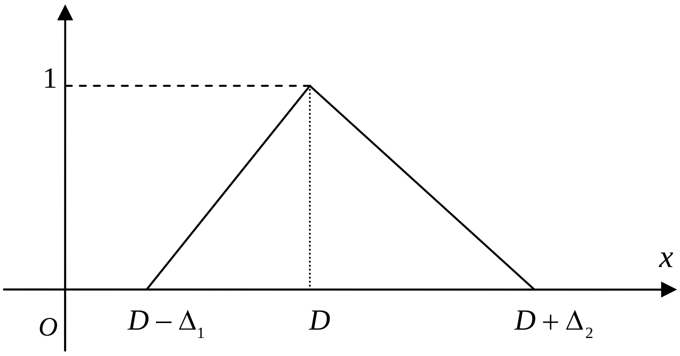

Fig.12 Triangular fuzzy number $\tilde{D}$

From fFig.12, we can see that $\tilde{D}$ has membership grade 1 at D. And become smaller as it apart from D and finally reaches 0 as it goes to $D-\Delta_1$ or $D+\Delta_2$. Therefore the membership grade has the same property as the confidence level. Therefore it is reasonable to define the confidence level to be the membership grade. Thus, it is also reasonable to define the triangular fuzzy number $\tilde{D}$ (38) for the interval (37). However , since the length of time is infinite, it is too idealistic always to have membership grade 1 at D. So we set the membership grade at D belongs to the interval $[\lambda,1]$, $0<\lambda<1$. And let the level λ triangular fuzzy number $\tilde{D}^L$ be

$$\tilde{D}^L=(D-\Delta_3, D, D+\Delta_4; \lambda) \tag{40}$$

The decision maker should suitably choose Δ_j, $j=1,2,3,4$ satisfy

$$0<\Delta_3<\Delta_1<D \text{ and } 0<\Delta_4<\Delta_2 \tag{41}$$

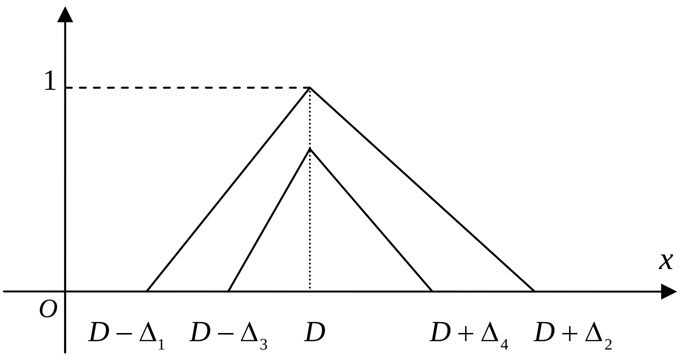

Fig.13 Level $(\lambda,1)$ interval-valued fuzzy number $\tilde{D}$

From (38), (40), we have level $(\lambda,1)$ interval-valued fuzzy number $\tilde{D}$,

$$\tilde{D}=[\tilde{D}^L, \tilde{D}^U] \tag{42}$$

Using (42), fuzzify $\sum_{j=1}^{n}\frac{D}{(1+k)^j}=\frac{1}{k}[1-(\frac{1}{1+k})^n]D$ in (36), we have the following $\tilde{Q}_n$.

Let $\tilde{Q}_n^L=\frac{1}{k}[1-(\frac{1}{1+k})^n]\tilde{D}^L=(\frac{1}{k}[1-(\frac{1}{1+k})^n](D-\Delta_3),\ \frac{1}{k}[1-(\frac{1}{1+k})^n]D,\ \frac{1}{k}[1-(\frac{1}{1+k})^n]D+\Delta_4;\lambda)$

$\tilde{Q}_n^U=\frac{1}{k}[1-(\frac{1}{1+k})^n]\tilde{D}^U=(\frac{1}{k}[1-(\frac{1}{1+k})^n](D-\Delta_1),\ \frac{1}{k}[1-(\frac{1}{1+k})^n]D,\ \frac{1}{k}[1-(\frac{1}{1+k})^n]D+\Delta_2;1)$

$\tilde{Q}_n=[\tilde{Q}_n^L,\tilde{Q}_n^U]$, $n=1,2,\cdots$.

The four α-cut of $\tilde{Q}_n^L,\tilde{Q}_n^U$ are

$$(\tilde{Q}_n^L)_l(\alpha)=\frac{1}{k}[1-(\frac{1}{1+k})^n][D-(1-\frac{\alpha}{\lambda})\Delta_3]$$

$$(\tilde{Q}_n^L)_r(\alpha)=\frac{1}{k}[1-(\frac{1}{1+k})^n][D+(1-\frac{\alpha}{\lambda})\Delta_4]$$

$$(\tilde{Q}_n^U)_l(\alpha)=\frac{1}{k}[1-(\frac{1}{1+k})^n][D-(1-\alpha)\Delta_1]$$

$$(\tilde{Q}_n^U)_r(\alpha)=\frac{1}{k}[1-(\frac{1}{1+k})^n][D+(1-\alpha)\Delta_2],\qquad 0\le\alpha\le 1$$

Let $\tilde{Q}_l^L(\alpha)=\frac{1}{k}[D-(1-\frac{\alpha}{\lambda})\Delta_3]$, $\tilde{Q}_r^L(\alpha)=\frac{1}{k}[D+(1-\frac{\alpha}{\lambda})\Delta_4]$,

$$\tilde{Q}_l^U(\alpha)=\frac{1}{k}[D+(1-\alpha)\Delta_1],\ \tilde{Q}_r^U(\alpha)=\frac{1}{k}[D+(1-\alpha)\Delta_2],\ 0\le\alpha\le 1 \qquad (43)$$

Then $\lim_{n\to\infty}(\tilde{Q}_n^L)_l(\alpha)=\tilde{Q}_l^L(\alpha)$.

Since for each $\alpha\in[0,1]$, $(\tilde{Q}_n^L)_l(\alpha)$ is an increasing function in n, and also a continuous function for $\alpha\in[0,1]$. Similarly, $\tilde{Q}_l^L(\alpha)$ is also a continuous function of α. By the theorem of DINI in Brand [1], we have $\lim_{n\to\infty}(\tilde{Q}_n^L)_l(\alpha)=\tilde{Q}_l^L(\alpha)$ is uniformly convergent in $0\le\alpha\le 1$.

By the same argument, $\lim_{n\to\infty}(\tilde{Q}_n^L)_r(\alpha)=\tilde{Q}_r^L(\alpha)$, $\lim_{n\to\infty}(\tilde{Q}_n^U)_l(\alpha)=\tilde{Q}_l^U(\alpha)$, and $\lim_{n\to\infty}(\tilde{Q}_n^U)_r(\alpha)=\tilde{Q}_r^U(\alpha)$ Are all uniformly convergent in $0\le\alpha\le 1$. So by Thm. 1,

$\lim_{n\to\infty}\tilde{Q}_n=\tilde{Q}$, where $\tilde{Q}=[\tilde{Q}^L,\tilde{Q}^U]$, and by (42),

$\tilde{Q}^L=(\frac{1}{k}(D-\Delta_3),\frac{1}{k}D,\frac{1}{k}(D+\Delta_4);\lambda)$, $\tilde{Q}^U=(\frac{1}{k}(D-\Delta_1),\frac{1}{k}D,\frac{1}{k}(D+\Delta_2);1)$

Applying the Appendix (A.1), defuzzify $\tilde{Q}$, we have

$$P_0 = \frac{D}{k} + \frac{\Delta_2^2 - \Delta_1^2 + \lambda(\Delta_4^2 - \Delta_3^2)}{3k[\Delta_2 + \Delta_1 + \lambda(\Delta_3 + \Delta_4)]} = P_0^* + \frac{\Delta_2^2 - \Delta_1^2 + \lambda(\Delta_4^2 - \Delta_3^2)}{3k[\Delta_2 + \Delta_1 + \lambda(\Delta_3 + \Delta_4)]} \quad \text{by (36).}$$

From (41), we can show that $P_0 > 0$, and P_0 is an expected present value at time 0 of an infinite stream of dividends. When $\Delta_1 = \Delta_2$, and $\Delta_3 = \Delta_4$, $P_0 = P_0^*$. i.e. the crisp case equals to the fuzzy case, or say, the crisp case is a special case of fuzzy case. If $\Delta_1 < \Delta_2$ and $\Delta_3 < \Delta_4$, then $P_0 > P_0^*$.

If $\Delta_2 < \Delta_1$, $\Delta_4 < \Delta_3$, then $P_0 < P_0^*$.

3. **Discussion**

Def. 13. (Pu and Liu [6]), $b \in R$, if the membership function of the fuzzy set $\tilde{b}$ is as follows, then we call $\tilde{b}$, a fuzzy point.

$$\mu_{\tilde{b}}(x) = \begin{cases} 1, & x = b \\ 0, & x \neq b \end{cases}$$

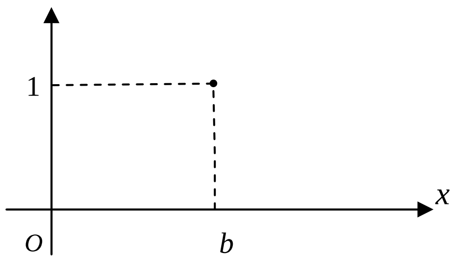

Fig. 14 Fuzzy point $\tilde{b}$

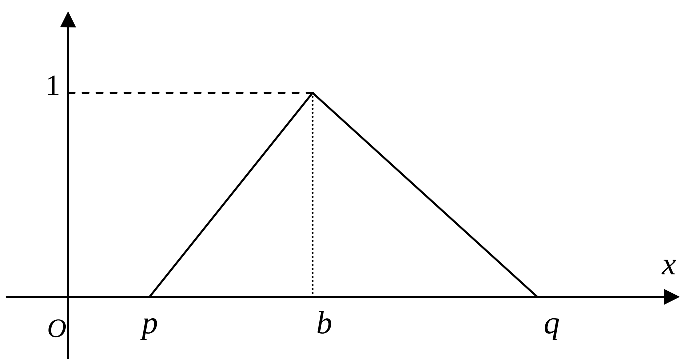

Fig.15 Triangular fuzzy number $(p, b, q; 1)$

Let F_p be the family of all fuzzy points on R, $b(\in R) \leftrightarrow \tilde{b}(\in F_p)$ is an one-to-one mapping from R to F_p. For $\tilde{A} = (p, b, q; 1) \in F_N (\subset F_S(1))$, let $p = q = b$, then $\tilde{A}$ becomes $(b, b, b; 1) = \tilde{b}$, is a fuzzy point. From Fig. 14, the left, right point of the α-cut of $\tilde{b}$ are $\tilde{b}_l(1) = \tilde{b}_r(1) = b$, $\tilde{b}_l(\alpha) = \tilde{b}_r(\alpha) = 0$ for all $\alpha \in [0,1)$. Then we have $\tilde{b} = \bigcup_{0 \leq \alpha \leq 1} [\tilde{b}_l(\alpha), \tilde{b}_r(\alpha); 1] = [b, b; 1]$. Hence Def. 11 becomes

Def. 14. For any $\varepsilon > 0$, there exists a natural number N such that $\forall\, n \ge N$, the following holds. If $[b_n, b_n;1] \subset (b-\varepsilon, b+\varepsilon;1)$, i.e. if $b_n \subset (b-\varepsilon, b+\varepsilon;1)$, then $\lim_{n\to\infty} \tilde{b}_n = \tilde{b}$

It is easy to prove that for $\tilde{a}, \tilde{b} \in F_p$, $\tilde{a}(+)\tilde{b} = \tilde{c}, c = a+b$; $\tilde{a}(\times)\tilde{b} = \tilde{d}$, $d = ab$

Thm 6 ~ Thm 8 become

Thm. 11 For $\tilde{b}_n, \tilde{b} \in F_p$, $n = 1,2,\cdots$; if $\lim_{n\to\infty} b_n = b$, then

(a) $\lim_{n\to\infty} \tilde{b}_n = \tilde{b}$

(b) $\lim_{n\to\infty} q\tilde{b}_n = q\tilde{b}, \quad q \in R$

(c) $\lim_{n\to\infty} V(\tilde{b}_n) = V(\tilde{b})$

Thm.12 For $\tilde{b}_{jn}, \tilde{b}_j \in F_p$, $j = 1,2$ and $n = 1,2,\cdots$; if $\lim_{n\to\infty} b_{jn} = b_j$ then

(a) $\lim_{n\to\infty} (\tilde{b}_{1n}(+)\tilde{b}_{2n}) = \tilde{b}_1(+)\tilde{b}_2$

(b) $\lim_{n\to\infty} (\tilde{b}_{1n}(\times)\tilde{b}_{2n}) = \tilde{b}_1(\times)\tilde{b}_2$.

Appendix

From Fig. 16, level $(\lambda,1)$ interval-valued fuzzy number $\tilde{E} = [\tilde{E}^L, \tilde{E}^U] = [(a,b,c;\lambda), (p.b.q;1)]$

$$\mu_{\tilde{E}^L}(x) = \begin{cases} \dfrac{\lambda(x-a)}{b-a}, & a \le x \le b \\ \dfrac{\lambda(c-x)}{c-b}, & b \le x \le c \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{E}^U}(x) = \begin{cases} \dfrac{x-p}{b-p}, & p \le x \le b \\ \dfrac{q-x}{q-b}, & b \le x \le q \\ 0 & otherwise \end{cases}$$

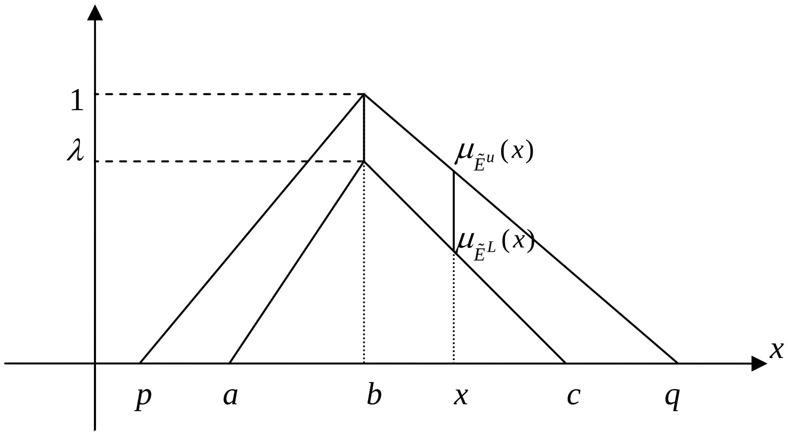

Fig. 16 Level (λ,ρ) interval-valued fuzzy number $\tilde{E}$

Let $f(x)=\mu_{\tilde{E}^U}(x)-\mu_{\tilde{E}^L}(x)$, $-\infty<x<\infty$ (see Fig.16), then the centroid of $f(x)$ is

$$c(f(x))=\frac{\int_{-\infty}^{\infty}xf(x)\,dx}{\int_{-\infty}^{\infty}f(x)\,dx}=\frac{(q-p)(p+b+q)+\lambda(c-a)(a+b+c)}{3[(q-p)+\lambda(c-a)]} \quad \text{(A.1)}$$

From Fig. 15, when $a=p, c=q, \lambda=0$, $\tilde{E}$ becomes a triangular fuzzy number $(p,b,q;1)=\tilde{E}^U$. In (A.1), let $a=p, c=q, \lambda=0$, then the centroid of $\tilde{E}^U$ is $c(f(x))=\frac{1}{3}(p+b+q)$, which is a special case of (A.1). So we can use (A.1) to defuzzify $\tilde{E}$.

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