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On Maps Preserving Zero Jordan Products

By

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Abstract. Let R be a ring, $A = M_n(R)$ and $\theta : A \to A$ a surjective additive map preserving zero Jordan products, i.e. if $x, y \in A$ are such that $xy + yx = 0$, then $\theta(x)\theta(y) + \theta(y)\theta(x) = 0$. In this paper, we show that if R contains $\frac{1}{2}$ and $n \geqslant 4$, then $\theta = \lambda\varphi$, where $\lambda = \theta(1)$ is a central element of A and $\varphi : A \to A$ is a Jordan homomorphism.

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1. Introduction

Let A be a ring. As usual we can define on A the Jordan multiplication by $x \circ y = xy + yx$ and the Lie multiplication by $[x, y] = xy - yx$ for $x, y \in A$. Let $x * y$ denote one of the ordinary product xy, the Jordan product $x \circ y$ or the Lie product $[x, y]$ on A. Suppose that $\theta : A \to B$ is an additive map from A into a ring B. We shall say that θ preserves zero $*$-products if $\theta(x) * \theta(y) = 0$ whenever $x * y = 0$. A natural possibility for θ to preserve zero $*$-products is to be of the form $\theta = \lambda\varphi$, where λ is a central element of B and $\varphi : A \to B$ is a $*$-homomorphism, that is, $\varphi(x * y) = \varphi(x) * \varphi(y)$ for all $x, y \in A$. An interesting question is for which rings A and B this natural possibility is the only possibility.

In case $*$ is the Lie multiplication, such kind of questions have been studied since the 1970s. In 1976, Watkins [29] described bijective maps which preserve zero Lie products on the matrix ring $M_n(F)$ over an algebraically closed field F of characteristic zero, where $n \geqslant 4$. He used a formula for the centralizer of a matrix due to Frobenius as well as a characterization of rank 1 preserving maps due to Marcus and Moyls [22]. Later it was sequentially studied for matrix algebras in [2, 25, 26] and for operator algebras in [19, 23, 24, 27, 32]. Applying the powerful techniques on functional identities (see surveys [10, 12]) the problem of character-

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izing maps preserving zero Lie products was solved for many classes of rings [4, 5, 8, 11, 14, 21], in particular prime rings [8, 11] and matrix rings over unital rings [4].

In case $*$ is the ordinary multiplication, a similar problem, namely that of describing maps preserving zero products, was studied in [1, 15, 18, 30, 31]. It turns out that the question of preservers on zero products is more difficult than that on zero Lie products. The difficulty is that in an arbitrary ring there are a lot of elements x and y such that $[x, y] = 0$ (for example, take $y = x^2$), while in many "good" rings there may exist too few elements x and y such that $xy = 0$. For example, in a division ring every additive map preserves zero products. So a reasonable approach to the question of preservers on zero products is to impose on the rings under consideration the existence of some nontrivial zero-divisors. In our recent papers maps preserving zero products were described for rings generated by idempotents [18] and for prime rings containing nontrivial idempotents [15].

In contrast to the existence of many results on maps preserving zero products or zero Lie products, there seem to be few results that can be found in the literature on maps preserving zero Jordan products. Actually, the only related paper we were aware of is the one by Šemrl [28] which deals with maps preserving square-zero matrices of sl_n, the set of $n \times n$ complex matrices with trace zero. Recently, we described maps preserving zero Jordan products on the Jordan algebra of all Hermitian operators [17] and maps preserving square-zero matrices over unital commutative rings [16].

In this paper we consider the case that A is the matrix ring over a unital ring in which 2 is invertible. Explicitly, we prove the following result.

Theorem 1.1. *Let R be a ring with $\frac{1}{2}$, $A = M_n(R)$ where $n \geqslant 4$, and $\theta : A \to A$ a surjective additive map preserving zero Jordan products. Then $\theta = \lambda\varphi$, where $\lambda = \theta(1)$ is a central element of A and $\varphi : A \to A$ is a Jordan homomorphism.*

To demonstrate the utility of the theorem, we apply it to study Clifford algebras. Clifford algebras are of great importance to quantum physics. Elements of the algebras correspond to fermionic creation or annihilation operators, and vectors in Fock spaces (some particular modules) over the algebras are physical states of fermions (see, e.g. [13] for a detailed discussion).

A Clifford algebra $C(E)$ over a field F is the universal algebra with 1 generated by a finite-dimensional vector space E over F associated to a quadratic form $f(x)$ which satisfies $x^2 = f(x)$ for all $x \in E$. Suppose that the characteristic of F is not 2. Then we have $\dim C(E) = 2^n$ where $n = \dim E$ [20, Theorem 4.12]. Suppose further that the quadratic form $f(x)$ is nondegenerate and F is algebraically closed. Let $\mathbf{C}_n$ be the Clifford algebra of dimension 2^n over F. Then $\mathbf{C}_n \cong M_{2^m}(F)$ if $n = 2m$ and $\mathbf{C}_n \cong M_{2^m}(F) \oplus M_{2^m}(F)$ if $n = 2m + 1$ [20, Theorem 4.12]. Also, let $\mathbf{C}_n^+$ be the even subalgebra of $\mathbf{C}_n$; then $\mathbf{C}_n^+ = \mathbf{C}_{n-1}$ [20, Theorem 4.13].

Taking $R = F$ or $R = F \oplus F$ one can apply Theorem 1.1 to obtain the following results for Clifford algebras.

Corollary 1.2. *Let $n \geqslant 4$. Then any surjective additive map $\theta : \mathbf{C}_n \to \mathbf{C}_n$ preserving zero Jordan products is of the form $\lambda\varphi$, where $\lambda = \theta(1)$ is a central element of $\mathbf{C}_n$ and $\varphi : \mathbf{C}_n \to \mathbf{C}_n$ is a Jordan homomorphism.*

Corollary 1.3. *Let $n \geqslant 5$. Then any surjective additive map $\theta : \mathbf{C}_n^+ \to \mathbf{C}_n^+$ preserving zero Jordan products is of the form $\lambda\varphi$, where $\lambda = \theta(1)$ is a central element of $\mathbf{C}_n^+$ and $\varphi : \mathbf{C}_n^+ \to \mathbf{C}_n^+$ is a Jordan homomorphism.*

We note that a map preserving zero Jordan products on the subalgebra $\mathbf{C}_n^+$ does not, generally speaking, preserve zero Jordan products on the whole algebra $\mathbf{C}_n$, since the center of $M_{2^m}(F) \oplus M_{2^m}(F)$ is bigger than that of $M_{2^{m+1}}(F)$.

In the proof of Theorem 1.1, we shall make use of some basic notions and results from a recently developed theory of rings, namely that of functional identities. For the readers' convenience, a very short summary on functional identities is provided in the last section as Appendix.

2. The Results

Under the setting of Theorem 1.1, the matrix ring $A = M_n(R)$ over the unital ring R is additively generated by the matrices ae_{ij} for $a \in R$ and $i, j \in \{1, 2, \ldots, n\}$, where e_{ij} is the matrix with 1 in the (i, j)-position and 0's elsewhere. We shall denote ae_{ij} by a_{ij} for simplicity. Certainly the additive map $\theta : A \to A$ is completely determined by the values $\theta(a_{ij})$ for $a \in R$ and $i, j \in \{1, 2, \ldots, n\}$.

In order to show that $\theta = \lambda\varphi$ for a central element λ and a Jordan homomorphism φ, we first show that the map θ also preserves equal Jordan products, that is, $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$ whenever $x, y, u, v \in A$ are such that $x \circ y = u \circ v$. This will be achieved by Lemma 2.1 where all possible products of the form $\theta(a_{ij}) \circ \theta(b_{jk})$ are investigated. Finally, applying the results on functional identities, we shall prove Theorem 1.1.

Lemma 2.1. *Let R and B be rings with $\frac{1}{2}$, $A = M_n(R)$ where $n \geqslant 3$, and $\theta : A \to B$ an additive map which preserves zero Jordan products. Then, for $a, b \in R$ and $i, j, k, l \in \{1, 2, \ldots, n\}$, we have*

(*a*) $\theta(a_{ij}) \circ \theta(b_{kl}) = 0$ *if* $i \neq l$ *and* $j \neq k$.
(*b*) $\theta(a_{ij}) \circ \theta(b_{jk}) = \theta((ab)_{ik}) \circ \theta(e_{kk}) = \theta((ab)_{ik}) \circ \theta(e_{ii})$ *if* $i \neq k$.
(*c*) $\theta(a_{ij}) \circ \theta(b_{ji}) = \frac{1}{2}\theta((ab)_{ii}) \circ \theta(e_{ii}) + \frac{1}{2}\theta((ba)_{jj}) \circ \theta(e_{jj})$.

Proof. (a) Since θ preserves zero Jordan products and $a_{ij} \circ b_{kl} = 0$ if $i \neq l$ and $j \neq k$, it follows that $\theta(a_{ij}) \circ \theta(b_{kl}) = 0$.

(b) Note first that $(ab)_{ik} \circ (e_{ii} - e_{kk}) = 0$ if $i \neq k$ and so $\theta((ab)_{ik}) \circ \theta(e_{kk}) = \theta((ab)_{ik}) \circ \theta(e_{ii})$.

Suppose that $j \neq k$. From $(a_{ij} + (ab)_{ik}) \circ (b_{jk} - e_{kk}) = 0$ for $i \neq k$, it follows that

$$(\theta(a_{ij}) + \theta((ab)_{ik})) \circ (\theta(b_{jk}) - \theta(e_{kk})) = 0. \tag{2.1}$$

Since $\theta(a_{ij}) \circ \theta(e_{kk}) = 0$ and $\theta((ab)_{ik}) \circ \theta(b_{jk}) = 0$ by (a), expansion of (2.1) yields $\theta(a_{ij}) \circ \theta(b_{jk}) = \theta((ab)_{ik}) \circ \theta(e_{kk})$.

Suppose that $j = k$; then $\theta(a_{ik}) \circ \theta(b_{kk}) = \theta((ab)_{ik}) \circ \theta(e_{ii})$ for $i \neq k$ can be derived from $(a_{ik} - e_{ii}) \circ ((ab)_{ik} + b_{kk}) = 0$ via a similar argument.

(c) Suppose first that $i \neq j$. Note that

$$\left(\frac{1}{2}(ab)_{ii} + a_{ij} - \frac{1}{2}(ba)_{jj}\right) \circ (-e_{ii} + b_{ji} + e_{jj}) = 0 \quad \text{for } i \neq j$$

and so

$$\left(\frac{1}{2}\theta((ab)_{ii}) + \theta(a_{ij}) - \frac{1}{2}\theta((ba)_{jj})\right) \circ (-\theta(e_{ii}) + \theta(b_{ji}) + \theta(e_{jj})) = 0. \tag{2.2}$$

Since $\theta((ab)_{ii}) \circ \theta(e_{jj}) = 0$ and $\theta((ba)_{jj}) \circ \theta(e_{ii}) = 0$ by (a), and $\theta((ab)_{ii}) \circ \theta(b_{ji}) = \theta((ba)_{jj}) \circ \theta(b_{ji}) = \theta((bab)_{ji}) \circ \theta(e_{ii})$ and $\theta(a_{ij}) \circ \theta(e_{ii}) = \theta(a_{ij}) \circ \theta(e_{jj})$ by (b), expansion of (2.2) yields $\theta(a_{ij}) \circ \theta(b_{ji}) = \frac{1}{2}\theta((ab)_{ii}) \circ \theta(e_{ii}) + \frac{1}{2}\theta((ba)_{jj}) \circ \theta(e_{jj})$.

Suppose next that $i = j$. We are going to show that $\theta(a_{ii}) \circ \theta(b_{ii}) = \frac{1}{2}\theta((a \circ b)_{ii}) \circ \theta(e_{ii})$. For $i \neq k$, expanding

$$\theta(a_{ii} - b_{ik} + b_{ki} - a_{kk}) \circ \theta(b_{ii} - a_{ik} + a_{ki} - b_{kk}) = 0$$

and using $\theta(a_{ii}) \circ \theta(b_{kk}) = 0$, $\theta(a_{kk}) \circ \theta(b_{ii}) = 0$, $\theta(a_{ii} - a_{kk}) \circ \theta(-a_{ik} + a_{ki}) = 0$, $\theta(b_{ik}) \circ \theta(a_{ik}) = 0$, $\theta(b_{ki}) \circ \theta(a_{ki}) = 0$ and $\theta(-b_{ik} + b_{ki}) \circ \theta(b_{ii} - b_{kk}) = 0$, we have

$$\theta(a_{ii}) \circ \theta(b_{ii}) + \theta(a_{kk}) \circ \theta(b_{kk}) = \frac{1}{2}(\theta((a \circ b)_{ii}) \circ \theta(e_{ii}) + \theta((a \circ b)_{kk}) \circ \theta(e_{kk})). \tag{2.3}$$

Now, take l with $l \notin \{i, k\}$. Then, by (2.3), we have

$$\begin{aligned}
&2\theta(a_{ii}) \circ \theta(b_{ii}) + (\theta(a_{kk}) \circ \theta(b_{kk}) + \theta(a_{ll}) \circ \theta(b_{ll}))\\
&\quad = (\theta(a_{ii}) \circ \theta(b_{ii}) + \theta(a_{kk}) \circ \theta(b_{kk})) + (\theta(a_{ii}) \circ \theta(b_{ii}) + \theta(a_{ll}) \circ \theta(b_{ll}))\\
&\quad = \frac{1}{2}(\theta((a \circ b)_{ii}) \circ \theta(e_{ii}) + \theta((a \circ b)_{kk}) \circ \theta(e_{kk}))\\
&\qquad + \frac{1}{2}(\theta((a \circ b)_{ii}) \circ \theta(e_{ii}) + \theta((a \circ b)_{ll}) \circ \theta(e_{ll}))\\
&\quad = \theta((a \circ b)_{ii}) \circ \theta(e_{ii}) + \frac{1}{2}(\theta((a \circ b)_{kk}) \circ \theta(e_{kk}) + \theta((a \circ b)_{ll}) \circ \theta(e_{ll}))\\
&\quad = \theta((a \circ b)_{ii}) \circ \theta(e_{ii}) + (\theta(a_{kk}) \circ \theta(b_{kk}) + \theta(a_{ll}) \circ \theta(b_{ll})).
\end{aligned}$$

Thus $2\theta(a_{ii}) \circ \theta(b_{ii}) = \theta((a \circ b)_{ii}) \circ \theta(e_{ii})$ and hence $\theta(a_{ii}) \circ \theta(b_{ii}) = \frac{1}{2}\theta((a \circ b)_{ii}) \circ \theta(e_{ii})$. $\square$

Now, we can show that a map preserving zero Jordan products preserves equal Jordan products.

Theorem 2.2. *Let R and B be rings with $\frac{1}{2}$, $A = M_n(R)$ where $n \geqslant 3$, and $\theta : A \to B$ an additive map which preserves zero Jordan products. Then, for $x_i, y_i \in A$ with $\sum_{i=1}^m x_i \circ y_i = 0$, we have $\sum_{i=1}^m \theta(x_i) \circ \theta(y_i) = 0$. In particular, for $x, y, u, v \in A$ with $x \circ y = u \circ v$, we have $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$.*

Proof. Let $W = \{a_{ij} \mid a \in R,\ 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant n\}$. For $x_1, y_1, \ldots, x_m, y_m \in A$, we see that $x_1 \circ y_1 + \cdots + x_m \circ y_m$ is a sum of elements of the form $x \circ y$ with $x, y \in W$, and that $\theta(x_1) \circ \theta(y_1) + \cdots + \theta(x_m) \circ \theta(y_m)$ is a sum of corresponding elements $\theta(x) \circ \theta(y)$. Note that, for $x, y \in W$, the element $x \circ y$ is of one of the following forms:

$$0, \quad c_{ij}, \quad c_{ii}, \quad c_{ii} + c'_{jj},$$

where $j \neq i$ and $c, c' \in R$.

For the terms $x \circ y$ with $x, y \in W$ and $x \circ y = 0$, that is, terms of the form (a) $a_{ij} \circ b_{kl}$ with $i \neq l$ and $j \neq k$, (b) $a_{ij} \circ b_{jk}$ with $i \neq k$ and $ab = 0$, (c) $a_{ij} \circ b_{ji}$ with $ab = ba = 0$, the corresponding terms $\theta(x) \circ \theta(y)$ also vanish by Lemma 2.1.

Since $x_1 \circ y_1 + \cdots + x_m \circ y_m = 0$, for any i and j with $i \neq j$, the sum of nonzero terms of the form c_{ij} is 0. These terms are resulted from terms of the form $a_{ik} \circ b_{kj}$ with $c = ab \neq 0$. The corresponding terms $\theta(a_{ik}) \circ \theta(b_{kj})$ can be written as $\theta(c_{ij}) \circ \theta(e_{ii})$ with $c = ab$ by (b) of Lemma 2.1 and hence the sum of them is 0.

Finally, for any fixed $i \in \{1, \ldots, n\}$, the sum of nonzero terms of the form c_{ii} is 0. These terms are resulted from terms of the form $a_{ij} \circ b_{ji} = c_{ii} + c'_{jj}$ with $j \neq i$, $c = ab \neq 0$ and $c' = ba$, or $a_{ii} \circ b_{ii} = c_{ii}$ with $c = a \circ b \neq 0$. By (c) of Lemma 2.1, the corresponding terms $\theta(a_{ij}) \circ \theta(b_{ji})$ can be written as $\frac{1}{2}(\theta(c_{ii}) \circ \theta(e_{ii}))+ \frac{1}{2}(\theta(c'_{jj}) \circ \theta(e_{jj}))$ with $c = ab$ and $c' = ba$, and $\theta(a_{ii}) \circ \theta(b_{ii})$ can be written as $\frac{1}{2}(\theta(c_{ii}) \circ \theta(e_{ii}))$ with $c = a \circ b$. Thus the sum of terms of the form $\theta(c_{ii}) \circ \theta(e_{ii})$ is 0. Therefore we have $\theta(x_1) \circ \theta(y_1) + \cdots + \theta(x_m) \circ \theta(y_m) = 0$. In particular, if $x, y, u, v \in M_n(A)$ are such that $x \circ y = u \circ v$, then $x \circ y + (-u) \circ v = 0$, and so we get $\theta(x) \circ \theta(y) - \theta(u) \circ \theta(v) = 0$. $\square$

In particular, Theorem 2.2 implies that θ satisfies the following nice property:

$$\theta\left(\frac{1}{2}\right) \circ \theta(x \circ y) = \theta(x) \circ (y) \quad \text{for all } x, y \in A,$$

that is, θ is "closed" to a Jordan homomorphism multiplied by a scalar. A natural possibility for an additive map $\theta : A \to B$ to preserve zero Jordan products is to be of the form $\theta = \lambda\varphi$, where λ is a central element in B and $\varphi : A \to B$ is a Jordan homomorphism. First we observe that this "natural possibility" depends heavily on B.

Example 2.3. Let $R = F[x]$ be the polynomial ring in an indeterminate x over a field F, $A = M_n(R)$ and B the (ring) direct sum of A and a ring N with trivial Jordan multiplication, that is, $n \circ n' = 0$ for all $n, n' \in N$ (e.g. N is a Boolean ring). Thus, by the definition of N and B, we have $n \circ b = 0$ for all $n \in N$, $b \in B$. Let $\theta : A \to B$ be an F-linear map. Then θ is uniquely determined by $\theta(x^k e_{ij})$ for all $i, j \in \{1, \ldots, n\}$ and nonnegative integers k. Define the map θ so that $\theta(e_{ij}) = e_{ij}$ and $\theta(x^k e_{ij}) \in N$ for all $k > 0$. It is easy to see that θ preserves zero Jordan products while is not of the form $\lambda\varphi$ for any central element λ in B and Jordan homomorphism $\varphi : A \to B$.

However, for some nice rings B, the natural possibility holds as we shall see in the next theorem. We shall use some notations and techniques of the theory of functional identities (see Appendix for some necessary definitions and results). Here we only mention that the class of 4-free subsets is quite wide.

Theorem 2.4. *Let B be a ring with center C containing $\frac{1}{2}$, R a ring with $\frac{1}{2}$, $A = M_n(R)$ where $n \geqslant 3$, and $\theta : A \to B$ an additive map preserving zero Jordan products such that $\theta(A)$ is a 4-free subset of B and the centralizer of $\theta(A)$ in B coincides with C. Then $\theta = \lambda\varphi$, where $\lambda = \theta(1) \in C$ and $\varphi : A \to B$ is a Jordan homomorphism.*

Proof. By Theorem 2.2 we have

$$\theta(xyx)\circ\theta(y)=\theta(yxy)\circ\theta(x) \tag{2.4}$$

for all $x,y\in A$. Linearization of (2.4) yields

$$\begin{aligned}&\theta(xyz)\circ\theta(u)+\theta(xuz)\circ\theta(y)+\theta(zyx)\circ\theta(u)+\theta(zux)\circ\theta(y)\\&\quad=\theta(yxu)\circ\theta(z)+\theta(uxy)\circ\theta(z)+\theta(yzu)\circ\theta(x)+\theta(uzy)\circ\theta(x),\end{aligned}$$

or, equivalently,

$$\begin{aligned}&(\theta(yzu)+\theta(uzy))\circ\theta(x)-(\theta(xuz)+\theta(zux))\circ\theta(y)\\&\quad+(\theta(yxu)+\theta(uxy))\circ\theta(z)-(\theta(xyz)+\theta(zyx))\circ\theta(u)=0,\end{aligned}$$

for all $x,y,z,u\in A$. Since $\theta(A)$ is a 4-free subset of B, applying [7, Theorem 2.6] we get

$$\begin{aligned}\theta(xyz)+\theta(zyx)&=\lambda_1'\theta(x)\theta(y)\theta(z)+\lambda_2'\theta(x)\theta(z)\theta(y)+\lambda_3'\theta(y)\theta(x)\theta(z)\\&\quad+\lambda_4'\theta(y)\theta(z)\theta(x)+\lambda_5'\theta(z)\theta(x)\theta(y)+\lambda_6'\theta(z)\theta(y)\theta(x)\\&\quad+\mu_1'(x)\theta(y)\theta(z)+\mu_2'(x)\theta(z)\theta(y)+\mu_3'(y)\theta(x)\theta(z)\\&\quad+\mu_4'(y)\theta(z)\theta(x)+\mu_5'(z)\theta(x)\theta(y)+\mu_6'(z)\theta(y)\theta(x)\\&\quad+\nu_1'(x,y)\theta(z)+\nu_2'(x,z)\theta(y)+\nu_3'(y,z)\theta(x)+\tau(x,y,z),\end{aligned}$$

for all $x,y,z\in A$, where $\lambda_i'\in C$, $\mu_i':A\to C$ are additive maps, $\nu_i':A^2\to C$ are biadditive maps and $\tau:A^3\to C$ is a triadditive map. In particular, we have

$$\begin{aligned}\theta(xyx)&=\lambda_1\theta(x)^2\theta(y)+\lambda_2\theta(x)\theta(y)\theta(x)+\lambda_3\theta(y)\theta(x)^2\\&\quad+\mu_1(x)\theta(x)\theta(y)+\mu_2(x)\theta(y)\theta(x)+\mu_3(y)\theta(x)^2\\&\quad+\nu_1(x,y)\theta(x)+\nu_2(x,x)\theta(y)+\frac{1}{2}\tau(x,y,x),\end{aligned} \tag{2.5}$$

for all $x,y\in A$, where $\lambda_i\in C$, $\mu_i:A\to C$ are additive maps and $\nu_i:A^2\to C$ are biadditive maps. Substituting (2.5) into (2.4), we have, on the left hand side,

$$\begin{aligned}&\theta(xyx)\circ\theta(y)\\&\quad=\lambda_1\big(\theta(x)^2\theta(y)^2+\theta(y)\theta(x)^2\theta(y)\big)+\lambda_2\big((\theta(x)\theta(y))^2+(\theta(y)\theta(x))^2\big)\\&\qquad+\lambda_3\big(\theta(y)\theta(x)^2\theta(y)+\theta(y)^2\theta(x)^2\big)+\mu_1(x)\big(\theta(x)\theta(y)^2+\theta(y)\theta(x)\theta(y)\big)\\&\qquad+\mu_2(x)\big(\theta(y)\theta(x)\theta(y)+\theta(y)^2\theta(x)\big)+\mu_3(y)\big(\theta(x)^2\theta(y)+\theta(y)\theta(x)^2\big)\\&\qquad+\nu_1(x,y)(\theta(x)\theta(y)+\theta(y)\theta(x))+2\nu_2(x,x)\theta(y)^2+\tau(x,y,x)\theta(y),\end{aligned} \tag{2.6}$$

and, on the right hand side,

$$\begin{aligned}&\theta(yxy)\circ\theta(x)\\&\quad=\lambda_1\big(\theta(y)^2\theta(x)^2+\theta(x)\theta(y)^2\theta(x)\big)+\lambda_2\big((\theta(y)\theta(x))^2+(\theta(x)\theta(y))^2\big)\\&\qquad+\lambda_3\big(\theta(x)\theta(y)^2\theta(x)+\theta(x)^2\theta(y)^2\big)+\mu_1(y)\big(\theta(y)\theta(x)^2+\theta(x)\theta(y)\theta(x)\big)\\&\qquad+\mu_2(y)\big(\theta(x)\theta(y)\theta(x)+\theta(x)^2\theta(y)\big)+\mu_3(x)\big(\theta(y)^2\theta(x)+\theta(x)\theta(y)^2\big)\\&\qquad+\nu_1(y,x)\big(\theta(y)\theta(x)+\theta(x)\theta(y)\big)+2\nu_2(y,y)\theta(x)^2+\tau(y,x,y)\theta(x),\end{aligned} \tag{2.7}$$

for all $x, y \in A$. Comparing (2.6) and (2.7) and applying [7, Corollary 2.12], we see that $\lambda_1 = \lambda_3 = 0$, $\mu_1 = \mu_2 = \mu_3 = 0$, $\nu_1(x,y) = \nu_1(y,x)$, $\nu_2(x,x) = 0$ and $\tau(x,y,x) = 0$ for all $x, y \in A$. Setting $\gamma = \lambda_2$ and $\nu = \nu_1$ in (2.5) we obtain

$$\theta(xyx) = \gamma\theta(x)\theta(y)\theta(x) + \nu(x,y)\theta(x) \tag{2.8}$$

for all $x, y \in A$. When $x = 1$, (2.8) reduces to

$$\theta(y) = \gamma\theta(1)\theta(y)\theta(1) + \nu(1,y)\theta(1) \tag{2.9}$$

for all $y \in A$. Setting $y = 1$ in (2.9) we obtain

$$(1 - \nu(1,1))\theta(1) = \gamma\theta(1)^3. \tag{2.10}$$

Commuting (2.9) with $\theta(1)^2$ and taking into account (2.10) we see that $\theta(1)^2 \in C$. Right multiplying (2.9) by $\theta(1)$ we obtain

$$\theta(y)\theta(1) - \gamma\theta(1)^3\theta(y) = \nu(1,y)\theta(1)^2 \in C$$

for all $y \in A$. By assumption, $\theta(A)$ is a 4-free, and hence 2-free, subset of B, so it follows from [7, Theorems 2.1 and 1.1] that $\gamma\theta(1)^3 = \theta(1) \in C$ and $\nu(1,y)\theta(1)^2 = 0$ for all $y \in A$. Thus

$$\nu(1,y)\theta(1) = \nu(1,y)\gamma\,\theta(1)^3 = 0$$

and we can rewrite (2.9) as

$$\theta(y) = \gamma\theta(1)^2\theta(y)$$

for all $y \in A$. By [7, Theorem 1.1] again, we get $\gamma\theta(1)^2 = 1$. Thus $\nu(x,1) = \nu(1,x) = \nu(1,x)\gamma\theta(1)^2 = 0$ for all $x \in A$. Setting $y = 1$ in (2.8) we get $\theta(x^2) = \gamma\theta(1)\theta(x)^2$ or, equivalently,

$$\theta(1)\theta(x^2) = \theta(x)^2 \quad \text{for all } x \in A.$$

Let $\lambda = \theta(1)$ and $\varphi = \lambda^{-1}\theta$. Then $\varphi(x^2) = \varphi(x)^2$ for all $x \in A$ and the proof is complete. □

In light of [4, Corollary 5.12], $B = M_n(R)$ is 4-free for any unital ring R and $n \geqslant 4$. Thus setting $B = A = M_n(R)$ for $n \geqslant 4$ in Theorem 2.4, we see that Theorem 1.1 follows immediately.

3. Appendix: Functional Identities

Let Q be a ring with center C containing 1, and R a nonempty subset of Q. For a positive integer m, we denote by R^m the m-th Cartesian power of R. For $x_1, x_2, \ldots, x_m \in R$, we denote by $\boldsymbol{x}_m$ the ordered m-tuple $(x_1, \ldots, x_m)$, by $\boldsymbol{x}_m^{\hat{i}}$ the ordered $(m-1)$-tuple obtained by dropping the i-th component of $\boldsymbol{x}_m$, and by $\boldsymbol{x}_m^{\widehat{ij}}$ or $\boldsymbol{x}_m^{\widehat{ji}}$ for $i \neq j$ the ordered $(m-2)$-tuple obtained by dropping both the i-th and the j-th components of $\boldsymbol{x}_m$. That is,

$$\boldsymbol{x}_m^{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

and

$$\boldsymbol{x}_m^{\widehat{ij}} = \boldsymbol{x}_m^{\widehat{ji}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m).$$

Let $I, J \subseteq \{1,2,\dots,m\}$ where $m \geqslant 2$, and for each $i \in I$, $j \in J$, let $E_i : R^{m-1} \to Q$ and $F_j : R^{m-1} \to Q$ be arbitrary maps. The basic functional identities of degree m are

$$\sum_{i \in I} E_i(\boldsymbol{x}_m^{\hat{i}})x_i + \sum_{j \in J} x_j F_j(\boldsymbol{x}_m^{\hat{j}}) = 0 \quad \text{for all } \boldsymbol{x}_m \in R^m, \tag{3.1}$$

and, a slightly more general one,

$$\sum_{i \in I} E_i(\boldsymbol{x}_m^{\hat{i}})x_i + \sum_{j \in J} x_j F_j(\boldsymbol{x}_m^{\hat{j}}) \in C \quad \text{for all } \boldsymbol{x}_m \in R^m. \tag{3.2}$$

It is understood that if the index set I or J is empty, then the corresponding sum is 0. Suppose that there exist maps $p_{ij} : R^{m-2} \to Q$, $i \in I, j \in J$, $i \neq j$, and $\lambda_k : R^{m-1} \to C$, $k \in I \cup J$, with $\lambda_k = 0$ for $k \notin I \cap J$, such that

$$E_i(\boldsymbol{x}_m^{\hat{i}}) = \sum_{\substack{j \in J,\\ j \neq i}} x_j p_{ij}(\boldsymbol{x}_m^{\widehat{ij}}) + \lambda_i(\boldsymbol{x}_m^{\hat{i}}) \quad \text{and} \quad F_j(\boldsymbol{x}_m^{\hat{j}}) = -\sum_{\substack{i \in I,\\ i \neq j}} p_{ij}(\boldsymbol{x}_m^{\widehat{ij}})x_i - \lambda_j(\boldsymbol{x}_m^{\hat{j}}) \tag{3.3}$$

for all $\boldsymbol{x}_m \in R^m$, $i \in I$, and $j \in J$. We make the convention that p_{ij} is just an element in Q in case $m = 2$. One can readily check that (3.3) implies (3.1) and *a fortiori* (3.2), and it does not depend on R. We shall refer to (3.3) as a *standard solution* of (3.1) and (3.2). Note that the functional identity $\sum_{i \in I} E_i(\boldsymbol{x}_m^{\hat{i}})x_i = 0$ has only one standard solution, namely $E_i = 0$ for all $i \in I$. It turns out that frequently the standard solutions are the only possible solutions. This is the reason why the following fundamental concept is introduced in [6].

For a positive integer d, a nonempty subset $R \subseteq Q$ is said to be d-free if, for any positive integer m and $I, J \subseteq \{1,2,\dots,m\}$, both of the following two conditions are satisfied:

(a) If $\max\{|I|,|J|\} \leqslant d$, then the functional identity (3.1) has only the solution (3.3).

(b) If $\max\{|I|,|J|\}| \leqslant d-1$, then the functional identity (3.2) has only the solution (3.3).

Roughly speaking, d-free subsets are those subsets R of Q such that any functional identity on R in "not too many" variables has only the standard solutions. Then, which sets are d-free after all? Algebras over fields, in particular the maximal right (or left) rings of quotients of prime rings, abound in d-free subsets.

For an element x in an algebra Q over a field C, we denote by $\deg_C(x)$ the degree of x over C if x is algebraic over C, or ∞ if x is not algebraic over C. And for a nonempty subset $R \subseteq Q$, we set

$$\deg_C(R) = \sup\{\deg(x) \mid x \in R\}.$$

In case A is a prime ring with maximal right quotient ring Q and extended centroid C (see the book [9] for definitions and basic properties), A is a d-free subset of Q if $\deg_C(A) \geqslant d$ [3, Theorem 1.2], a noncentral Lie ideal of A is d-free if $\deg_C(A) \geqslant d+1$ [3, Theorem 1.2], and the sets of symmetric elements and of skew-symmetric elements in A are both d-free if A is equipped with an involution and $\deg_C(A) \geqslant 2d+2$ [6, Theorem 2.4]. Some other interesting examples of d-free subsets are presented in [4]. For example, for any ring R with unity it holds that $Q = M_n(R)$ is n-free (see [4, Corollary 5.12]).

For applications we need more involved functional identities than (3.1) and (3.2). Let S be a set and let $\alpha : S \to Q$, $E_i, F_j : S^{m-1} \to Q$, $i \in I$, $j \in J$, be maps of sets. We are interested in the following two identities:

$$\sum_{i\in I} E_i(\boldsymbol{x}_m^{\hat{i}})\alpha(x_i) + \sum_{j\in J} \alpha(x_j)F_j(\boldsymbol{x}_m^{\hat{j}}) = 0 \quad \text{for all } \boldsymbol{x}_m \in S^m, \tag{3.4}$$

and

$$\sum_{i\in I} E_i(\boldsymbol{x}_m^{\hat{i}})\alpha(x_i) + \sum_{j\in J} \alpha(x_j)F_j(\boldsymbol{x}_m^{\hat{j}}) \in C \quad \text{for all } \boldsymbol{x}_m \in S^m. \tag{3.5}$$

It is easy to see that in case $S = R \subseteq Q$ and α is the identity map, the functional identities (3.4) and (3.5) are exactly identities (3.1) and (3.2). The *standard solutions* of the functional identities (3.4) and (3.5) are of the forms

$$E_i(\boldsymbol{x}_m^{\hat{i}}) = \sum_{\substack{j\in J,\\ j\neq i}} \alpha(x_j)p_{ij}(\boldsymbol{x}_m^{\widehat{ij}}) + \lambda_i(\boldsymbol{x}_m^{\hat{i}}) \quad \text{and} \quad F_j(\boldsymbol{x}_m^{\hat{j}}) = -\sum_{\substack{i\in I,\\ i\neq j}} p_{ij}(\boldsymbol{x}_m^{\widehat{ij}})\alpha(x_i) - \lambda_j(\boldsymbol{x}_m^{\hat{j}}) \tag{3.6}$$

for all $\boldsymbol{x}_m \in S^m$, where $p_{ij} : S^{m-2} \to Q$, $i \in I$, $j \in J$, $i \neq j$, $\lambda_k : S^{m-1} \to C$, $k \in I \cup J$, with $\lambda_k = 0$ for $k \notin I \cap J$. In light of [6, Theorem 2.6], if $\alpha(S)$ is a d-free subset of Q, then the functional identities (3.4) and (3.5) have only the standard solutions (3.6).

Sometimes, we can describe the solutions $E_i(\boldsymbol{x}_m^{\hat{i}})$ and $F_j(\boldsymbol{x}_m^{\hat{j}})$ more explicitly in terms of *Beidar polynomials*, a concept in honor of late K. I. Beidar who made an extremely important contribution to the theory of functional identities. Here we give this concept in a loose manner and refer the reader to [7] for details.

First, we say that a map $E : S \to Q$ is a Beidar polynomial of degree 1 in α if there exist an element $\lambda \in C$ and a map $\mu : S \to C$ such that

$$E(x) = \lambda\alpha(x) + \mu(x) \quad \text{for all } x \in S.$$

Next, a map $E : S^2 \to Q$ is said to be a Beidar polynomial of degree 2 in α if there exist elements $\lambda_1, \lambda_2 \in C$, maps $\mu_1, \mu_2 : S \to C$ and a map $\nu : S^2 \to C$ such that

$$E(x,y) = \lambda_1\alpha(x)\alpha(y) + \lambda_2\alpha(y)\alpha(x) + \mu_1(x)\alpha(y) + \mu_2(y)\alpha(x) + \nu(x,y)$$

for all $x, y \in S$. In this way, we can define a Beidar polynomial of degree m in α which involves summands such as

$$\lambda\alpha(x_1)\cdots\alpha(x_m),$$
$$\mu(x_1)\alpha(x_2)\cdots\alpha(x_m), \ldots, \mu(x_m)\alpha(x_1)\cdots\alpha(x_{m-1}),$$
$$\nu(x_1,x_2)\alpha(x_3)\cdots\alpha(x_m), \ldots, \nu(x_{m-1},x_m)\alpha(x_1)\cdots\alpha(x_{m-2}),$$

and so on. In view of [7, Theorem 2.6], the solutions $E_i(\boldsymbol{x}_m^{\hat{i}})$ and $F_j(\boldsymbol{x}_m^{\hat{j}})$ can be expressed as Beidar polynomials.

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赴大陸地區研究心得報告

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工作記要：7 月 1 日赴廣州華南師範大學參加 ”代數及其應用研討會”（International Workshop in Algebras and Applications），會議由 7 月 2 日至 4 日進行 3 天，在會中發表論文 ”A note on polynomial rings over nil rings”。7 月 5 日轉赴北京航空航天大學參加 ”第二屆代數與組合國際會議”（The Second International Congress in Algebras and Combinatorics），會議由 7 月 6 日至 11 日進行 6 天，在會中發表論文 ”On the Behrens radical of matrix rings and polynomial rings”。