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This note tries to prove the non-existence of a correction map, whose conjectural existence has been believed by Kontsevich and some of his followers.

Some notation first. For x, y in $H^*(X, \wedge^* TX)$, we will denote by $x \circ y$ the "new" product, which is the one inherited from Ext group via R map (or its inverse), and by $x.y$ the standard product in $H^*(\wedge^* TX)$. A correction map, denoted by T, has the effect that via R map (or its inverse), we demand $T(x).T(y) = x \circ y$. More precisely, we assume that $T(x) = x + t_1(x) + t_2(x) + ...$ such that the $\wedge TX$-degrees are decreasing by $1, 2...$ (however $t_i(x)$ are allowed to involve terms of different Cech degree). We make the following

Assumption, or to be proved later. i) T can be explicitly given by local formulas. ii) There are no correction terms of higher $\wedge TX$-degree, i.e. $t_{-i}(x) = 0$, for $i > 0$.

To prove the nonexistence of T, we will argue by contradiction. Suppose there is a universal correction map T on dim=4. For $x \in H^0(\wedge^2 TX)$ with X arbitrary, dim=4, $T(x) = x + t_1^{0,2}(x) + t_2^{0,2}(x) + ...$, where the superscript in the preceding $t_i^{p,q}$ refers to the bi-degree of the class x, which, for the sake of convenience, will be dropped from now on. For $y \in H^0(\wedge^2 TX)$, $T(y) = y + t_1(y) + t_2(y) + t_2(y) + ...$, so that $T(x).T(y) = x.y + t_1(x).y + x.t_1(y) + ...$ Consider the terms with $\wedge TX$-degree 2 in $T(x).T(y)$: these can only be

$$t_1(x).t_1(y) + t_2(x).y + x.t_2(y). \tag{0.1}$$

Since T is universal and X arbitrary, the above formula applies to the case where $M = X_1 \times X_2$ with $X_1 = X_2$ being K3 surfaces. Let $x' = y' \in H^0(\wedge^2 TX_2)$ and $x = 1 * x'$ be the image of x' in $H^0(\wedge^2 TM)$ under Kunneth formula. We shall argue that for the case of M

$$t_2(x) \neq 0. \tag{0.2}$$

Note that the preceding non-vanishing (0.2) is exactly the same as Conjecture 1 (specialized to dim=4 case) in our previous discussion.

To see this, note first of all that by the definition of T and the computation of extra terms in K3 case, it is known, by using "Hochschild...", that $T(x').T(x') - x'.x' \neq 0$ ($x'.x' = 0$ in this case) denoted by $E(x', x')$, indeed $E(x', x')$ contains only one term of pure bidegree $(2, 2)$. Furthermore, it is known by Conjecture 2 in our previous discussion or by local formulas, that one has $T(x').T(x') = T(x).T(x)$, i.e. $E(x, x) = E(x', x')$. As $E(x, x)$ is now of bidegree $(2, 2)$, it follows that

$$t_1(x).t_1(x) + t_2(x).x + x.t_2(x) \tag{0.3}$$

can only be of bidegree $(2,2)$. We shall throw away those terms appearing in the sum (0.3) which are not of bidegree $(2,2)$; we denote the resulting sum still by $t_1(x).t_1(x) + t_2(x).x + x.t_2(x)$. We shall analyze this sum (0.3) in the following.

We shall prove that the only non-trivial term in $t_1(x).t_1(x)$ of (0.3) is the product of two terms of pure bidegree $(1,1)$. To reach this, we should eliminate the product terms of bidegrees $(0,1)$ with $(2,1)$, and $(2,1)$ with $(0,1)$ possibly existent in $t_1(x).t_1(x)$. It suffices to prove only that $(0,1)$ term is 0, so that its product with any others becomes zero. This vanishing result follows from the fact that there is no global vector field on K3, hence on the product M neither. Thus, we have proved that $t_1(x).t_1(x)$ is only a product of two terms, denoted by $s(x)$, of $(1,1)$ type, i.e. we have proved

$$t_1(x).t_1(x) = s(x).s(x), s(x) \in H^1(TM). \tag{0.4}$$

Next we shall argue that $s(x)$ lies in $1 \otimes H^1(TX_2)$ via the Kunneth formula. Note the Kunneth formula that $H^1(TM) = (1 \otimes H^1(TX_2)) + (H^1(TX_1) \otimes 1)$. Since $s(x) \in H^1(TM)$, by the above Kunneth formula it needs only to prove that the tangent index in $s(x)$ comes from X_2. For this, note that by using the local formulas of T or $t_1(x)$ from i) of Assumption above, the 2-vector field x can only be reduced to 1-vector in $s(x)$ by some contraction off one of its two tangent indices. Since x is already in X_2, after the contraction the remaining index in $s(x)$ is still seated in X_2. Thus we have proved

$$s(x) \in 1 \otimes H^1(TX_2) \subset H^1(TM). \tag{0.5}$$

Next consider the term $x.t_2(x) + t_2(x).x$ in (0.3). Since it is of bidegree $(2,2)$ and x is of $(0,2)$, it follows that $t_2(x) \in H^2(\mathcal{O}_M)$. By Kunneth formula and $H^1(\mathcal{O}_X) = 0$ for any K3 surface X, one has $H^2(\mathcal{O}_M) = (1 \otimes H^2(\mathcal{O}_{X_2})) + (H^2(\mathcal{O}_{X_1}) \otimes 1)$. We can now write

$$t_2(x) = (u_1(x) \otimes 1) + (1 \otimes u_2(x)) \tag{0.6}$$

where $u_i(x) \in H^2(\mathcal{O}_{X_i})$, $i = 1, 2$. So, one has

$$x.t_2(x) + t_2(x).x$$
$$= (u_1(x) \otimes x') + (1 \otimes x'.u_2(x)) + (u_1(x) \otimes x') + (1 \otimes u_2(x).x')$$
$$2(u_1(x) \otimes x') + (1 \otimes x'.u_2(x)) + (1 \otimes u_2(x).x') \tag{0.7}$$

in $H^2(\wedge^2 TM)$.

Therefore we have found

$$t_1(x).t_1(x) + t_2(x).x + x.t_2(x)$$
$$= s(x).s(x) + 2(u_1(x) \otimes x') + (1 \otimes x'.u_2(x)) + (1 \otimes u_2(x).x') \tag{0.8}$$

Since the above (0.8) equals $E(x, x)$, which is non-zero and which lies in $1 \otimes H^2(\wedge^2 TX_2)$ by Conjecture 2, this implies in particular that $u_2(x) \neq 0$ or $s(x) \neq 0$ by comparing the Kunneth components.

Case i): $u_2(x) \neq 0$. This is what the Conjecture 1 claims. So we have finished the proof.

Case ii): $u_2(x) = 0$, yet $s(x) \neq 0$.

Assuming that we are in Case ii), we can still derive a contradiction the same way the Conjecture 1 does for us.

To get a contradiction, we take $x" = x'$ in $X_1 (= X_2)$, and write $x'' \otimes 1 := x_1$ and the original $x = 1 \otimes x' := x_2$. Since the local formulas for T depends only on the bidegree of the class x_1, x_2, one still has

$$T(x_1).T(x_2) - x_1.x_2 =$$
$$t_1(x_1).t_2(x_2) + x_1.t_2(x_2) + t_2(x_1).x_2 + ... \tag{0.9}$$

It has been known that $E(x_1, x_2) = 0$. Consider only the correction terms of $\wedge TM$-degree 2 in (0.9). These can only be $t_1(x_1).t_2(x_2) + x_1.t_2(x_2) + t_2(x_1).x_2$ with possibly varying Cech degree in the sum. Since it equals $E(x_1, x_2)$, it gives zero after the summation. For the sum (0.9) we now single out the product terms of Cech degree 2, hence of bidegree (2,2). The (2,2) terms in (0.9) can only come from the following (cf. (0.8)):

$$0 = s(x_1).s(x_2) + \tag{0.10}$$
$$(x".u_1(x_2) \otimes 1) + (x" \otimes u_2(x_2)) + (u_1(x_1) \otimes x') + (1 \otimes u_2(x_1).x').$$

By Kunneth formula $s(x_1).s(x_2) \in H^1(TX_1) \otimes H^1(TX_2)$ would never be zero provided $s(x_1) \neq 0 \in H^1(TX_1)$ and $s(x_2) \neq 0 \in H^1(TX_2)$ (cf. (0.5)). Since x_1 and x_2 are the pull-back by the same class on $X_1 = X_2$, one has $s(x_1) \neq 0$ and $s(x_2) \neq 0$ by the assumption of Case ii). (Note the local formula $s(x)$ depends only on the bidegree of x. So also $s(x_1) \neq 0$, by Case ii).) By comparing the Kunneth components in (0.10), we get $(0.10) \neq 0$, a contradiction.

Report on the Workshop "Differential Equations and Exact WKB Analysis"

This workshop "Differential Equations and Exact WKB Analysis" was held in Research Institute of Mathematical Sciences in Kyoto University, dated from October 9 to October 12. The scope of the workshop appears very broad, ranging from Microlocal study, algebraic geometry constructions, Hamiltonian systems, up to contact and symplectic geometry. Scholars from Japan, England, France, are present in the Workshop, among whom there are M. Ruzhansky from Imperial college, on "Strichartz estimates for hyperbolic equations", M. Kashiwara from RIMS, on "Quantization of symplectic manifolds and rational Cherednik algebras" and Lours B. de Monvel from Paris 6, on "Asymptotic equivariant index of Toeplitz operators, and Atiyah-Weinsten conjecture" etc.All of these talks are quite interesting. In what follows I will only talk about what seems of most interesting to me from my personal point of views and specialization.

In two fundamental works published in Ann. of Math. early 2000, entitled "Construction of boundary invariants and the logarithmic singularity of the Bergman kernel" and "Logarithmic singularity of the Szego kernel and a global invariant of stictly pseudoonvex domains", K.Harachi showed that the logarithmic trace of the Szego projector is an invariant of the CR structure. It was extended by Boutet de Monvel for the generalized Szego projectors associated to a contact structure, so that it is a contact invariant, and vanishesin the case where the base manifold is a 3-sphere. Recently, it was furthered shown that the logarithmic trace vanishes always. To understand this, the analogue of Szego kernel on a strictly pseudoconvex boundary with the Toeplitz projectors on a compact conatct manifold was identified, so that the kernel of a Toeplitz projector as the Szego kernel, has a holonomic singularity including a logrithmic term. The coefficient of this logrithmic term is well defined, so as its trace.The 1st result in this direction isthat this trace is a contact structure invariant. Turning to the vanishing result as above, one found that the Toeplitz algebra associated to a contact structure can be embedded in that of a sphere, from which the desired vanishing follows. This vanishing result looks negative, but it is not completely trivial. Though this holds for the Toeplitz algebras associated to a CR or a contact structure, but there are many other star algebras which are locally isomorphic to the Toeplitz algebras. Any such algebra A carries a canonical trace, so that the local trace may glue together. If the contact base is compact, the global trace becomes well-defined, it is found that there are easy examples showing that the global trace is not always vanishing.