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k-tuple domination in graphs

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Abstract

In a graph G, a vertex is said to dominate itself and all of its neighbors. For a fixed positive integer k, the k-tuple domination problem is to find a minimum sized vertex subset in a graph such that every vertex in the graph is dominated by at least k vertices in this set. The current paper studies k-tuple domination in graphs from an algorithmic point of view. In particular, we give a linear-time algorithm for the k-tuple domination problem in strongly chordal graphs, which is a subclass of chordal graphs and includes trees, block graphs, interval graphs and directed path graphs. We also prove that the k-tuple domination problem is NP-complete for split graphs (a subclass of chordal graphs) and for bipartite graphs.

Keywords: Algorithms; k-tuple domination; Domination; Strongly chordal graph; Chordal graph; Split graph; Bipartite graph

1. Introduction

The concept of domination in graph theory is a natural model for many location problems in operations research. In a graph G, a vertex is said to *dominate* itself and all of its neighbors. A *dominating set* of a graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex in V is dominated by at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Domination and its variations have been extensively studied in the literature, see [3,10,11].

Among the variations of domination, the k-tuple domination was introduced in [9], also see [10, p. 189]. For a fixed positive integer k, a *k-tuple dominating set* of a graph $G = (V, E)$ is a subset D of V such that every vertex in V is dominated by at least k vertices of D. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k-tuple dominating set of G. In the case where there is no k-tuple dominating sets, $\gamma_{\times k}(G)$ is defined to be ∞. The special case when $k = 1$ is the usual domination. The case when $k = 2$ was called *double domination* in [9], where exact values of the double domination numbers for some special graphs are obtained. The same paper also gave various bounds of the double and the k-tuple domination numbers in terms of other parameters. Nordhaus–Gaddum type inequality for double domination was given in [8]. For algorithmic results, [17] gave a linear-time algorithm for double domination in trees.

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In this paper we give a linear-time algorithm for the k-tuple domination problem in strongly chordal graphs, which is a subclass of chordal graphs and includes trees, block graphs, interval graphs and directed path graphs. We also prove that the k-tuple domination problem is NP-complete for split graphs (a subclass of chordal graphs) and for bipartite graphs. Concepts related to k-tuple domination are also discussed.

2. Notation and definitions

In a graph $G = (V, E)$, the *neighborhood* of a vertex v is $N_G(v) = \{u \in V: uv \in E\}$. The *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of v is $\deg_G(v) = |N_G(v)|$. We use $\delta(G)$ to denote the minimum degree of a vertex in G. Notice that not every graph has a k-tuple dominating set. In fact, a graph G has a k-tuple dominating set if and only if $\delta(G) + 1 \geqslant k$.

To establish an efficient algorithm for the k-tuple domination in strongly choral graphs, we use the notion of M-domination introduced in [17]. This is a labeling approach which were used for variations of domination in tree-type graphs, see [2,5,13,15,17,19,21,23]. Suppose $G = (V, E)$ is a graph in which every vertex v is associated with a label $M(v) = (t(v), k(v))$, where $t(v) \in \{\mathrm{B}, \mathrm{R}\}$ and $k(v)$ is a nonnegative integer. The interpretation of the label is that we want to find a dominating set D containing all vertices v with $t(v) = \mathrm{R}$ (called *required* vertices) such that each vertex v is dominated by at least $k(v)$ vertices in D. More precisely, an *M-dominating set* of $G = (V, E)$ is a subset D of V satisfying the following two conditions.

(MD1) If $t(v) = \mathrm{R}$, then $v \in D$.
(MD2) $|N_G[v] \cap D| \geqslant k(v)$ for all vertices $v \in V$.

The *M-domination number* $\gamma_M(G)$ is the minimum cardinality of an M-dominating set in G. Note that k-tuple domination is M-domination with $M(v) = (\mathrm{B}, k)$ for all vertices v in V. Also, G has an M-dominating set, i.e., $\gamma_M(G)$ is finite, if and only if $|N_G[v]| \geqslant k(v)$ for all vertices v in V. For instance, if G contains exactly one vertex x, then $\gamma_M(G) = 0$ when $M(x) = (\mathrm{B}, 0)$, $\gamma_M(G) = 1$ when $M(x) \in \{(\mathrm{B}, 1), (\mathrm{R}, 0), (\mathrm{R}, 1)\}$, and $\gamma_M(G) = \infty$ otherwise.

Finally, we introduce the classes of graphs discussed in this paper. In a graph, a *stable set* is a pairwise non-adjacent vertex subset, and a *clique* is a pairwise adjacent vertex subset. A graph is *bipartite* if its vertex set can be partitioned into two stable sets. A graph is *split* if its vertex set can be partitioned into a stable set and a clique. Note that a split graph is *chordal*, that is, every cycle of length greater than three has two non-consecutive vertices that are adjacent.

Strongly chordal graphs were introduced by several authors [4,6,12] in the study of domination. In particular, most of the variations of the domination problem are solvable in this class of graphs. There are many equivalent ways to define them. Here we adapt the notation from Farber's paper [6]. A vertex x is *simple* if $N_G[x] = \{x_1, x_2, \ldots, x_r\}$, where $x = x_1$, satisfies $N_G[x_i] \subseteq N_G[x_j]$ for $1 \leqslant i \leqslant j \leqslant r$. A graph $G = (V, E)$ is *strongly chordal* if every (vertex) induced subgraph has a simple vertex. It is also the case that $G = (V, E)$ is strongly chordal if and only it admits a *strong* (*elimination*) *ordering* which is an ordering $[v_1, v_2, \ldots, v_n]$ of V such that the following condition holds.

(SEO) If $i \leqslant j \leqslant k$ and $v_j, v_k \in N_i[v_i]$,
then $N_i[v_j] \subseteq N_i[v_k]$,

where $N_i[v_j] = \{v_p \in N_G[v_j]: p \geqslant i\}$. Notice that v_i is a simple vertex of the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_n\}$. Strongly chordal graphs is a subclass of chordal graphs, and they include many interesting classes of graphs such as trees, block graphs, interval graphs and directed path graphs. The recognition problem for strongly chordal graphs has the following progress. First, $\mathrm{O}(|V|^3)$-time algorithms for testing if a graph $G = (V, E)$ is strongly chordal were presented in [1,12]. Improvements to an $\mathrm{O}(L(\log L)^2)$-time algorithm was given in [18], where $L = |V| + |E|$, to an $\mathrm{O}(L \log L)$-time algorithm in [20], and to an $\mathrm{O}(|V|^2)$-time algorithm in [22]. These algorithms also give a strong ordering in case the answer is positive.

3. k-tuple domination in strongly chordal graphs

In this section we establish a linear-time algorithm for the k-tuple domination problem in strongly chordal graphs if a strong ordering is provided. We in fact give the algorithm for M-domination. The following lemmas are the base of the algorithm.

Lemma 1. *If v is a vertex in a graph G with $k(v) > |N_G[v]|$, then $\gamma_M(G) = \infty$.*

Having this lemma, we may check the condition $k(v) > |N_G[v]|$ for all vertices v at the beginning of the algorithm. If $k(v) > |N_G[v]|$ for some vertex v, we may stop the processing, otherwise we will have the condition that $k(v) \leqslant |N_G[v]|$ for all vertices v during the algorithm.

For the following lemmas, we assume that G is a strongly chordal graph in which x is a simple vertex with $N_G[x] = \{x_1, x_2, \ldots, x_r\}$, where $x_1 = x$ and $N_G[x_i] \subseteq N_G[x_j]$ for $1 \leqslant i \leqslant j \leqslant r$. Let $s = k(x) - |\{x_i \in N_G[x]\colon\ t(x_i) = \mathrm{R}\}|$ and $\widehat{B} = \{x_i \in N_G[x]\colon t(x_i) = \mathrm{B}\} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_b}\}$ where $i_1 > i_2 > \cdots > i_b$.

Lemma 2. *If $s > 0$, then $\gamma_M(G) = \gamma_{M'}(G)$, where M' is defined by setting $k'(v) = k(v)$ and $t'(v) = t(v)$ for all vertices v in G except that $t'(x_{i_j}) = \mathrm{R}$ for $1 \leqslant j \leqslant s$.*

Proof. We first notice that $s \leqslant b$ as $k(x) \leqslant |N_G[x]|$.

Since M' is the same as M except resetting some values $t'(x_{i_j})$ to R, any M'-dominating set of G is also an M-dominating set of G. Consequently, $\gamma_M(G) \leqslant \gamma_{M'}(G)$.

On the other hand, suppose D is a minimum M-dominating set of G. By the condition for M-domination, $|N_G[x] \cap D| \geqslant k(x)$ and so $|\widehat{B} \cap D| \geqslant k(x) - |\{x_i \in N_G[x]\colon\ t(x_i) = \mathrm{R}\}| = s$. Let $\widehat{B} \cap D$ include some $x_{i'_1}, x_{i'_2}, \ldots, x_{i'_s}$, where $i'_1 > i'_2 > \cdots > i'_s$. Then $i_j \geqslant i'_j$ and so $N_G[x_{i'_j}] \subseteq N_G[x_{i_j}]$ for $1 \leqslant j \leqslant s$. Consider the set $D' = (D - \{x_{i'_1}, x_{i'_2}, \ldots, x_{i'_s}\}) \cup \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$. It is then straightforward to check that D' is an M'-dominating set of G. Therefore, $\gamma_M(G) \geqslant \gamma_{M'}(G)$. □

By resetting the values of $t(x_{i_j})$ to R for $1 \leqslant j \leqslant s$, we may now assume that $s \leqslant 0$.

Lemma 3. *Suppose $s \leqslant 0$. Let $G' = (V', E')$ be the graph obtained from G by deleting the vertex x.*

(1) *If $t(x) = \mathrm{R}$ or $k(x_{i^*}) = |N_G[x_{i^*}]|$ for some $1 \leqslant i^* \leqslant r$, then $\gamma_M(G) = \gamma_{M'}(G') + 1$, where M' is the restriction of M on V' with the modification on $k'(x_i) = \max\{k(x_i) - 1, 0\}$ for $2 \leqslant i \leqslant r$.*

(2) *If $t(x) = \mathrm{B}$ and $k(x_i) < |N_G[x_i]|$ for all $1 \leqslant i \leqslant r$, then $\gamma_M(G) = \gamma_{M'}(G')$, where M' is the restriction of M on V'.*

Proof. (1) Suppose D' is a minimum M'-dominating set of G'. Let $D = D' \cup \{x\}$. As M' is the restriction of M on V' with the modification on $k'(x_i) = \max\{k(x_i) - 1, 0\}$ for $2 \leqslant i \leqslant r$, the conditions (MD1) and (MD2) for G follows from that for G' except condition (MD2) for all $x_i \in N_G[x]$. However, $k(x_i) \leqslant k'(x_i) + 1$ and $D \backslash D' = \{x\} \subseteq N_G[x_i]$ imply that condition (MD2) for G holds for all $x_i \in N_G[x]$. Hence D is an M-dominating set of G and so $\gamma_M(G) \leqslant |D| = |D'| + 1 = \gamma_{M'}(G') + 1$.

On the other hand, suppose D is a minimum M-dominating set of G. By the assumption that $t(x) = \mathrm{R}$ or $|N_G[x_{i^*}]| = k(x_{i^*})$ for some $1 \leqslant i^* \leqslant r$, we have $x \in D$. Let $D' = D \backslash \{x\}$. Again, as M' is the restriction of M on V' with the modification on $k'(x_i) = \max\{k(x_i) - 1, 0\}$ for $2 \leqslant i \leqslant r$, D' is an M'-dominating set of G'. Hence $\gamma_{M'}(G') \leqslant |D'| = |D| - 1 = \gamma_M(G) - 1$.

Both inequalities prove part (1) of the lemma.

(2) Suppose D' is a minimum M'-dominating set of G'. As M' is the restriction of M on V' and $s \leqslant 0$, D' is also an M-dominating set of G. Hence $\gamma_M(G) \leqslant |D'| = \gamma_{M'}(G')$.

On the other hand, suppose D is a minimum M-dominating set of G. For the case when $x \notin D$, we have that D is also an M'-dominating set of G' and so $\gamma_{M'}(G') \leqslant |D| = \gamma_M(G)$. For the case when $x \in D$, choose a minimum index $i > 1$ with some vertex $y \in N_G[x_i] \backslash D$. Let $D' = (D - \{x\}) \cup \{y\}$ if there is such y and $D' = D - \{x\}$ otherwise. It is again the case that D' is an M'-dominating set of G' and so $\gamma_{M'}(G') \leqslant |D'| \leqslant |D| = \gamma_M(G)$.

Both inequalities prove part (2) of the lemma. □

Based on the lemmas above, we have Algorithm MDSC for the M-domination problem in strongly chordal graphs.

Algorithm MDSC. Find a minimum M-dominating set of a strongly chordal graph.

Input. A strongly chordal graph G with a strong ordering $v_1, v_2, \ldots, v_n$, in which each vertex v has a label $M(v) = (t(v), k(v))$ where $t(v) \in \{\mathrm{B}, \mathrm{R}\}$ and $k(v)$ is a nonnegative integer.

Output. A minimum M-dominating set D of G.

Method.

for $i = 1$ **to** n **do**
  if $(k(v_i) > |N_G[v_i]|)$ **then stop** and return the infeasibility of the problem;
$D \leftarrow \emptyset$;
for $i = 1$ **to** n **do**
  { let $s = k(v_i) - |\{y \in N_i[v_i]\colon t(y) = \mathrm{R}\}|$;
   if $(s > 0)$ **then**
    { let $\{y \in N_i[v_i]\colon t(y) = \mathrm{B}\} = \{v_{i_1}, v_{i_2}, \ldots, v_{i_b}\}$ where $i_1 > i_2 > \cdots > i_b$;
     for $j = 1$ **to** s **do** $t(v_{i_j}) = \mathrm{R}$; }
   if $(t(v_i) = R$ **or** $|N_i[v_{i^*}]| = k(v_{i^*})$ for some $i^* \in N_i[v_i])$ **then**
    { $k(v) = \max\{k(v) - 1, 0\}$ for all $v \in N_i[v_i]$;
     $D \leftarrow D \cup \{v_i\}$; }
  }

Theorem 4. *Algorithm MDSC produces a minimum M-dominating set of a strongly chordal graph if a strong ordering is provided.*

4. NP-completeness results

This section establishes NP-complete results for the k-tuple domination problem in split graphs (a subclass of chordal graphs) and in bipartite graphs. The transformation is from the vertex cover problem, which is known to be NP-complete. The *vertex cover problem* is for a given nontrivial graph and a positive integer k to answer if there is a vertex set of size at most k such that each edge of the graph has at least one end vertex in this set.

Theorem 5. *For any fixed positive integer k, the k-tuple domination problem is* NP*-complete for split graphs.*

Proof. The k-tuple domination problem for split graphs is NP-complete as we may transform the vertex cover problem to it as follows.

Given a nontrivial graph $G = (V, E)$, construct the graph $G' = (V', E')$ with vertex set

$$V' = V \cup S \cup E$$

where $S = \{s_1, s_2, \ldots, s_{k-1}\}$, and edge set

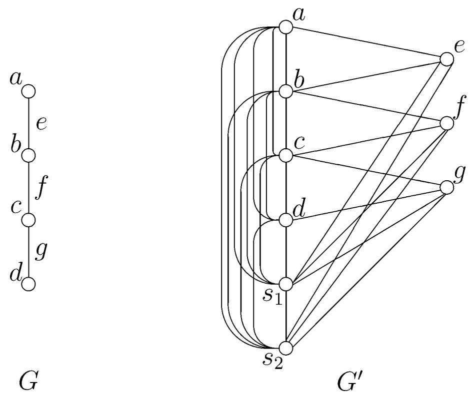

Fig. 1. A transformation to a split graph when $k = 3$.

$$E' = \{uv\colon u \neq v \text{ in } V \cup S\} \cup \{ve\colon v \in V,\ e \in E,\ v \in e\} \cup \{s_i e\colon s_i \in S,\ e \in E\}.$$

Notice that G' is a split graph whose vertex set is the disjoint union of the clique $V \cup S$ and the independent set E. Fig. 1 shows an example of the transformation.

It is straightforward to show that G has a vertex cover of size α if and only if G' has a k-tuple dominating set of size $\alpha + k - 1$. For the detail of the proof, see [16]. □

We also have the NP-complete result for bipartite graphs.

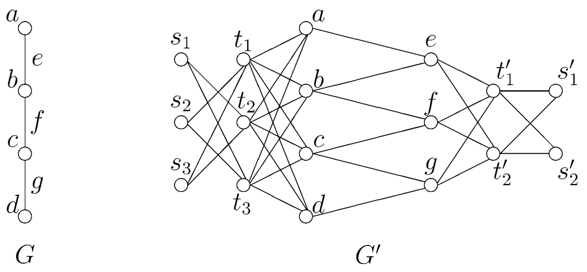

Fig. 2. A transformation to a bipartite graph when $k = 3$.

Theorem 6. *For any fixed positive integer k, the k-tuple domination problem is* NP*-complete for bipartite graphs.*

Proof. Again, we may transform the vertex cover problem to the k-tuple domination problem for bipartite graphs as follows.

Given a nontrivial graph $G = (V, E)$, construct the graph $G' = (V', E')$ with vertex set

$$V' = S \cup T \cup V \cup E \cup T' \cup S'$$

where $S = \{s_1, s_2, \ldots, s_k\}$, $T = \{t_1, t_2, \ldots, t_k\}$, $T' = \{t'_1, t'_2, \ldots, t'_{k-1}\}$, $S' = \{s'_1, s'_2, \ldots, s'_{k-1}\}$; and edge set

$$\begin{aligned} E' = & \left\{s_i t_j\colon\ s_i \in S,\ t_j \in T,\ i \neq j\right\} \\ & \cup \left\{t_j v\colon\ t_j \in T,\ v \in V\right\} \\ & \cup \left\{ve\colon\ v \in V,\ e \in E,\ v \in e\right\} \\ & \cup \left\{e t'_j\colon\ e \in E,\ t'_j \in T'\right\} \\ & \cup \left\{t'_j s'_i\colon\ t'_j \in T',\ s'_i \in S'\right\}. \end{aligned}$$

Notice that G' is a bipartite graph whose vertex set V' is the disjoint union of two independent sets $S \cup V \cup T'$ and $T \cup E \cup S'$. Fig. 2 shows an example of the transformation.

It can be shown that G has a vertex cover of size α if and only if G' has a k-tuple dominating set of size $\alpha + 4k - 2$. For the detail, see [16]. □

5. Conclusion

The main purpose of this paper is to establish a linear-time algorithm for the k-tuple domination problem in strongly chordal graphs. NP-complete results for the problem are also shown for split graphs (a subclass of chordal graphs) and for bipartite graphs.

A slightly different concept called k-total domination was studied in [14]. A vertex set D is called a *k-total dominating set* of $G = (V, E)$ if every vertex x in V is dominated by at least k vertices in $D \setminus \{x\}$. Note that a k-total dominating set is a k-tuple dominating set but not the converse. The arguments for the k-tuple domination problem can be modified to solve the k-total domination problem for strongly chordal graphs, by replacing closed neighbors in most of the statements concerning neighbors. In addition, Fink and Jacobson [7] introduced the concept of k-domination. A *k-dominating set* is a vertex set D of V such that every vertex in $V \setminus D$ is dominated by at least k vertices in D. The method in this paper is not able to modified straightforwardly into one for k-domination. A new approach is desirable.

We close the paper with two questions concerning multiple domination. First, what are the complexities of the k-tuple domination for other subclasses of perfect graphs. Secondly, characterize the relationship between the three above mentioned multiple domination problems.

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