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Note

A note on the Gallai–Roy–Vitaver Theorem ☆

Gerard J. Chang[a,*], Li-Da Tong[b], Jing-Ho Yan[c], Hong-Gwa Yeh[d]

[a] Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
[b] Department of Applied Mathematics, National Sun Yet-sen University, Kaohsiung 804, Taiwan
[c] Department of Mathematics, Aletheia University, Tamsui 251, Taiwan
[d] Department of Applied Mathematics, National University of Kaoshiung, Kaohsiung 811, Taiwan

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Abstract

The well-known theorem by Gallai–Roy–Vitaver says that every digraph G has a directed path with at least $\chi(G)$ vertices; hence this holds also for graphs. Li strengthened the digraph result by showing that the directed path can be constrained to start from any vertex that can reach all others. For a graph G given a proper $\chi(G)$-coloring, he proved that the path can be required to start at any vertex and visit vertices of all colors. We give a shorter proof of this. He conjectured that the same holds for digraphs; we provide a strongly connected counterexample. We also give another extension of the Gallai–Roy–Vitaver Theorem on graphs.

Keywords: Coloring; k-Coloring; Chromatic number; Path; Tournament

In this note we consider graphs and digraphs. Notation and terminology used here are standard and can be found in [5].

Let G be a graph or digraph with vertex set $V(G)$ and edge set $E(G)$. A *k-coloring* of G is a mapping $f: V(G) \to \{1, 2, \ldots, k\}$. A k-coloring f is *proper* if $xy \in E(G)$ implies $f(x) \neq f(y)$. A graph or digraph is *k-colorable* if it has a proper k-coloring. The *chromatic number* $\chi(G)$ is the minimum number k such that G is k-colorable.

The problems of determining chromatic numbers and longest paths are important in graph theory and have been well studied. The following well-known result, due to

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* Corresponding author. Tel.: +886-223633860; fax: +886-223677981.
E-mail addresses: gjchang@math.ntu.edu.tw (G.J. Chang), ldtong@math.nsysu.edu.tw (L.-D. Tong), jhyan@email.au.edu.tw (J.-H. Yan), hgyeh@nuk.edu.tw (H.-G. Yeh).

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Gallai [1], Roy [3] and Vitaver [4] independently, gives a relation between the longest paths and the chromatic numbers.

Theorem 1 (Gallai [1], Roy [3] and Vitaver [4]). *Every digraph G has a directed path with at least $\chi(G)$ vertices.*

Corollary 2. *Every graph G has a path with at least $\chi(G)$ vertices.*

Li [2] generalized the Gallai–Roy–Vitaver Theorem by specifying the starting vertex of the directed path.

Theorem 3 (Li [2]). *If G is a digraph in which v is a vertex that can reach all other vertices, then G has a directed path starting at v with at least $\chi(G)$ vertices.*

Corollary 4. *For any vertex v in a connected graph G, there is a path starting at v with at least $\chi(G)$ vertices.*

Corollary 4 gives an affirmative answer to the conjecture in Problem 748 of Written on the Wall (Graffiti). For graphs, Li in fact not only specified the starting vertex of the path, but also considered the colors used in the path. The following theorem answers a question in Graffiti Problem 748.

Theorem 5 (Li [2]). *For any proper $\chi(G)$-coloring of a connected graph G and any vertex $v \in V(G)$, there is a path starting at v whose vertices use all $\chi(G)$ colors.*

At the end of his paper [2], Li gave a conjecture for the digraph version.

Conjecture (Li [2]). For any proper $\chi(G)$-coloring of a digraph G and any vertex $v \in V(G)$ that can reach all other vertices, there is a directed path starting at v whose vertices use all $\chi(G)$ colors.

We first give counterexamples to the conjecture as follows. For every even integer $n \geqslant 6$, consider the digraph G_n with $V(G_n) = \{x_1, x_2, \ldots, x_n\}$ and

$$\begin{aligned} E(G_n) = {} & \{x_1x_2\} \cup \{x_ix_{i-1}, x_ix_{i+1}\colon\ 3 \leqslant i \leqslant n-3, i \text{ odd}\} \cup \{x_{n-1}x_{n-2}, x_{n-1}x_1\} \\ & \cup \{x_nx_j\colon\ 1 \leqslant j \leqslant n-1, j \text{ odd}\} \cup \{x_kx_n\colon\ 1 \leqslant k \leqslant n-1, k \text{ even}\}. \end{aligned}$$

It is easy to see that $\chi(G_n) = 4$, and we may define a proper 4-coloring f_n by

$$f_n(x_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } 4 \leqslant i \leqslant n-1 \text{ even}, \\ 2 & \text{if } i = 2 \text{ or } 4 \leqslant i \leqslant n-1 \text{ odd}, \\ 3 & \text{if } i = 3, \\ 4 & \text{if } i = n. \end{cases}$$

Fig. 1 shows the digraph G_6 and the proper 4-coloring f_6. Although x_n can reach all other vertices, there is no directed path starting at x_n having vertices of all four colors.

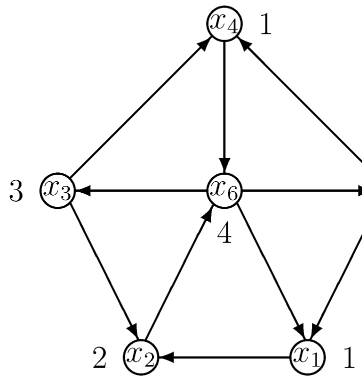

Fig. 1. A counterexample.

Note that these digraphs are strongly connected. Our feeling is that for the conjecture to be true, we need very strong properties for the digraph G. For instance, according to Theorem 3, tournaments have the desired properties.

Next we give a shorter proof for Theorem 5.

Alternative proof for Theorem 5. We shall prove the theorem by induction on $\chi(G)$. The theorem is trivial for $\chi(G)=1$.

Consider a vertex v of a general graph G with $\chi(G)\geqslant 2$. Let f be a proper $\chi(G)$-coloring, and suppose $f(v)=i$. Consider the graph $G-f^{-1}(i)$, which is the disjoint union of components $G_1,G_2,\ldots,G_r$. At least one component G_j has $\chi(G_j)=\chi(G)-1$. The restriction f_j of f on $V(G_j)$ is then a proper $\chi(G_j)$-coloring of G_j. Let P be a shortest path from v to $V(G_j)$, and let w be the vertex at which P arrives in G_j. By the induction hypothesis, G_j has a path Q starting at w and visiting vertices of all colors in $\{1,2,\ldots,\chi(G)\}-\{i\}$ under f_j. The path consisting of P follows by Q is now a path of the desired form. $\square$

The idea used in the proof above for Theorem 5 can be applied to prove the following.

Theorem 6. *Let f be a proper $\chi(G)$-coloring of a connected graph G. For any r distinct colors $c_1,c_2,\ldots,c_r$, there is a path $v_1,v_2,\ldots,v_r$ such that $f(v_i)=c_i$ for $1\leqslant i\leqslant r$.*

Proof. We shall prove the theorem by induction on r. The theorem is trivial for $r=1$. Consider $r\geqslant 2$. A path $v_1,v_2,\ldots,v_r$ with $f(v_i)=c_i$ for $1\leqslant i\leqslant r$ is called a $(c_1,c_2,\ldots,c_r)$-*path*. Now define

$$A=\{y\in V(G)\colon \text{ there is a } (c_2,c_3,\ldots,c_r)\text{-path starting at } y\}.$$

We may assume that $f^{-1}(c_1)\cup A$ is independent, for otherwise an edge xy with $x\in f^{-1}(c_1)$ and $y\in A$ would yield a $(c_1,c_2,\ldots,c_r)$-path starting at x. Now $G-(f^{-1}(c_1)\cup A)$ has a component G' with $\chi(G')=\chi(G)-1$. Since f is a proper coloring, it uses colors $c_2, \ldots, c_r$ on G'. By the induction hypothesis, G' has a $(c_2, c_3, \ldots, c_r)$-path whose first vertex is in A, which contradicts the definition of A. $\square$

It is obvious that Theorem 6 cannot be strengthened by also specifying the starting vertex of the path, since this would require that in every optimal coloring of every connected graph, every vertex has neighbors of all other colors. The 5-cycle is a counterexample, as is the graph consisting of a triangle and a pendant edge.

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