Mathematics Division, National Center for Theoretical Sciences at Taipei

NCTS/TPE-Math Technical Report 2008-014

$L(h, k)$-labelings of Hamming graphs*

Lian-Hwao Huang[1†] and Gerard J. Chang[123‡]

[1]Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

[2]Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan

[3]National Center for Theoretical Sciences, Taipei Office, Taiwan

May 25, 2007 (revision February 10 and April 20, 2008)

Abstract

Given integers $c \geq 0$ and $h \geq k \geq 1$, a c-$L(h, k)$-labeling of a graph G is a mapping $f : V(G) \to \{0, 1, 2, \ldots, c\}$ such that $|f(u) - f(v)| \geq h$ if $d_G(u, v) = 1$ and $|f(u) - f(v)| \geq k$ if $d_G(u, v) = 2$. The $L(h, k)$-number $\lambda_{h,k}(G)$ of G is the minimum c such that G has a c-$L(h, k)$-labeling. The Hamming graph is the Cartesian product of complete graphs. In this paper, we study $L(h, k)$-labeling numbers of Hamming graphs. In particular, we determine $\lambda_{h,k}(K_n^q)$ for $2 \leq q \leq p$ with $h/k \leq n - q + 1$ or $2 \leq q \leq p$ with $h/k \geq qn - 2q + 2$ or $q = p + 1$ with $h/k \leq n/p$, where p is the minimum prime factor of n.

Keywords. $L(h, k)$-labeling, $L(h, k)$-number, Hamming graph, Cartesian product, complete graph.

*Supported in part by the National Science Council under grant NSC945-2221-M002-013-MY3.

†E-mail: r90221003@ntu.edu.tw

‡E-mail: gjchang@math.ntu.edu.tw.

1 Introduction

The problem of labeling vertices with a condition at distance two was first studied by Griggs and Yeh [10]. It arose from a variation of the channel assignment problem introduced by Hale [11]. Given a number of transmitters (or stations), the object is to assign a channel to each transmitter such that the interference is avoided. In order to reduce the interference, any two "close" transmitters must receive channels at least k apart, and any two "very close" transmitters must receive channels by at least h apart, where $h \geq k$. One can construct an interference graph for this problem so that the transmitters are represented by the vertices of a graph and there is an edge between two "very close" transmitters, while two transmitters are "close" if the corresponding vertices are of distance two.

More precisely, for a given graph G and integers $h \geq k \geq 1$, an $L(h,k)$-*labeling* of G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq h$ if $d_G(u,v) = 1$ and $|f(u) - f(v)| \geq k$ if $d_G(u,v) = 2$, where $d_G(u,v)$ is the distance between u and v in G. For a nonnegative integer c, a c-$L(h,k)$-*labeling* is an $L(h,k)$-labeling such that no label is greater than c. The $L(h,k)$-*labeling number* of G, denoted by $\lambda_{h,k}(G)$, is the smallest c such that G has a c-$L(h,k)$-labeling. A c-$L(h,k)$-labeling f is *optimal* if $c = \lambda_{h,k}(G)$, and in this case f is also called a $\lambda_{h,k}$-*labeling*.

The $L(h,k)$-labeling problem has been studied in [3, 4, 6, 7, 8, 9, 10, 12, 14]. For more information, see the surveys [1, 2, 13]. The purpose of this paper is to study $L(h,k)$-numbers of Hamming graphs described below. The *Cartesian product* of q graphs $G_1, G_2, \ldots, G_q$ is the graph $G_1 \Box G_2 \Box \ldots \Box G_q$ whose vertex set is $\prod_{i=1}^{q} V(G_i)$ and two vertices $(u_1, u_2, \ldots, u_q)$ and $(v_1, v_2, \ldots, v_q)$ are adjacent if and only if $u_j v_j \in E(G_j)$ for some j and $u_i = v_i$ for all other $i \neq j$. See Figure 1 for an example of Cartesian product of two graphs. For the case when $G_i = G$ for all i, we use G^q to denote $G_1 \square G_2 \square \ldots \square G_q$. A *Hamming graph* is the Cartesian product $K_{n_1} \square K_{n_2} \square \ldots \square K_{n_q}$ of complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_q}$.

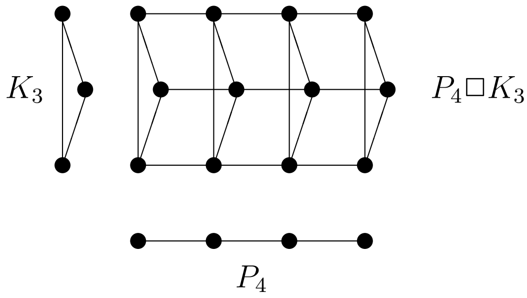

Figure 1: $P_4 \square K_3$.

Georges, Mauro and Stein [9] used a group techniques to determine the $\lambda_{2,1}$-numbers of some special Hamming graphs: If $2 \le q$ and p is a prime, then

$$\lambda_{2,1}(K^q_{p^r}) = p^{2r} - 1 \text{ if} \begin{cases} r \ge 2 \text{ and } q \le p, & (1.1) \\ r = 1 \text{ and } q < p. & (1.2) \end{cases}$$

They also obtained that if p is prime then

$$\lambda_{1,1}(K_p^p) = \lambda_{1,1}(K_p^{p+1}) = p^2 - 1. \tag{2}$$

For the general $L(h,k)$-labelings they proved that for integer $h \ge k \ge 1$,

$$\lambda_{h,k}(K_n \square K_m) = \begin{cases} (mn-1)k, & \text{if } 2 \le n < m \text{ and } h/k \le n; & (3.1) \\ (m-1)h + (n-1)k, & \text{if } 2 \le n < m \text{ and } h/k > n; & (3.2) \\ (n^2-1)k, & \text{if } 2 \le n = m \text{ and } h/k \le n-1; & (3.3) \\ (n-1)h + (2n-2)k, & \text{if } 2 \le n = m \text{ and } h/k > n-1. & (3.4) \end{cases}$$

Besides some interesting results for $\lambda_{1,1}(G)$ on some Hamming graphs G, Georges and Mauro [8] also established that for integers $h \ge k \ge 1$,

$$\lambda_{h,k}(K_n^3) \begin{cases} = (n^2-1)k, & \text{if } n \text{ is odd and } h/k \le n-2; & (4.1) \\ \le (n-1)(h+3k), & \text{if } n \text{ is odd and } h/k \ge n-2; & (4.2) \\ = (n-1)(h+3k), & \text{if } n \text{ is odd and } h/k \ge 3n-4; & (4.3) \\ = (n^2-1)k, & \text{if } n \text{ is even and } h/k \le n/2; & (4.4) \\ \le (n-1)h + n(2n-1)k, & \text{if } n \text{ is even.} & (4.5) \end{cases}$$

Erwin et al. [6] determined the $\lambda_{h,k}$-number of the Hamming graph $K_{n_1}\square K_{n_2}\square\ldots\square K_{n_q}$ $q \geq 3$ and relatively prime $n_1, n_2, \ldots, n_q$. Results for $\lambda_{h,k}$-numbers of the Hamming graphs are also obtained when $(h,k) = (2,0)$, $(2,1)$ and (1.1) in [5].

In this paper, we extend these results more generally. In particular, we determine $\lambda_{h,k}(K_n^q)$ for $2 \leq q \leq p$ with $h/k \leq n-q+1$ or $2 \leq q \leq p$ with $h/k \geq qn-2q+2$ or $q = p+1$ with $h/k \leq n/p$, where p is the minimum prime factor of n.

2 $\lambda_{h,k}$-numbers of Hamming graphs K_n^q

In this section, we establish results for the $\lambda_{h,k}$-numbers of Hamming graphs K_n^q, see Theorems 2 and 4. Suppose p is the minimum prime factor of n. Theorem 2 considers the case where $2 \leq q \leq p$, and Theorem 4 considers the case where $q = p+1$ and $h/k \leq n/p$.

To fix notation, let $[n] = \{0, 1, \ldots, n-1\}$. We use $[n]$ as the vertex set of the complete graph K_n, and $[n]^q = \{(a_1, a_2, \ldots, a_q) : a_i \in [n] \text{ for } 1 \leq i \leq q\}$ the vertex set of the Hamming graph K_n^q.

We first establish a lower bound for the result in Theorem 2 in the next lemma.

Lemma 1 *If $n \geq 2$ and $q \geq 2$, then $\lambda_{h,k}(K_n^q) \geq (n-1)(h+qk)$ whenever $h/k \geq qn-2q+2$.*

Proof. Suppose to the contrary that K_n^q has an $L(h,k)$-labeling f using labels only in $[0, (n-1)(h+qk)-1]$. Partition $[0, (n-1)(h+qk)-1]$ into segments $X_1, X_2, \ldots, X_{(n-1)(q+1)}$ such that $|X_{iq+1}| = h$ for $0 \leq i \leq n-2$ and all other $|X_r| = k$. Namely, with $w = h+(q-1)k$ we have

$$X_{iq+j} = \begin{cases} [iw, iw+h-1], & \text{if } 0 \leq i \leq n-2 \text{ and } j = 1; \\ [iw+h+(j-2)k, iw+h+(j-1)k-1], & \text{if } 0 \leq i \leq n-2 \text{ and } 2 \leq j \leq q; \\ [(n-1)w+(j-1)k, (n-1)w+jk-1], & \text{if } i = n-1 \text{ and } 1 \leq j \leq n-1. \end{cases}$$

A segment X_r with $|X_r| = h$ is called a *h-segment*, and with $|X_r| = k$ a *k-segment*.

We claim that no label under f is in a k-segment X_r with $r \le (n-1)q$. Suppose to the contrary that there is a vertex v with $f(v) \in X_r$ for some k-segment X_r with $r \le (n-1)q$. We may assume that r is such a minimum index. Write $r = sq+t$ where $0 \le s \le n-2$ and $2 \le t \le q$. Notice that K_n^q has q subgraphs isomorphic to K_n, each containing v. Each h-segment X_{iq+1} with $0 \le i \le s-1$ contains at most one label used by each copy of K_n and the h-segment X_{sq+1} contains one label from each of at most $t-1$ copies of K_n. Therefore, there are at least $q-t+1$ copies of K_n each contains at least $n-s-1$ vertices with label greater than $f(v)$. Thus, the largest label among those vertices is at least $f(v)+(n-s-1)h+(q-t)k \ge s(h+(q-1)k)+h+(t-2)k+(n-s-1)h+(q-t)k = nh+((s+1)(q-1)-1)k \ge nh+(q-2)k \ge (n-1)(h+qk)$, contradicting that all labels are in $[0,(n-1)(h+qk)-1]$. Hence, no label under f is in a k-segment X_r with $r \le (n-1)q$.

Now, $f^{-1}(X_r)$ contains at most n^{q-1} vertices for each h-segment X_r, and $f^{-1}(X_r)$ contains at most n^{q-2} vertices for each k-segment X_r with $r > (n-1)q$. Therefore, $|V(K_n^q)| \le (n-1)n^{q-1} + (n-1)n^{q-2} = n^q - n^{q-2} < n^q$, a contradiction. $\square$

We now come to the first main result. Notice that (5.1) generalizes (1.1), (1.2), first part of (2), (3.3) and (4.1); (5.2) generalizes (4.2); (5.3) generalizes (4.3). For instance, to see (5.1) implying (1.1), we let $h = 2$, $k = 1$ and $n = p^r$ with $r \ge 2$. In this case, condition "$h/k \le n-q+1$" of (5.1) and condition "$2 \le q \le p$" of Theorem 2 give condition "$2 \le q \le p$" of (1.1). For (5.1) implying (1.2), we choose $h = 2$, $k = 1$ and $n = p^2$ with $r = 1$. In this case, condition "$h/k \le n-q+1$" of (5.1) is just condition "$q < p$" of (1.1). Similarly, we can check implications. We also notice that (5.2) is only an upper bound. The exact value of $\lambda_{h,k}(K_n^q)$ for $n-q+1 < h/k < qn-2q+2$ is desirable.

Theorem 2 *If p is the minimum prime factor of $n \geq 2$ and $2 \leq q \leq p$, then*

$$\lambda_{h,k}(K_n^q) \begin{cases} = (n^2-1)k, & \text{if } h/k \leq n-q+1; \quad (5.1) \\ \leq (n-1)(h+qk), & \text{if } h/k > n-q+1; \quad (5.2) \\ = (n-1)(h+qk), & \text{if } h/k \geq qn-2q+2. \quad (5.3) \end{cases}$$

Proof. For $a = (a_1, a_2, \ldots, a_q) \in V(K_n^q)$, define

$$x(a) = \left(\sum_{i=1}^{q} a_i\right) \bmod n, \quad y(a) = \left(\sum_{i=1}^{q-1} -ia_i\right) \bmod n.$$

For the case of $h/k \leq n-q+1$, consider the mapping f on $V(K_n^q)$ defined by

$$f(a) = (x(a)n + y(a))k.$$

See Figure 2 for an example with $h=3, k=1, n=4, q=2$.

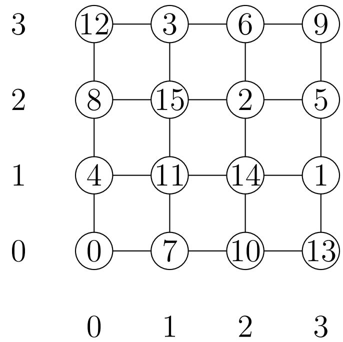

Figure 2: The mapping f on K_4^2.

We shall check that f is an $L(h,k)$-labeling of K_n^q. Suppose $a = (a_1, a_2, \ldots, a_q)$ and $b = (b_1, b_2, \ldots, b_q)$ are two adjacent vertices in K_n^q, say they differ only at position r. It is then the case that $x(a) \neq x(b)$, say $x(a) > x(b)$. For the case of $x(a) \geq x(b)+2$, we have $f(a) - f(b) \geq (n+1)k \geq h$. For the case of $x(a) = x(b)+1$, we have $a_r = (b_r + 1) \bmod n$. If $r \leq q-1$, then $y(a) = (y(b) - r) \bmod n$ and so $f(a) - f(b) \geq (n-r)k \geq (n-q+1)k \geq h$. If $r = q$, then $y(a) = y(b)$, which gives that $f(a) - f(b) = nk \geq h$. We next consider the case when $a = (a_1, a_2, \ldots, a_q)$ and $b = (b_1, b_2, \ldots, b_q)$ are of distance two in K_n^q, say they differ only at positions r and

s with $1 \leq r < s \leq q$. In order to check the distance two condition, we only have to verify that $x(a) \neq x(b)$ or $y(a) \neq y(b)$. Suppose to the contrary that $x(a) = x(b)$ and $y(a) = y(b)$. If $s \leq q-1$, then $a_r + a_s \equiv b_r + b_s \pmod{n}$ and $ra_r + sa_s \equiv rb_r + sb_s \pmod{n}$ lead to $(s-r)a_r \equiv (s-r)b_r \pmod{n}$. Since $1 \leq (s-r) < p$, so $a_r = b_r$, which is impossible. If $s = q$, then $ra_r \equiv rb_r \pmod{n}$. Since $1 \leq r < p$, so $a_r = b_r$, also impossible. Therefore, f is an $L(h,k)$-labeling of K_n^q. This and the fact that the n^2 vertices whose r-th coordinate are 0 for $3 \leq r \leq q$ are pairwise of distance two gives that $(n^2-1)k \leq \lambda_{h,k}(K_n^q) \leq (n^2-1)k$ and so $\lambda_{h,k}(K_n^q) = (n^2-1)k$.

For the case of $h/k > n-q+1$, consider the mapping g on $V(K_n^q)$ defined by

$$g(a) = x(a)(h + (q-1)k) + y(a)k.$$

We shall check that g is an $L(h,k)$-labeling of K_n^q. Suppose $a = (a_1, a_2, \ldots, a_q)$ and $b = (b_1, b_2, \ldots, b_q)$ are two adjacent vertices in K_n^q, say they differ only at position r. It is then the case that $x(a) \neq x(b)$, say $x(a) > x(b)$. For the case of $x(a) \geq x(b) + 2$, we have $g(a) - g(b) \geq 2(h + (q-1)k) - (n-1)k = 2h - (n-1-2q+2) \geq h$. For the case of $x(a) = x(b) + 1$, we have $a_r = (b_r + 1) \bmod n$. If $r \leq q-1$, then $y(a) = (y(b) - r) \bmod n$ and so $g(a) - g(b) \geq (h + (q-1)k) - rk) \geq h$. If $r = q$, then $y(a) = y(b)$, which give that $g(a) - g(b) = h + (q-1)k \geq h$. Since $h + (q-1)k \geq nk$, precisely the same argument as in the last case gives that $|g(a) - g(b)| \geq k$ for vertices a and b of distance two in K_n^q. Hence, $\lambda_{h,k}(K_n^q) \leq (n-1)(h+qk)$ as desired.

The last equality follows from Lemma 1 and the inequality above. □

For the second result, we need the following useful lemma.

Lemma 3 *If p is the minimum prime factor of $n \geq 2$, $1 \leq r \leq p-1$, $1 \leq s \leq n-1$ and $rs \equiv 1 \pmod{n}$, then $s \leq n - n/p$.*

Proof. Suppose to the contrary that $s = n - n/p + t$ for some $1 \leq t < n/p$. Then,

$rs = rn - rn/p + rt \equiv 1 \pmod{n}$ and so $r(n/p - t) + 1 \equiv 0 \pmod{n}$, contradicting $1 \le r(n/p - t) + 1 \le (p-1)(n/p - t) + 1 < n$. □

Then, we have the second main result. Notice that it generalizes the second part of (2) and (4.4).

Theorem 4 *If p is the minimum prime factor of $n \ge 2$, then $\lambda_{h,k}(K_n^{p+1}) = (n^2 - 1)k$ whenever $h/k \le n/p$.*

Proof. For $a = (a_1, a_2, \ldots, a_{p+1}) \in V(K_n^{p+1})$, define

$$x(a) = \left(a_{p+1} + \sum_{i=1}^{p} i a_i\right) \bmod n, \quad y(a) = \left(\sum_{i=1}^{p} -a_i\right) \bmod n.$$

We consider the mapping f on $V(K_n^{p+1})$ defined by

$$f(a) = (x(a)n + y(a))k.$$

We shall check that f is an $L(h,k)$-labeling of K_n^{p+1}. Suppose $a = (a_1, a_2, \ldots, a_{p+1})$ and $b = (b_1, b_2, \ldots, b_{p+1})$ are two adjacent vertices in K_n^{p+1}, say they differ only at position r and let $\Delta_r = (a_r - b_r) \bmod n$. We may assume that $x(a) \ge x(b)$. For the case of $x(a) \ge x(b) + 2$, we have $f(a) - f(b) \ge (n+1)k \ge h$. For the case of $x(a) = x(b) + 1$, we have $r \ne p$ and $s\Delta_r \equiv 1 \pmod{n}$ for some $1 \le s \le p - 1$. By Lemma 3, $f(a) - f(b) \ge (n - n + n/p)k \ge nk/p \ge h$. For the case of $x(a) = x(b)$, we have $r = p$ and $\Delta_r = sn/p \bmod n$ for some $1 \le s < n/p$. Therefore, $f(a) - f(b) \ge nk/p \ge h$.

We next consider the case when $a = (a_1, a_2, \ldots, a_{p+1})$ and $b = (b_1, b_2, \ldots, b_{p+1})$ are of distance two in K_n^{p+1}, say they differ only at positions r and s with $1 \le r < s \le p + 1$. In order to check the distance two condition, we only have to verify that $x(a) \ne x(b)$ or $y(a) \ne y(b)$. Suppose to the contrary that $x(a) = x(b)$ and $y(a) = y(b)$. If $s \le p$, then $a_r + a_s \equiv b_r + b_s \pmod{n}$ and $ra_r + sa_s \equiv rb_r + sb_s \pmod{n}$ lead to

$(s-r)a_r \equiv (s-r)b_r \pmod{n}$. Since $1 \le (s-r) < p$, so $a_r = b_r$, which is impossible. If $s = p+1$, then $a_r = b_r$, also impossible.

Therefore, f is an $L(h,k)$-labeling of K_n^{p+1}. This and the fact that the n^2 vertices whose r-th coordinate are 0 for $3 \le r \le p+1$ are pairwise of distance two gives that $(n^2-1)k \le \lambda_{h,k}(K_n^{p+1}) \le (n^2-1)k$ and so $\lambda_{h,k}(K_n^{p+1}) = (n^2-1)k$. □

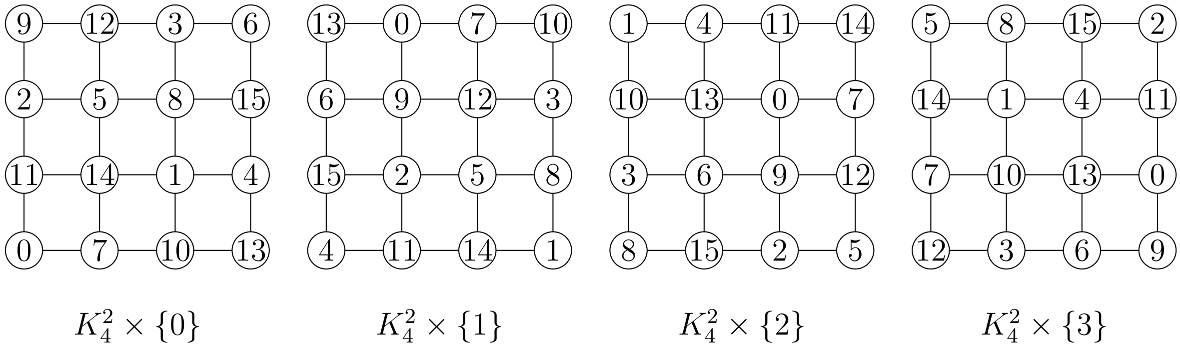

Figure 3: A $\lambda_{2,1}$-labeling of K_4^3.

3 Conclusion remarks and open problem

The purpose of this paper is to study $L(h,k)$-labeling numbers of Hamming graphs. In particular, we determine $\lambda_{h,k}(K_n^q)$ for $2 \le q \le p$ with $h/k \le n-q+1$ or $2 \le q \le p$ with $h/k \ge qn-2q+2$ or $q = p+1$ with $h/k \le n/p$, where p is the minimum prime factor of n. Although these generalize previous results at some degree, results for more general cases are still desirable and to be studied in further.

For instance, in the case when $2 \le q \le p$ and $n-q+1 < h/k < qn-2q+2$, we only have the upper bound $\lambda_{h,k}(K_n^q) \le (n-1)(h+qk)$. Although we believe that this is the exact value, an argument stronger than that in the proof of Lemma 1 is needed to establish the lower bound.

Having Theorem 4, we are still lack of the value of $\lambda_{h,k}(K_n^q)$ for the case when $q = p+1$ and $h/k > n/p$. More generally, results for the case of $q > p+1$ are desirable although it is expected not easy.

Acknowledgements. The authors thank the referees for many constructive suggestions.

References

[1] K. I. Aardal, S. P. M. van Hoesel, A. M. C. A. Koster, C. Mannino and A. Sassano, *Models and solution techniques for frequency assignment problems*, Annals Operations Research 153 (2007) 79–129.

[2] T. Calamoneri, *The L(h,k)-labeling problem: A survey and annotated bibliography*, Comput. J. 49 (2006) 585–608.

[3] G. J. Chang and D. Kuo, *The $L(2,1)$-labeling problem on graphs*, SIAM J. Discretre Math. 9 (1996) 309–316.

[4] G. J. Chang, W.-T. Ke, D. Kuo, D. D.-F. Liu and R. K. Yeh, *On $L(d,1)$-labelings of graphs*, Discrete Math. 220 (2000) 57–66.

[5] G. J. Chang, C. Lu and S. Zhou, *Minimum spans of Hamming graphs under distance-two labellings*, submitted.

[6] D. J. Erwin, J. P. Georges and D. W. Mauro, *On labeling the vertices of products of complete graphs with distance constraints*, Naval Research Logistics 52 (2005) 138–141.

[7] J. P. Georges and D. W. Mauro, *Generalized vertex labelings with a condition at distance two*, Congr. Numer. 109 (1995) 141–159.

[8] J. P. Georges and D. W. Mauro, *Some results on λ_k^j-numbers of the products of complete graphs*, Congr. Numer. 140 (1999) 141–160.

[9] J. P. Georges, D. W. Mauro and M. I. Stein, *Labeling products of complete graphs with a condition at distance two*, SIAM J. Discrete Math. 14 (2000) 28–35.

[10] J. R. Griggs and R. K. Yeh, *Labelling graphs with a condition at distance* 2, SIAM J. Discrete Math. 5 (1992) 586–595.

[11] W. K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE 68 (1980) 1497–1514.

[12] M. A. Whittlesey, J. P. Georges and D. W. Mauro, *On the λ-number of Q_n and related graphs*, SIAM J. Discrete Math. 8 (1995) 499–506.

[13] R. K. Yeh, *A survey on labeling graphs with a condition at distance two*, Discrete Math. 306 (2006) 1217–1231.

[14] S. Zhou, *Labelling Cayley graphs on Abelian groups*, SIAM J. Discrete Math. 19 (2006) 985–1003.