FACTORING THREEFOLD DIVISORIAL CONTRACTIONS TO POINTS

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ABSTRACT. We show that terminal 3-fold divisorial contraction to a point of index > 1 with non-minimal discrepancy may be factored into a sequence of flips, flops and divisorial contractions to a point with minimal discrepancies.

1. INTRODUCTION

In minimal model program, the elementary birational maps consists of flips, flops and divisorial contractions. In dimension three, after the milestone work of Mori (cf. [13]), these maps are reasonably well understood while there are many recent progresses in describing these birational maps explicitly. The geometry of flips and flops in dimension three can be found in the seminal papers of Kollár and Mori (cf. [10, 11, 14]). Divisorial contractions to a curve was studies by Cutkosky and intensively by Tziolas (cf. [2, 15, 16, 17]). Divisorial contractions to points are most well-understood. By results of Hayakawa, Kawakita, and Kawamata(cf. [3, 4, 6, 7, 8, 9]), it is now known that divisorial contractions to higher index points in dimension three are weighted blowups (under suitable embedding) and completely classified. It is expected that all divisorial contractions to points can be realized as weighted blowups.

Let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index $n > 1$ in dimension three. We say that f has minimal discrepancy if the discrepancy of f is the minimal possible $1/n$ (cf. w-morphism in [1]). Divisorial contractions to higher index points with minimal discrepancies play a very interesting role for the following two reasons.

(1) For any terminal singularities $P \in X$ of index $n > 1$, there exists a partial resolution $X_n \to \dots X_0 := X$ such that each X_n has only terminal Gorenstein singularities and each $X_{i+i} \to X_i$

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is a divisorial contraction to a point with minimal discrepancy (cf. [4]).

(2) For any flipping contraction or divisorial contraction to a curve, by taking a divisorial extraction over the highest index point with minimal discrepancy, one gets a factorization into "simpler" birational maps (cf. [1]).

On the other hand, divisorial contractions to points with non-minimal discrepancies are rather special. For example, if $P \in X$ is of type cAx/2, cAx/4 or cD/3, then there is no divisorial contraction with non-minimal discrepancy. The purpose of this note is to show that divisorial contractions to a higher index point with non-minimal discrepancies can be factored into divisorial contractions of minimal discrepancies, flips and flops (cf. [1]).

In fact, let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index $n > 1$. Suppose that the discrepancy of f is $a/n > 1/n$. If Y has only Gorenstein singularities, then by the classification of [13, 2], one has that X is Gorenstein unless $f : Y \to X$ is a divisorial contraction to a quotient singularity $P \in X$ of type $\frac{1}{2}(1,1,1)$ with discrepancy $\frac{1}{2}$. Therefore, we may and do assume that Y has some non-Gorenstein point $Q \in Y$ of index p. We thus consider a divisorial contraction over Q with minimal discrepancy.

Theorem 1.1. *Let $f : Y \to X$ be an extremal contraction to a point $P \in X$ of index $n > 1$ with exceptional divisor E. Let $Q \in Y$ be a point of highest index p in $E \subset Y$ and $g : Z \to Y$ be an extremal extraction with discrepancy $\frac{1}{p}$. Then the relative canonical divisor $-K_{Z/X}$ is nef.*

Notice that the relative Picard number $\rho(Z/X) = 2$. Therefore, we are able to play the so called 2-ray game. As a consequence, there is a flip or flop $Z \dashrightarrow Z^+$. By running the minimal model program of Z^+/X, we have $Z \dashrightarrow Z^\sharp \xrightarrow{g^\sharp} Y^\sharp \xrightarrow{f^\sharp} X$, where $Z \dashrightarrow Z^\sharp$ consists of a sequence of flips and flops, $Z^\sharp \to Y^\sharp$ is a divisorial contraction. Let $F_{Y^\sharp}$ (resp., $F_{Z^\sharp}, E_{Z^\sharp}$) be the proper transform of F (resp. F, E) in $Y^\sharp$ (resp. $Z^\sharp$).

In fact, we have the following more precise description.

Theorem 1.2. *Keep the notation as above. We have that $f^\sharp$ is a divisorial contraction to $P \in X$ with discrepancy $\frac{a'}{n} < \frac{a}{n}$. Moreover, $g^\sharp$ is a divisorial contraction to a singular point $Q' \in F_{Y^\sharp}$ of index p' with discrepancy $\frac{q'}{p'}$. We may rite ${g^\sharp}^* F_{Y^\sharp} = F_{Z^\sharp} + \frac{\mathfrak{q}}{p'} E_{Z^\sharp}$, then*

$$\frac{a}{n} = \frac{a'}{n} \cdot \frac{q'}{p'} + \frac{\mathfrak{q}}{p'}.$$

More specifically, exactly one of the following holds.

(1) *If $P \in X$ is of type other than cE/2, then Q' is a point of index n, and $g^\sharp$ has discrepancy $\frac{a''}{n}$ with $a' + a'' = a$.*

(2) *If $P \in X$ is of type cE/2, then Q' is a point of index $p' = 3$, and $g^\sharp$ has minimal discrepancy $\frac{1}{3}$.*

As an immediate corollary by induction on discrepancy a, we have:

Corollary 1.3. *For any divisorial contraction $Y \to X$ to a point $P \in X$ of index $n > 1$ with discrepancy $\frac{a}{n} > \frac{1}{n}$. There exsits a sequence of birational maps*

$$Y =: X_n \dashrightarrow \dots \dashrightarrow X_0 =: X$$

such that each map $X_{i+1} \dashrightarrow X_i$ is one of the following:

(1) *a divisorial extraction over a point of index $r_i > 1$ with minimal discrepancy $\frac{1}{r_i}$;*
(2) *a divisorial contraction to a point of index $r_i > 1$ with minimal discrepancy $\frac{1}{r_i}$;*
(3) *a flip or flop.*

We now briefly explain the idea. According the 2-ray game, we have the following diagram of birational maps.

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, "g"'] & Z^\sharp \arrow[d, "g^\sharp"] \\ Y \arrow[r] \arrow[d, "f"'] & Y^\sharp \arrow[d, "f^\sharp"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

Notice that in this diagram the order of exceptional divisors of the tower $Z^\sharp \to Y^\sharp \to X$ and $Z \to Y \to X$ are reversed. The usual difficulty to understand the diagram explicitly is that we need to determine the center of $E_{Z^\sharp}$ in $Y^\sharp$.

On the other hand, since $f : Y \to X$ is a weighted blowup, one can embed X into a toric variety $\mathcal{X}_0$ and understand $f : Y \to X$ as the proper transform of a toric weight blowup $\mathcal{X}_1 \to \mathcal{X}_0$, which is nothing but a subdivision of a cone along a vector v_1. If $Z \to Y$ can be realized as the proper transform of a toric weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ over the origin of the standard coordinate charts, then we can view $\mathcal{X}_2 \to \mathcal{X}_1$ as a toric weighted blowup along a vector v_2. Therefore, the tower $\mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$ is obtained by subdivision along vectors v_1 and then v_2.

We may reverse the ordering of v_1, v_2 (under mild combinatorial condition) by considering a tower $\mathcal{X}_2' \to \mathcal{X}_1' \to \mathcal{X}_0$ of toric weighted blowup by subdivision along v_2 and then v_1. The proper transforms of X in this tower then gives $Z' \to Y' \to X$. Clearly, this is a tower reversing the order of exceptional divisors of $Z \to Y \to X$ by construction. Notice that the proper transform Y' and Z' is not necessarily terminal, a priori.

In section 2, we recall and generalize the construction of weighted blowup. We also derive a criterion for $-K_{Z/X}$ being nef. Moreover, we show that if the tower $Z \to Y \to X$ can be embedded into a tower of weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$ and $-K_{Z/X}$ is nef, then the output of 2-ray game coincides with the output by "reversing order of vectors" of the tower of weighted blowups.

In Section 3, we study divisorial contractions with non-minimal discrepancies case by case. We see that the divisorial extraction $Z \to Y$ over a point of index > 1 usually give a tower $Z \to Y \to X$ such that $-K_{Z/X}$ is nef and it can be embedded into a tower of weighted blowups $\mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$ with vectors v_1, v_2. Indeed this is always the case if $Z \to Y$ is a contraction over a point of highest index. The theorems then follows easily.

We always work over complex number field $\mathbb{C}$ and in dimension three. We assume that threefold X, Y are $\mathbb{Q}$-factorial. We freely use the standard notions in minimal model program such as terminal singularities, divisorial contractions, flips, and flops. For the precise definition, we refer to [12].

2. Preliminaries

2.1. **weighted blowups.** We recall the construction of weighted blowups by using the toric language.

Let $N = \mathbb{Z}^d$ be a free abelian group of rank d with standard basis $\{e_1, ..., e_d\}$. Let $v = \frac{1}{n}(a_1, ..., a_d) \in \mathbb{Q}^d$ be a vector. We may assume that $gcd(n, a_1, ..., a_d) = 1$. We consider $\overline{N} := N + \mathbb{Z}v$. Clearly, $N \subset \overline{N}$. Let M (resp. $\overline{M}$) be the dual lattice of N (resp. $\overline{N}$).

Let σ be the cone of first quadrant, i.e. the cone generated by the standard basis $e_1, ..., e_d$ and Σ be the fan consists of σ and all the subcones of σ. We have

$$\begin{aligned} &\mathcal{X}_{N,\Sigma} := \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}^d, \\ &\mathcal{X}_{\overline{N},\Sigma} := \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap \overline{M}] = \mathbb{C}^d/\tfrac{1}{n}(a_1, ..., a_d), \end{aligned}$$

Let $v_1 = \frac{1}{r_1}(b_1, ..., b_d)$ be a primitive vector in $\overline{N}$. We assume that $b_i \in \mathbb{Z}_{>0}$ and $gcd(r_1, b_1, ..., b_d) = 1$. We are interested in the weighted blowup of $o \in \mathbb{C}^d/\frac{1}{n}(b_1, ..., b_d) = \mathcal{X}_{\overline{N},\Sigma} =: \mathcal{X}_0$ with weights $v_1 = \frac{1}{r_1}(b_1, ..., b_d)$ which we describe now.

Let $\overline{\Sigma}$ be the fan obtained by subdivision of Σ along v_1. One thus have a toric variety $\mathcal{X}_{\overline{N},\overline{\Sigma}}$ together with the natural map $\mathcal{X}_{\overline{N},\overline{\Sigma}} \to \mathcal{X}_{\overline{N},\Sigma}$. More concretely, let σ_i be the cone generated by $\{e_1, ..., e_{i-1}, v_1, e_{i+1}, ..., e_d\}$, then

$$\mathcal{X}_1 := \mathcal{X}_{\overline{N},\overline{\Sigma}} = \cup_{i=1}^d \mathcal{U}_i,$$

where $\mathcal{U}_i = \mathcal{X}_{\overline{N},\sigma_i} = \mathrm{Spec}\mathbb{C}[\sigma_i^\vee \cap \overline{M}]$. We always denote the origin of $\mathcal{U}_i$ as Q_i.

2.2. **tower of toric weighted blowups.** Let us look at $\mathcal{U}_i$, which is $\mathcal{X}_{\overline{N},\sigma_i}$. Suppose that there is a primitive vector $v_2 = \sum \frac{c_i}{r_2} e_i \in \overline{N}$ such that v_2 is in the interior of σ_i.

We can write

$$\begin{aligned} v_2 &= \tfrac{1}{n}(c_1e_1 + ... + c_de_d) \\ &= \tfrac{1}{p}(q_1e_1 + q_2e_2 + . + q_iv_1 + .. + q_de_d), \end{aligned}$$

for some $q_i \in \mathbb{Z}_{>0}$.

We denote $w_2 = \frac{1}{p}(q_1, ..., q_d)$ to be the weight of v_2 is the cone σ_i, or simply the weight of v_2 if no confusion is likely. It is convenient to introduce $\widehat{w_2} := \frac{1}{p}(q_1, ..., q_{i-1}, 0, q_{i+1}, ..., q_d)$. Then we have

$$v_2 = \frac{q_i}{p} v_1 + \widehat{w_2}. \qquad \sharp$$

Observation. Keep the notation as above. Notice that if

$$\overline{N} \text{ is generated by } \{v_1, v_2, e_1, ..., \hat{e_i}, ..., e_d\}, \qquad \dagger$$

then $\mathcal{X}_{\overline{N},\sigma_i} \cong \mathbb{C}^d/w_2 = \mathbb{C}^d/\frac{1}{p}(q_1, q_2, ..., q_d)$ has only quotient singularity at Q_i.

We can consider the second weighted blowup with vector v_2. Let $\overline{\overline{\Sigma}}$ be the fan obtained by subdivision of σ_i along v_2. One thus have a toric variety $\mathcal{X}_{\overline{N},\overline{\overline{\Sigma}}}$. Similarly, let τ_j be the cone generated by

$$\begin{cases} \{e_1, ..., e_{j-1}, v_2, e_{j+1}, ..., e_{i-1}, v_1, e_{i+1}, ..., e_d\}, & \text{if } j \neq i \\ \{e_1, ..., e_{i-1}, v_2, e_{i+1}, .., e_d\}, & \text{if } j = i \end{cases}$$

Then

$$\mathcal{X}_2 := \mathcal{X}_{\overline{N},\overline{\overline{\Sigma}}} = (\cup_{j=1}^d \mathcal{V}_j) \bigcup (\cup_{k\neq i}\mathcal{U}_k),$$

where $\mathcal{V}_j = \mathrm{Spec}\mathbb{C}[\tau_j^\vee \cap \overline{M}]$. Let $w_2 = \frac{1}{p}(q_1, ..., q_d)$. Then the weighted blowup $\cup_{j=1}^d \mathcal{V}_j \to \mathcal{U}_i$ with vector v_2 can be conseidered as a weighted blowup with weights w_2.

Definition 2.1. We say that $\mathcal{X}_1 \to \mathcal{X}_0$ is the weighted blowup with with vector v_1 or we say that $\mathcal{X}_1 \to \mathcal{X}_0$ is the weighted blowup with weights $w_1 = \frac{1}{r_1}(c_1, ..., c_d)$. Similarly, we say that $\mathcal{X}_2 \to \mathcal{X}_1$ is the weighted blowup with vector v_2 or with weights $w_2 = \frac{1}{p}(q_1, ..., q_d)$.

Notice that by construction $v_2 = \frac{1}{r_2}(c_1e_1 + ... + c_de_d)$ with $c_i > 0$ for all i. We can consider $\mathcal{X}'_1 \to \mathcal{X}_0$ the weighted blowup with vector v_2, then we have that $\mathcal{X}' = \cup \mathcal{U}'_i = \cup \mathrm{Spec}\mathbb{C}[{\sigma'_1}^\vee \cap \overline{M}]$ with σ'_i the cone generated by $\{e_1, ..., e_{i-1}, v_2, e_{i+1}, ..., e_d\}$. Then clearly,

$$\mathcal{U}'_i = \mathrm{Spec}\mathbb{C}[{\sigma'_i}^\vee \cap \overline{M}] = \mathrm{Spec}\mathbb{C}[\tau_i^\vee \cap \overline{M}] = \mathcal{V}_i.$$

Notice also that the exceptional divisor $\mathcal{F}$ of $\mathcal{X}_2 \to \mathcal{X}_1$ and the exceptional divisor $\mathcal{F}'$ of $\mathcal{X}'_1 \to \mathcal{X}_0$ defines the same valuation given by the cone generated by v_2.

Suppose furthermore that v_1 is in the interior of σ'_k for some k. Then we can consider a weighted blowup $\mathcal{X}'_2 \to \mathcal{X}'_1$ with vector v_1. Notice also that the exceptional divisor $\mathcal{E}$ of $\mathcal{X}_2 \to \mathcal{X}_1$ and the exceptional divisor $\mathcal{E}'$ of $\mathcal{X}'_1 \to \mathcal{X}_0$ defines the same valuation given by the cone generated by v_1.

Remark 2.2. We say that v_1 and v_2 are *interchangeable* if v_2 is in the interior of σ_i for some i and v_1 is in the interior of σ'_k for some k. It is easy to see that v_1, v_2 are interchangeable if $b_j c_l \neq b_l c_j$ for all $j \neq l$.

In this situation, we say that the tower of weighted blowups $\mathcal{X}'_2 \to \mathcal{X}'_1 \to \mathcal{X}_0$ (with vectors v_1, v_2 successively) is obtained by reversing the order of the tower of weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$ with vectors v_2, v_1. We have the following diagram

$$\begin{tikzcd}
\mathcal{X}_2 \arrow[r, dashed] \arrow[d, "v_2"'] & \mathcal{X}'_2 \arrow[d, "v_1"] \\
\mathcal{X}_1 \arrow[d, "v_1"'] & \mathcal{X}'_1 \arrow[d, "v_2"] \\
\mathcal{X}_0 \arrow[r, "="] & \mathcal{X}_0.
\end{tikzcd}$$

2.3. **complete intersections.** The toric variety $\mathcal{X}_0 \cong \mathbb{C}^d/v$ is a quotient by $\mathbb{Z}_n$-action with weights $\frac{1}{n}(a_1, ..., a_d)$. For any semi-invariant $\varphi = \sum \alpha_{i_1,...,i_d} x_1^{i_1} ... x_d^{i_d}$, we define

$$wt_v(\varphi) := \min\{\sum_{j=1}^{d} \frac{a_j}{n} i_j | \alpha_{i_1,...,i_d} \neq 0\}.$$

For any vector $v' \in \overline{N}$, we define $wt_{v'}$ similarly.

Given a cyclic quotient of complete intersection variety, i.e. an embedding $X = (\varphi_1 = \varphi_2 = ... = \varphi_k = 0) \subset \mathbb{C}^d/v = \mathcal{X}_0$, where each φ_i is a semi-invariant. Let $\mathcal{X}_1 \to \mathcal{X}_0$ be a weighted blowup with vector v_1 and exceptional divisor $\mathcal{E}$. Let Y be the proper transform of X in $\mathcal{X}_1$. Then we say that the induce map $\phi : Y \to X$ is the weighted blowup with vector v_1. Note that its exceptional set is $E = \mathcal{E} \cap Y$.

Quite often, we need to embed X into a another ambient space. For example, write $\varphi_k = f_0 + f_1 f_2$ with f_1 being a semi-invariant. We set $v' := v + wt_v(f_1)e_{d+1} = (\frac{a_1}{n}, \ldots, \frac{a_d}{n}, wt_v(f_1))$.

We then consider $X' \subset \mathcal{X}'_0 := \mathbb{C}^{d+1}/v'$ by setting:

$$\begin{cases} \varphi'_j := \varphi_j, & \text{for } 1 \leq j \leq k-1, \\ \varphi'_k := f_0 + x_{d+1} f_2, \\ \varphi'_{k+1} := x_{d+1} - f_1. \end{cases}$$

It is easy to see that $X \cong X'$.

Let $\mathcal{X}_1 \to \mathbb{C}^d/v := \mathcal{X}_0$ be a weighted blowup with weights $v_1 = \frac{1}{r_1}(c_1, ..., c_d)$. We set $v_1' = (\frac{c_1}{r_1}, ..., \frac{c_d}{r_1}, wt_{v_1}(f_1))$ and let $\mathcal{X}_1' \to \mathcal{X}_0'$ be the weighted blowup with weights v_1'. Let Y, Y' be their proper transform in $\mathcal{X}_1, \mathcal{X}_1'$ respectively. Then it is straightforward to check that $Y \cong Y'$ canonically. Indeed, the isomorphism follows from the canonical isomorphism of $Y \cap \mathcal{U}_j \cong Y' \cap \mathcal{U}_j'$ for $j \le d$ and $Y' \cap \mathcal{U}_{d+1}' = \emptyset$.

Definition 2.3. The weighted blowups $Y \to X$ with weights v_1 and $Y' \to X'$ with weight v_1' are said to be compatible if the equations and weights are defined as above.

2.4. **2-ray game.** Turning back to the study of terminal threefolds. We may write $P \in X$ as $(\varphi = 0) \subset \mathbb{C}^4/v_0$ (resp. $\varphi_1 = \varphi_2 = 0 \subset \mathbb{C}^5/v_0$). By a weighted blowup $f : Y \to X$ with weights $v_1 = \frac{1}{r_1}(b_1, ..., b_4)$ (resp. $v_1 = \frac{1}{r_1}(b_1, ..., b_5)$), we denote the standard coordinate chart as $U_i = \mathcal{U}_i \cap Y$, $i = 1, \ldots, 4$ (resp. $i =, \ldots, 5$) and let Q_i be the origin of $\mathcal{U}_i$.

Given a divisor D on any of the birational model, adding a subscript, e.g. D_X, D_Y, will denote its proper transform in X, Y respectively (if its center is a divisor). Similarly for a 1-cycle l.

Let us consider a divisorial contraction $f : Y \to X$ to a point of index r with discrepancy $\frac{a}{n} > \frac{1}{n}$. Let E be the exceptional divisor of f.

Suppose that $Q_i \subset Y$ is point of index $p > 1$. We consider $g : Z \to Y$ be a divisorial contraction with discrepancy $\frac{1}{p}$. Let F be the exceptional divisor of g. We may write $g^*E = E_Z + \frac{\mathfrak{q}}{p}F$.

Let $D_0 \neq E$ be a divisor on Y passing through Q_i such that $l_0 := D_0 \cdot E$ is irreducible (possibly non-reduced). Let $D_{0,X}, D_{0,Z}$ be its proper transform on X, Z respectively. Notice that we have

$$f^*D_{0,X} = D_0 + \frac{c_0}{n}E, \quad g^*D_0 = D_{0,Z} + \frac{q_0}{p}F$$

for some $c_0, q_0 \in \mathbb{Z}_{>0}$.

Notice also that $l_{0,Z} = D_{0,Z} \cdot E_Z$. Clearly, we have

$$g^*l_0 = l_{0,Z} + \frac{q_0}{p}l_F,$$

as a 1-cycle, where $l_{0,Z}$ is the proper transform and $l_F := F \cdot E_Z$.

It is easy to see that

$$l_0 \cdot K_Y = D_0 \cdot E \cdot K_Y = \frac{-ac_0}{n^2}E^3 < 0.$$

We also have

$$l_{0,Z} \cdot K_Z = D_{0,Z} \cdot E_Z \cdot K_Z = D_j \cdot E \cdot K_Y + \frac{\mathfrak{q}q_0}{p^3}F^3 = \frac{-ac_0}{n^2}E^3 + \frac{\mathfrak{q}q_0}{p^3}F^3. \quad (1)$$

Now for any curve $l \subset E$. Since $\rho(Y/X) = 1$, we have that l is proportional to l_0 as a 1-cycle. In other words, for any divisor D on Y,

$$l \cdot D = \alpha l_0 \cdot D,$$

for some α. We set $c = \alpha c_0$ (not necessarily an integer). Therefore, $l \cdot K_Y = \frac{-ac}{n^2} E^3$.

We can write $g^* l = l_Z + \frac{q}{p} l_F$ for $\rho(Z/X) = 2$ and the cone of curves clearly generated by l_Z and l_F (note that we did not assume that q is an integer here). Similar computation shows that

$$l_Z \cdot K_Z = l \cdot K_Y + \frac{q_i q}{p^3} F^3 = \frac{-ac}{n^2} E^3 + \frac{\mathfrak{q} q}{p^3} F^3. \tag{2}$$

Notice that

$$l \cdot_E l_0 = l \cdot_Y D_0 = \frac{c}{c_0} l_0 \cdot D_0 = \frac{c}{c_0} D_0^2 E = \frac{c}{c_0} \frac{c_0^2}{r^2} E^3 = \frac{cc_0}{r^2} E^3.$$

Also this quantity can be computed by

$$g^* l \cdot_{E_Z} g^* l_0 = g^* l \cdot_Z g^* D_0 = l_Z \cdot D_{0,Z} + \frac{q \mathfrak{q} q_0}{p^3} F^3 = l_Z \cdot_{E_Z} l_{0,Z} + \frac{q \mathfrak{q} q_0}{p^3} F^3.$$

If $l \neq l_0$, then $l_Z \cdot_{E_Z} l_{0,Z} \geq 0$. So we have

$$\frac{cc_0}{n^2} E^3 \geq \frac{q \mathfrak{q} q_0}{p^3} F^3.$$

Compare with (2), we have that for $l \neq l_0$,

$$q_0 l_Z \cdot K_Z \leq \frac{c}{n^2} (c_0 - a q_0) E^3. \tag{3}$$

We thus conclude the following criterion.

Proposition 2.4. *Let $D_0 \neq E$ be a divisor on Y passing through a point Q_i of index p such that $l_0 := D_0 \cdot E$ is irreducible (possibly non-reduced). Let $g : Z \to Y$ be a extremal contraction to Q_i. Let $D_{0,X}, D_{0,Z}$ be the proper transform of D_0 on X, Z respectively. We write*

$$f^* D_{0,X} = D_0 + \frac{c_0}{n} E, \quad g^* D_0 = D_{0,Z} + \frac{q_0}{p} F, \quad g^* E = E_Z + \frac{\mathfrak{q}}{p} F$$

for some $c_0, q_0, \mathfrak{q} \in \mathbb{Z}_{>0}$. Then $-K_{Z/X}$ is nef if the following inequalities holds:

$$\begin{cases} T(f, g, D_0) := \frac{-ac_0}{n^2} E^3 + \frac{\mathfrak{q} q_0}{p^3} F^3 \leq 0, \\ c_0 - a q_0 \leq 0. \end{cases}$$

Indeed, one has a more effective way of calculation by using the "general elephant", if its restriction is irreducible. Let $\Theta \in |-K_Y|$ be an elephant and $\theta = \Theta|_E$. We have

$$\begin{aligned} &g^* \Theta = \Theta_Z + \tfrac{1}{p} F, \\ &f^* \Theta_X = \Theta + \tfrac{a}{n} E, g^* E = E_Z + \tfrac{\mathfrak{q}}{p} F. \end{aligned}$$

Suppose that θ is irreducible, then one has that $-K_{Z/X}$ is nef if

$$T(f, g) := \theta_Z \cdot K_Z = \frac{-a^2}{n^2} E^3 + \frac{\mathfrak{q}}{p^3} F^3 \leq 0$$

since the second inequality holds automatically.

Suppose now that $-K_{Z/X}$ is nef, then we can play the so-called "2-ray game" as in [1]. We have $Z \dashrightarrow Z^\sharp \to Y^\sharp \to X$, where $Z \dashrightarrow Z^\sharp$ consists of a sequence of flips and flops, $g^\sharp : Z^\sharp \to Y^\sharp$ is a divisorial contraction.

Proposition 2.5. *Keep the notation as above. We have $g^\sharp$ contracts $E_{Z^\sharp}$ and $f^\sharp$ is a divisorial contraction to $P \in X$ contracting $F_{Y^\sharp}$.*

Proof. Since there are only two exceptional divisors $E_{Z^\sharp}$ and $F_{Z^\sharp}$ on $Z^\sharp$ over X. Suppose on the contrary that $g^\sharp$ contracts $F_{Z^\sharp}$. Then $E_{Y^\sharp}$ is the only exceptional on $Y^\sharp/X$. Moreover, $\rho(Y^\sharp/X) = 1$. We thus have $Y^\sharp \cong Y$ for E and $E_{Y^\sharp}$ clearly defines the same valuation. Then one sees that Z/Y has exceptional divisor F and $Z^\sharp/Y$ has exceptional divisor $F_{Z^\sharp}$ which again defines the same valuation. Hence $Z \cong Z^\sharp$, which is absurd.

Notice that $\rho(Y^\sharp/X) = 1$, $Y^\sharp$ is terminal $\mathbb{Q}$-factorial and $F_{Y^\sharp}$ is the support of the exceptional set. It suffices to show that $K_{Y^\sharp/X}$ is $-f^\sharp$-ample. Let $\gamma \subset F_{Y^\sharp}$ be a curve. Pick any very ample divisor H on $Y^\sharp$, then we have ${f^\sharp}^* H_X = H + \mu F_{Y^\sharp}$ for some $\mu > 0$. Intersect with γ, we have

$$0 = \gamma \cdot {f^\sharp}^* H_X = \gamma \cdot H + \mu\gamma \cdot F_{Y^\sharp}.$$

Hence $\gamma \cdot F_{Y^\sharp} < 0$. Now

$$\gamma \cdot K_{Y^\sharp} = \gamma \cdot a(F_{Y^\sharp}, X) F_{Y^\sharp} = \gamma \cdot a(F, X) F_{Y^\sharp} < 0,$$

for the discrepancy of F over X is positive and depends only on the its valuation. $\square$

2.5. **weighted blowups and 2-ray game.** We fix an embedding $P \in X \hookrightarrow \mathcal{X}_0$ such that the divisorial contraction $f : Y \to X$ is given by the weighted blowup $\mathcal{X}_1 \to \mathcal{X}_0$ with weights v_1. That is, Y is the proper transform of X in $\mathcal{X}_1$. Let $g : Z \to Y$ be a divisorial contraction with minimal discrepancy over a point Q_i of index $p > 1$

Suppose that, under such embedding, the following hypotheses holds.

Hypothesis $\flat$.

(1) The divisorial extraction $g : Z \to Y$ is given by a weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ over a point Q_i with vector v_2.
(2) The vectors v_1, v_2 are interchangeable (cf. Remark 2.2).
(3) $-K_{Z/X}$ is nef.

Then we have the following diagram.

$$\begin{tikzcd}
Z^\sharp \arrow[d, "g^\sharp"'] & Z \arrow[l, dashed] \arrow[r, hook] \arrow[d, "g"'] & \mathcal{X}_2 \arrow[r, dashed] \arrow[d, "v_2"'] & \mathcal{X}_2' \arrow[d, "v_1"] & Z' \arrow[l, hook'] \arrow[d, "g'"] \\
Y^\sharp \arrow[d, "f^\sharp"'] & Y \arrow[r, hook] \arrow[d, "f"'] & \mathcal{X}_1 \arrow[d, "v_1"'] & \mathcal{X}_1' \arrow[d, "\eta"', "v_2"] & Y' \arrow[l, hook'] \arrow[d, "f'"] \\
X & X \arrow[l, "="'] \arrow[r, hook] & \mathcal{X}_0 \arrow[r, "="] & \mathcal{X}_0 & X, \arrow[l, hook']
\end{tikzcd}$$

where $Z^\sharp \to Y^\sharp \to X$ is the output of the two-rays game and Z', Y' are proper transform of X in $\mathcal{X}'_2, \mathcal{X}'_1$ respectively.

Theorem 2.6. *Keep the notation as above and suppose that Hypothesis $\flat$ holds. Then $Y^\sharp \cong Y'$ and $Z^\sharp \cong Z'$. In particular, both $f^\sharp$ and $g^\sharp$ are weighted blowups and both f' and g' are divisorial contractions to a point.*

Proof. Let $\mathcal{F}$ be the exceptional divisor of $\mathcal{X}_2 \to \mathcal{X}_1$. It is the excetional divisor induced by the vector v_2. Hence its proper transform $\mathcal{F}'$ in $\mathcal{X}'_1$ is the exceptional divisor of $\eta : \mathcal{X}'_1 \to \mathcal{X}_0$. Recall that by the construction in Subsection 2.2, there is a canonical isomorphism $\mathcal{V}_i \cong \mathcal{U}'_i$ for some i, where $\mathcal{V}_i \subset \mathcal{X}_2$ and $\mathcal{U}'_i \subset \mathcal{X}'_1$ are coordinate charts. Surely, we have an induced isomorphism $Z \cap \mathcal{V}_i \cong Y' \cap \mathcal{U}'$. Since F is irreducible and

$$F \cap \mathcal{V}_i = (\mathcal{F} \cdot Z) \cap \mathcal{V}_i \cong (\mathcal{F}' \cdot Y') \cap \mathcal{U}'_i.$$

It follows that $F_{Y'} := \mathcal{F}' \cdot Y'$ is irreducible, which coincides with the exceptional set. On the other hand, the proper transform of F in $Y^\sharp$ is $F_{Y^\sharp}$, which is the exceptional divisor of $f^\sharp$. One sees immediately that $F_{Y'}$ and $F_{Y^\sharp}$ define the same valuation in the function field.

Note that $-\mathcal{F}'$ is clearly η-ample. It follows that $-F_{Y'}$ is f'-ample. Hene we have

$$Y^\sharp = \mathrm{Proj}(\oplus_{m\geq 0} f^\sharp_* \mathcal{O}(-mF_{Y^\sharp})) \cong \mathrm{Proj}(\oplus_{m\geq 0} f'_* \mathcal{O}(-mF_{Y'})) = Y'.$$

The proof for $Z^\sharp \cong Z'$ is similar. $\square$

3. CASE STUDIES

In this section we study divisorial contractions to a higher index point with non-minimal discrepancy case by case. For each case, we consider the extraction over a higher index point. We shall show that the Hypothesis $\flat$ holds for all tower by extracting over a highest index point and for some other extraction over another higher index point. Hence, in particular, Theorem 1.1 follows.

Moreover the output of 2-ray game and interchanging vectors of weighted blowups coincide. Hence we end up with a diagram for each case, where every vertical map is a weighted blowup. Theorem 1.2 then follows by checking the diagram for each case.

3.1. **discrpancy=4/2 over a cD/2 point.** Let $Y \to X$ be a divisorial contraction to a cD/2 point $P \in X$ with discrepancy 2. By Kawakita's work (cf. [8]), it is known that there exists an embedding

$$\begin{cases} \varphi_1 : x_1^2 + x_4 x_5 + p(x_2, x_3, x_4) = 0, \\ \varphi_2 : x_2^2 + q(x_1, x_3, x_4) + x_5 = 0 \end{cases}$$

with $v = \frac{1}{2}(1,1,1,0,0)$. Also f is the weighted blowup with weights $v_1 = (4l+1, 4l, 2, 1, 8l+1)$ or $(4l, 4l-1, 2, 1, 8l-1)$.

We treat this case in greater detail. The remaining cases can be treated similarly. Note that we can write $p(x_2,x_3,x_4) = x_4 p_1(x_2,x_3,x_4) + p_0(x_2,x_3)$. Therefore, replacing x_5 by $x_5 + p_1(x_2,x_3,x_4)$, we may assume that $\varphi_1 = x_1^2 + x_4x_5 + p(x_2,x_3)$.
Case 1. $v_1 = (4l+1, 4l, 2, 1, 8l+1)$.
Note that $wt_{v_1}(p(x_2,x_3)) \geq 8l+1$, $wt_{v_1}(q(x_1,x_3,x_4)) \geq 8l$.
Step 1. We search for points in Y with index > 1. This can only happen over Q_i. Clearly, $Q_1, Q_2 \notin Y$.

We first look at Q_3. By computation of local charts, one sees that $Q_3 \in \mathcal{X}_1$ is a quotient singularity of type $\frac{1}{4}(1,2,1,3,3)$.
Claim 1. $Q_3 \notin Y$ and $x_3^{4l} \in \varphi_2$.
To see this, according to Kawakita's description, there is only one non-hidden non-Gorenstein singularity and also the hidden singularities has index at most 2. Hence $Q_3 \notin Y$. In other words, one must have either $x_3^{4l+1} \in \varphi_1$ or $x_3^{4l} \in \varphi_2$. Note that $x_3^{4l+1} \notin \varphi_1$ otherwise φ_1 is not a semi-invariant. We thus conclude that $x_3^{4l} \in \varphi_2$.

We can see that $Q_5 \in \mathcal{X}_1$ is a quotient singularity of type $\frac{1}{2(8l+1)}(6l+1, 10l+1, 1, 12l+2, 4l)$ with index $2(8l+1)$. We set $w_2 = \frac{1}{2(8l+1)}(6l+1, 10l+1, 1, 12l+2, 4l)$ so that $v_2 = \frac{1}{2}(2l+1, 2l+1, 1, 2, 4l)$.
Remark. The point $Q_4 \in Y$ is a "hidden" cD/2 point (see [7, p.68]). By the classification of Hayakawa (cf. [4]), any divisorial contraction $g: Z \to Y$ has the property that $g^*E = E_Z + \frac{t}{2}F$ with $t > 0$ even. Therefore, $a(F,X) = \frac{t}{2}2 + \frac{1}{2} > 2$. Hence our theorem does not hold for arbitrary extraction over a point Q of index $r > 1$.
Step 2. The weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ with weights w_2 gives a divisorial contraction $g: Z \to Y$ of discrepancy $\frac{1}{2(8l+1)}$.
To see this, note that the local equation of Q_5 is given by

$$\begin{cases} \overline{\varphi_1} : \overline{x_1}^2 + \overline{x_4} + \overline{p}(\overline{x_2}, \overline{x_3}) = 0, \\ \overline{\varphi_2} : \overline{x_2}^2 + \overline{q}(\overline{x_1}, \overline{x_3}, \overline{x_4}) + \overline{x_5} = 0. \end{cases}$$

We have natural isomorphism between $o \in \mathbb{C}^3/\frac{1}{2(8l+1)}(6l+1, 10l+1, 1) =: \mathcal{Y}_1$ and $Q_5 \in \mathbb{C}^5/w_2$. The only extremal extraction over o with discrepancy $\frac{1}{2(8l+1)}$ is the Kawamata blowup $\mathcal{Y}_2 \to \mathcal{Y}_1$, which is the weighted blowup with weights $\overline{w_2} = \frac{1}{2(8l+1)}(6l+1, 10l+1, 1)$. Since $\overline{x_3}^{4l} \in \overline{\varphi_2}$, one sees that

$$\begin{cases} wt_{w_2}(\overline{x_4}) = wt_{\overline{w_2}}(\overline{x_1}^2) = wt_{\overline{w_2}}(\overline{x_1}^2 + \overline{p}(\overline{x_2}, \overline{x_3})), \\ wt_{w_2}(\overline{x_5}) = wt_{\overline{w_2}}(\overline{x_3}^{4l}) = wt_{\overline{w_2}}(\overline{x_2}^2 + \overline{q}(\overline{x_1}, \overline{x_3}, \overline{x_4})). \end{cases}$$

Therefore, the weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ with weights w_2 and $\mathcal{Y}_2 \to \mathcal{Y}_1$ are compatible (cf. Subsection 2.3). In particular, the only divisorial contraction $g: Z \to Y$ of discrepancy $\frac{1}{2(8l+1)}$ is obtained by weighted blowup with weights w_2 (with vector v_2). This verifies Hypothesis $\flat(1)$. The hypothesis $\flat(2)$ can be verified trivially.

Step 3. We now checked the numerical conditions for 2-ray game. By Kawakita's Table (cf. [7, Table 1,2,3]), we have

$$E^3 = \frac{2}{2(8l+1)}, \quad F^3 = \frac{(2(8l+1))^2}{(6l+1)(10l+1)}.$$

Note that the exceptional divisor E can be realized as a $\mathbb{Z}_2$-quotient of complete intersection

$$\tilde{E} := (\varphi_{1,8l+2} = \varphi_{2,8l} = 0) \subset \mathbb{P}(4l+1, 4l, 2, 1, 8l+1),$$

where $\varphi_{i,k}$ denotes the homogeneous part of φ_i of v_1-weight $k/2$. Indeed, if we pick $D_{0,X} = (x_3 = 0)$, which is an elephant in $|-K_X|$, we have that $E \cap D_0$ is defined by $\mathbb{Z}_2$-quotient of the complete intersection

$$\begin{cases} x_3 = 0, \\ \varphi_{1,8l+2|x_3=0} = x_1^2 + x_4x_5, \\ \varphi_{2,8l|x_3=0} = x_2^2 + q_{8l}(x_1, 0, x_4). \end{cases}$$

If $q_{8l}(x_1, 0, x_4)$ is not a perfect square, then this is clearly irreducible. If $q_{8l}(x_1, 0, x_4)$ is a perfect square, then this is reducible on $\tilde{E}$ but irreducible on E after the $\mathbb{Z}_2$-quotient.

Therefore, we can simply check

$$T(f,g) = \frac{1}{2(8l+1)}(-8 + \frac{4l}{(6l+1)(10l+1)}) < 0$$

to conclude that $-K_Z/X$ is nef. This verifies Hypothesis $\flat(3)$.

Step 4. The weighted blowup $\mathcal{X}' \to \mathcal{X}_0$ with vector v_2 gives a divisorial contraction $f' : Y' \to X$ of discrepancy $\frac{1}{2}$.

This follows from Theorem 2.6. In fact, we can check this directly as well by considering a re-embedding $\overline{X} \subset \mathbb{C}^4/\frac{1}{2}(1,1,1,0)$ defined by

$$\varphi : x_1^2 + x_2^2x_4 + q(x_1, x_3, x_4)x_4 + p(x_2, x_3, x_4)$$

with $x_3^{4l}x_4 \in \varphi$. Let $\overline{v_2} = \frac{1}{2}(2l+1, 2l+1, 1, 2)$, then one sees that the weighted blowup $Y' \to X$ with weight v_2 is compatible with weighted blowup of $\overline{Y} \to \overline{X}$ with weigh $\overline{v_2}$. It is easy to see that $wt_{v_1}(p) \geq 8l+1$ implies that $wt_{\overline{v_2}}(p) > 2l$ and $wt_{v_1}(q) \geq 8l$ implies that $wt_{\overline{v_2}}(qx_4) \geq 2l+1$. Therefore, the weighted blowup $\overline{Y} \to \overline{X}$ with weight $\overline{v_2}$ is indeed the weighted blowup given in Proposition 5.8 of [4], which is a divisorial contraction with minimal discrepancy $\frac{1}{2}$. Hence so is $Y' \to X$.

Step 5. One sees that $v_1 = \frac{6l+1}{2}e_1 + \frac{6l-1}{2}e_2 + \frac{3}{2}e_3 + v_2 + \frac{12l+2}{2}e_5$. Therefore, one consider the weighted blowup $\mathcal{X}'_2 \to \mathcal{X}'_1$ with weights $w'_2 = \frac{1}{2}(6l+1, 6l-1, 3, 2, 12l+2)$ over $Q'_4 \in \mathcal{X}'_1$. Let Z' be the proper transform in $\mathcal{X}'_2$. Notice that $Z' \to Y'$ is a divisorial contraction over Q'_4 with discrepancy $\frac{3}{2}$. This is indeed the map in Case 1 of Subsection 3.2 (after re-embedding into $\mathbb{C}^4/v$ as in Step 4.)

We summarize this case into following diagram.

$$\begin{tikzcd}
Z \arrow[rr, "\dashrightarrow"] \arrow[d, "wt=w_2", "\frac{1}{2(8l+1)}"'] & & Z' \arrow[d, "wt=w_2'", "\frac{3}{2}"'] \\
Q_5 \in Y \arrow[d, "wt=w_1", "\frac{4}{2}"'] & & Y' \ni Q_4' \arrow[d, "wt=w_1'", "\frac{1}{2}"'] \\
X \arrow[rr, "="] & & X
\end{tikzcd}$$

Where

$$\begin{array}{ll}
w_1 = v_1 = (4l+1, 4l, 2, 1, 8l+1), & w_1' = v_2 = \frac{1}{2}(2l+1, 2l+1, 1, 2, 4l), \\
w_2 = \frac{1}{2(8l+1)}(6l+1, 10l+1, 1, 12l+2, 4l), & w_2' = \frac{1}{2}(6l+1, 6l-1, 3, 2, 12l+2).
\end{array}$$

It is easy to verify the condition $\sharp$ that

$$v_2 = \frac{4l}{2(8l+1)} v_1 + \widehat{w_2}, \quad v_1 = \widehat{w_2'} + v_2.$$

Case 2. $v_1 = (4l, 4l-1, 2, 1, 8l-1)$.
We first look at Q_3, which is a quotient singularity of type $\frac{1}{4}(2, 3, 1, 3, 1)$ in $\mathcal{X}_1$.
Claim. $Q_3 \notin Y$ and $x_3^{4l} \in \varphi_1$.
To see this, according to Kawakita's description, there is only one non-hidden non-Gorenstein singularity and also the hidden singularities has index at most 2. Hence $Q_3 \notin Y$. In other words, one must have either $x_3^{4l} \in \varphi_1$ or $x_3^{4l-1} \in \varphi_2$. Note that $x_3^{4l-1} \notin \varphi_2$ otherwise φ_2 is not a semi-invariant. We thus conclude that $x_3^{4l} \in \varphi_1$.

Next notice that $Q_5 \in \mathcal{X}_1$ is a quotient singularity of type $\frac{1}{2(8l-1)}(10l-1, 6l-1, 1, 4l, 12l-2)$. We set $w_2 = \frac{1}{2(8l-1)}(10l-1, 6l-1, 1, 4l, 12l-2)$ so that $v_2 = \frac{1}{2}(6l+1, 6l-1, 3, 2, 12l-2)$.

As before, the weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ with vector v_2 gives a divisorial contraction $g : Z \to Y$ of discrepancy $\frac{1}{2(8l-1)}$, which is compatible with the Kawamata blowup. This can be seen by examining the local equation at Q_5 and the weights as in Case 1.

$$\left\{ \begin{array}{l} \overline{x_1}^2 + \overline{x_4} + \overline{p}(\overline{x_2}, \overline{x_3}, \overline{x_4}) = 0, \\ \overline{x_2}^2 + \overline{q}(\overline{x_1}, \overline{x_3}, \overline{x_4}) + \overline{x_5} = 0 \end{array} \right.$$

We now checked the numerical conditions for 2-ray game. We have

$$E^3 = \frac{2}{2(8l-1)}, \quad F^3 = \frac{(2(8l-1))^2}{(6l-1)(10l-1)},$$

and

$$T(f, g) = \frac{1}{2(8l-1)}(-8 + \frac{2}{10l-1}) < 0.$$

We pick $D_{0,X} = (x_3 = 0)$, which is an elephant in $|-K_X|$. One sees that $E \cap D_0$ is defined by $\mathbb{Z}_2$-quotient of the complete intersection

$$\begin{cases} x_3 = 0, \\ \varphi_{1,8l|x_3=0} = x_1^2 + x_4x_5, \\ \varphi_{2,8l-2|x_3=0} = x_2^2 + q_{8l-2}(x_1, 0, x_4). \end{cases}$$

Same argument as in Case 1 shows that $D_0 \cap E$ is irreducible. Therefore, we can simply check

$$T(f,g) = \frac{1}{2(8l-1)}(-8 + \frac{2}{10l-1}) < 0.$$

to conclude that $-K_Z/X$ is nef. This verifies Hypothesis $\flat(3)$.

Hence $-K_Z/X$ is nef.

The weighted blowup $\mathcal{X}_1' \to \mathcal{X}_0$ with vector v_2 gives a divisorial contraction $f' : Y' \to X$ of discrepancy $\frac{3}{2}$. This can be seen to be a compatible re-embedding of Kawakita's description by eliminating x_5.

One sees that $v_1 = \frac{2l-1}{2}e_1 + \frac{2l-1}{2}e_2 + \frac{1}{2}e_3 + v_2 + \frac{4l}{2}e_5$. Therefore, one consider the weighted blowup $\mathcal{X}_2' \to \mathcal{X}_1'$ with weights $w_2' = \frac{1}{2}(2l-1, 2l-1, 1, 2, 4l)$ over $Q_4' \in \mathcal{X}_1'$. Let Z' be the proper transform in $\mathcal{X}_2'$, then one can easily check that $Z' \to Y'$ is a divisorial contraction over Q_4' with discrepancy $\frac{1}{2}$.

We summarize this case into following diagram.

$$\begin{array}{ccc} Z & \overset{--\to}{\longrightarrow} & Z' \\ {\scriptstyle \frac{1}{2(8l-1)}}\Big\downarrow{\scriptstyle wt=w_2} & & {\scriptstyle \frac{1}{2}}\Big\downarrow{\scriptstyle wt=w_2'} \\ Q_5 \in Y & & Y' \ni Q_4' \\ {\scriptstyle \frac{4}{2}}\Big\downarrow{\scriptstyle wt=w_1} & & {\scriptstyle \frac{3}{2}}\Big\downarrow{\scriptstyle wt=w_1'} \\ X & \overset{=}{\longrightarrow} & X \end{array}$$

Where

$$\begin{aligned} w_1 &= v_1 = (4l, 4l-1, 2, 1, 8l-1), \\ w_2 &= \tfrac{1}{2(8l-1)}(10l-1, 6l-1, 1, 1, 4l, 12l-2), \\ w_1' &= v_2 = \tfrac{1}{2}(6l+1, 6l-1, 3, 2, 12l-2), \\ w_2' &= \tfrac{1}{2}(2l-1, 2l-1, 1, 2, 4l). \end{aligned}$$

3.2. **discrepancy=$a/2$ over a cD/2 point.** Let $Y \to X$ be a divisorial contraction to a cD/2 point $P \in X$ with discrepancy $\frac{a}{2}$. This was classified by Kawakiata into two cases (cf. [7, Theorem 1.2.ii]).

Case 1. In the case (a), the local equation is given by

$$\varphi : x_1^2 + x_2^2x_4 + x_1x_3q(x_3^2, x_4) + \lambda x_2x_3^{2\alpha-1} + p(x_3^2, x_4) = 0 \subset \mathbb{C}^4/v$$

with $v = \frac{1}{2}(1,1,1,0)$ and f is the weighted blowup with weights $v_1 = \frac{1}{2}(r+2, r, a, 2)$, where $r+1 = 2ad$ and both a, r are odd. Notice that $wt_{v_1}(\varphi) = r+1$ and as observed in [1], we have that $x_3^{4d} \in p(x_3^2, x_4)$.

There are two quotient singularities Q_1, Q_2 of index $r+2, r$ respectively.

Subcase 1. We first take $g : Z \to Y$ the Kawamata blowup at Q_1, which is of type $\frac{1}{r+2}(4d, 1, r+2-4d)$. We set $w_2 = \frac{1}{r+2}(4d, 4d, 1, r+2-4d)$ so that the weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ with weights w_2 is compatible with g.

One has

$$E^3 = \frac{4(r+1)}{ar(r+2)}, \quad F^3 = \frac{(r+2)^2}{4d(r+2-4d)}.$$

In this case, the naive choose of $D_{0,X} = (x_3 = 0) \in |-K_X|$ is reducible. We therefore pick $D_{0,X} = (x_4 = 0)$ instead. It is elementary to check that $\varphi_{2l+2|x_4=0} = x_3^{4d}$. Hence $E \cap D_0$ is irreducible. We have $c_0 = 2, q_0 = r+2-4d$, hence $c_0 - 4q_0 < 0$ and

$$T(f, g, D_0) = \frac{1}{r+2}(-\frac{2(r+1)}{r} + 1) < 0.$$

Therefore $-K_{Z/X}$ is nef and Hypothesis $\flat$ holds.

We summarize this case into following diagram.

$$\begin{array}{ccc} Z & \xrightarrow{\dashrightarrow} & Z' \\ {\scriptstyle \frac{1}{r+2}}\big\downarrow {\scriptstyle wt=w_2} & & {\scriptstyle \frac{a-2}{2}}\big\downarrow {\scriptstyle wt=w_2'} \\ Q_1 \in Y & & Y' \ni Q_4' \\ {\scriptstyle \frac{a}{2}}\big\downarrow {\scriptstyle wt=w_1} & & {\scriptstyle \frac{2}{2}}\big\downarrow {\scriptstyle wt=w_1'} \\ X & \xrightarrow{=} & X \end{array}$$

Where

$$\begin{array}{ll} w_1 = v_1 = \frac{1}{2}(r+2, r, a, 2), & w_1' = v_2 = (2d, 2d, 1, 1) \\ w_2 = \frac{1}{r+2}(4d, 4d, 1, r+2-4d), & w_2' = \frac{1}{2}(r+2-4d, r-4d, a-2, 2). \end{array}$$

Notice also that f' is a divisorial contraction of the same type over a cD/2 point with smaller discrepancy $\frac{a-2}{2}$, where $r+1-4d = 2d(a-2)$. The map g' is a contraction with discrepancy 1 which is in Case 1 of Subsection 3.4.

Subcase 2. If we take $g : Z \to Y$ to be the awamata blowup at Q_2, which is a quotient singularity of type $\frac{1}{r}(4d, r-4d, 1)$. We set $w_2 = \frac{1}{r}(4d, r-4d, 1, 4d)$ so that the weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ with weights w_2 is compatible with g.

One has

$$E^3 = \frac{4(r+1)}{ar(r+2)}, \quad F^3 = \frac{r^2}{4d(r-4d)}.$$

We pick $D_{0,X} = (x_4 = 0)$ as in Subcase 1, then we have $c_0 = 2, q_0 = 4d$ and

$$T(f, g) = \frac{1}{r}(-\frac{2(r+1)}{r+2} + 1) < 0.$$

Therefore $-K_{Z/X}$ is nef and hence Hypotheis $\flat$ hold.

We summarize this case into following diagram.

$$\begin{array}{ccc} Z & \stackrel{--\rightarrow}{\longrightarrow} & Z' \\ {\scriptstyle \frac{1}{r}}\Big\downarrow{\scriptstyle wt=w_2} & & {\scriptstyle \frac{2}{2}}\Big\downarrow{\scriptstyle wt=w'_2} \\ Q_2 \in Y & & Y' \ni Q'_4 \\ {\scriptstyle \frac{a}{2}}\Big\downarrow{\scriptstyle wt=w_1} & & {\scriptstyle \frac{a-2}{2}}\Big\downarrow{\scriptstyle wt=w'_1} \\ X & \stackrel{=}{\longrightarrow} & X \end{array}$$

Where

$$\begin{array}{ll} w_1 = v_1 = \frac{1}{2}(r+2, r, a, 2), & w'_1 = v_2 = \frac{1}{2}(r+2-4d, r-4d, a-2, 2) \\ w_2 = \frac{1}{r}(4d, r-4d, 1, 4d), & w'_2 = (2d, 2d, 1, 1). \end{array}$$

Notice also that g' is a divisorial contraction of the same type over a cD/2 point with smaller discrepancy $\frac{a-2}{2}$, where $r+1-4d = 2d(a-2)$. The map f' is a contraction with discrepancy 1 which is in Case 1 of Subsection 3.4.

Case 2. In the case (b), the local equation is given by

$$\left\{ \begin{array}{l} \varphi_1 = x_4^2 + x_2x_5 + p(x_1, x_3) = 0 \\ \varphi_2 = x_2x_3 + x_1^{2d+1} + q(x_1, x_3)x_1x_3 + x_5 = 0. \end{array} \right\} \subset \mathbb{C}^5/v,$$

with $v = \frac{1}{2}(1,1,0,1,1)$ and f is a weighted blowup with weights $v_1 = \frac{1}{2}(a, r, 2, r+2, r+4)$ with $r+2 = (2d+1)a$. Notice that a is allowed to be even in this case.

There are quotient singularities Q_2, Q_5 of index $r, r+4$ respectively.

Subcase 1. We first consider the extraction $Z \to Y$ over Q_5, which is a quotient singularity of type $\frac{1}{r+4}(1, r-2d+3, 2d+1)$. We set $w_2 := \frac{1}{r+4}(1, 4d+2, r-2d+3, 2d+1, 2d+1)$, then its give rise to a weighted blowup compatible with Kawamata blowup $g : Z \to Y$.

We check that

$$E^3 = \frac{4(r+2)}{ar(r+4)}, \quad F^3 = \frac{(r+4)^2}{(2d+1)(r-2d+3)}.$$

We pick $D_{0,X} = (x_3 = 0)$ in this case. Then it is elementary to check that $D_0 \cap E$ is irreducible. We have $c_0 = 2, q_0 = r-2d+3, q_i := q_5 = 2d+1$ and

$$T(f, g, D_0) = \frac{1}{r+4}(-\frac{2(r+2)}{r} + 1) < 0.$$

We summarize this case into following diagram.

$$\begin{array}{ccc} Z & \xrightarrow{\dashrightarrow} & Z' \\ {\scriptstyle\frac{1}{r+4}}\Big\downarrow{\scriptstyle wt=w_2} & & {\scriptstyle\frac{a-1}{2}}\Big\downarrow{\scriptstyle wt=w_2'} \\ Q_5 \in Y & & Y' \ni Q_3' \\ {\scriptstyle\frac{a}{2}}\Big\downarrow{\scriptstyle wt=w_1} & & {\scriptstyle\frac{1}{2}}\Big\downarrow{\scriptstyle wt=w_1'} \\ X & \xrightarrow{=} & X \end{array}$$

Where

$$\begin{aligned} w_1 &= v_1 = \tfrac{1}{2}(a, r, 2, r+2, r+4), \\ w_2 &= \tfrac{1}{r+4}(1, 4d+2, r-2d+3, 2d+1, 2d+1), \\ w_1' &= v_2 = \tfrac{1}{2}(1, 2d+1, 2, 2d+1, 2d+1), \\ w_2' &= \tfrac{1}{2}(a-1, r-2d-1, 2, r-2d+1, r-2d+3). \end{aligned}$$

Notice also that g' is a divisorial contraction of the same type over a cD/2 point with smaller discrepancy $\frac{a-1}{2}$, where $r-2d+1 = (2d+1)(a-1)$. The map f' is a contraction with discrepancy $\frac{1}{2}$ which is a compatible weighted blowup of [4, Proposition 5.8] by eliminating x_5.

Subcase 2. One can also consider $Z \to Y$ be the Kawamata blowup over Q_2, which is a quotient singularity of type $\frac{1}{r}(1, r-2d-1, 2d+1)$. We set $w_2 := \frac{1}{r}(1, r-2d-1, 2d+1, 2d+1, 4d+2)$ so that the weighted blowup is compatible with the Kawamata blowup g.

We check that

$$E^3 = \frac{4(r+2)}{ar(r+4)}, \quad F^3 = \frac{r^2}{(2d+1)(r-2d-1)}.$$

We still pick $D_{0,X} = (x_3 = 0)$ in this case which is known to be irreducible. We have $c_0 = 2, q_0 = 2d+1, q_i := r-2d-1$ and hence

$$T(f,g) = \frac{1}{r}\left(-\frac{2(r+2)}{r+4} + 1\right) < 0.$$

Therefore, Hypothsis $\flat$ holds.

We summarize this case into following diagram.

$$\begin{array}{ccc} Z & \xrightarrow{\dashrightarrow} & Z' \\ {\scriptstyle\frac{1}{r}}\Big\downarrow{\scriptstyle wt=w_2} & & {\scriptstyle\frac{1}{2}}\Big\downarrow{\scriptstyle wt=w_2'} \\ Q_2 \in Y & & Y' \ni Q_3' \\ {\scriptstyle\frac{a}{2}}\Big\downarrow{\scriptstyle wt=w_1} & & {\scriptstyle\frac{a-1}{2}}\Big\downarrow{\scriptstyle wt=w_1'} \\ X & \xrightarrow{=} & X \end{array}$$

Where

$$\begin{aligned}
&w_1 = v_1 = \tfrac{1}{2}(a, r, 2, r+2, r+4),\\
&w_2 = \tfrac{1}{r}(1, r-2d-1, 2d+1, 2d+1, 4d+2),\\
&w_1' = v_2 = \tfrac{1}{2}(a-1, r-2d-1, 2, r-2d+1, r-2d+3),\\
&w_2' = \tfrac{1}{2}(1, 2d+1, 2, 2d+1, 2d+1).
\end{aligned}$$

3.3. **discrepancy 2/2 to a cE/2 point.** In this case, by [5, Theorem 1.2], the local equation is

$$\varphi : x_4^2 + x_1^3 + x_2^4 + x_3^8 + ... = 0 \subset \mathbb{C}^4/v,$$

with $v = \frac{1}{2}(0, 1, 1, 1)$.

By Hayakawa's result [5], we know that $Y \to X$ is given by weighted blowup with vector $v_1 = (3, 2, 1, 4)$. There is a quotient singularity Q_1 of index 6.

Remark. There is another quotient singularity R_3 of index 2 in the fixed locus of $\mathbb{Z}_2$ action on U_3, which is not Q_3.

We can take $w_2 = \frac{1}{6}(2, 5, 1, 1)$, then $v_2 = \frac{1}{2}(2, 3, 1, 3)$. We pick $D_{0,X} = (x_3 = 0) \in |-K_X|$ and it is easy to see that $D_0 \cap E$ is $\mathbb{Z}_2$ quotient of $(x_4^2 + x_2^4 = 0) \subset \mathbb{P}(3, 2, 1, 4)$, which is irreducible. We also checked that

$$E^3 = \frac{1}{6}, \quad F^3 = \frac{36}{5}, \quad T(f, g) = \frac{-1}{10} < 0.$$

We summarize this case into following diagram.

$$\begin{array}{ccc}
Z & \overset{-\,-\,\rightarrow}{\longrightarrow} & Z' \\
{\scriptstyle \frac{1}{6}}\big\downarrow {\scriptstyle wt=w_2} & & {\scriptstyle \frac{1}{3}}\big\downarrow {\scriptstyle wt=w_2'} \\
Q_1 \in Y & & Y' \ni Q_2' \\
{\scriptstyle \frac{2}{2}}\big\downarrow {\scriptstyle wt=w_1} & & {\scriptstyle \frac{1}{2}}\big\downarrow {\scriptstyle wt=w_1'} \\
X & \overset{=}{\longrightarrow} & X
\end{array}$$

Where

$$\begin{aligned}
&w_1 = v_1 = (3, 2, 1, 4), \quad && w_1' = v_2 = \tfrac{1}{2}(2, 3, 1, 3)\\
&w_2 = \tfrac{1}{6}(2, 5, 1, 1), && w_2' = \tfrac{1}{3}(5, 4, 1, 6).
\end{aligned}$$

Notice that $f' : Y' \to X$ is the weighted blowup with vector v_2 with discrepancy $\frac{1}{2}$ as in [5, Theorem 10.41]. The point $Q_2' \in Y'$ is a cD/3 point with local equation

$$\overline{x_4}^2 + \overline{x_1}^3 + \overline{x_2}^3 + \overline{x_3}^8\overline{x_2} + ... = 0 \subset \mathbb{C}^4/v,$$

with $v = \frac{1}{3}(2, 1, 1, 0)$. Hence $Z' \to Y'$ is the weighted blowup with weights w_2' with discrepancy $\frac{1}{3}$ as in [5, Theorem 9.25].

3.4. **discrepancy 2/2 to a cD/2 point.** There are three cases to consider according Hayakawa's classification [5, Theorem 1.1]. Note that the case of Theorem 1.1.(iii) was treated in Subsection 3.2 already.
Case 1. The case of Theorem 1.1.(i) in [5].
In this case, the local equation is

$$x_1^2 + x_2^2 x_4 + s(x_3,x_4)x_2x_3x_4 + r(x_3)x_2 + p(x_3,x_4) = 0 \subset \mathbb{C}^4/v,$$

with $v = \frac{1}{2}(1,1,1,0)$. The map $f : Y \to X$ is given by weighted blowup with vector $v_1 = (2l, 2l, 1, 1)$. Moreover, $wt_{v_1}(\varphi) = 2l$ and $x_3^{4l} \in p(x_3, x_4)$.

There is a singularity Q_2 of type $cA/4l$ with $aw = 2$. The local equation in U_2 is given by

$$\overline{x_1}^2 + \overline{x_2x_4} + \overline{x_3}^{4l} + ... = 0 \subset \mathbb{C}^4/\frac{1}{4l}(0, 2l-1, 1, 2l+1).$$

Since $\overline{x_3}^{4l}$ appears in the equation, in terms of the terminology as in [3, §6], one has $\tau - wt(\overline{x_3}^{4l}) = 1$. This implies that there is only one weighted blowup $Z \to Y$ with minimal discrepancy $\frac{1}{4l}$ which is given by the weight $w_2 = \frac{1}{4l}(4l, 2l-1, 1, 2l+1)$.

We pick $D_{0,X} = (x_3 = 0) \in |-K_X|$ and it is easy to see that $D_0 \cap E$ is $\mathbb{Z}_2$ quotient of $(x_1^2 + a_{0,4l}x_4^{4l} = 0) \subset \mathbb{P}(3,2,1,4)$, where $a_{0,4l}$ denotes the coefficient. In any event, this is irreducible.

We checked that

$$E^3 = \frac{2}{4l}, \quad F^3 = \frac{(4l)^2}{(2l+1)(2l-1)}, \quad T(f,g) = \frac{1}{4l}(-2 + \frac{1}{2l+1}) < 0.$$

Hence Hypothesis $\flat$ holds.

Hence we can summarize this case into following diagram.

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, "\frac{1}{4l}"', "wt=w_2"] & Z' \arrow[d, "\frac{1}{2}"', "wt=w'_2"] \\ Q_2 \in Y \arrow[d, "\frac{2}{2}"', "wt=w_1"] & Y' \ni Q'_4 \arrow[d, "\frac{1}{2}"', "wt=w'_1"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

Where

$$\begin{array}{ll} w_1 = v_1 = (2l, 2l, 1, 1), & w'_1 = v_2 = \frac{1}{2}(2l+1, 2l-1, 1, 2) \\ w_2 = \frac{1}{4l}(4l, 2l-1, 1, 2l+1), & w'_2 = \frac{1}{2}(2l-1, 2l+1, 1, 2). \end{array}$$

In this case, both f' and g' are divisorial contractions to a cD/2 point as in [4, Proposition 5.8].
Case 2. The case of Theorem 1.1.(i') in [5].
In this case, the local equation is

$$\varphi : x_1^2 + x_2x_3x_4 + x_2^4 + x_3^{2b} + x_4^c = 0 \subset \mathbb{C}^4/v,$$

with $b \geq 2, c \geq 4$ and $v = \frac{1}{2}(1,1,1,0)$. The map $f : Y \to X$ is given by weighted blowup with vector $v_1 = (2,2,1,1)$. Moreover, $wt_{v_1}(\varphi) = 4$.

There is a singularity Q_2 of type $cA/4$ with local equation in U_2 is given by

$$\overline{x_1}^2 + \overline{x_3x_4} + \overline{x_2}^4 + \overline{x_3}^{2b}\overline{x_2}^{2b-4} + \overline{x_4}^c\overline{x_2}^{c-4} = 0 \subset \mathbb{C}^4/\frac{1}{4}(0,1,1,3).$$

Since $\overline{x_2}^4$ appears in the equation, one has $\tau - wt = 1$. This implies that there is only one weighted blowup $Z \to Y$ with minimal discrepancy $\frac{1}{4}$ which is given by the weight $w_2 = \frac{1}{4}(4,1,1,3)$.

We pick $D_{0,X} = (x_3 = 0) \in |-K_X|$ again and it is easy to see that $D_0 \cap E$ is $\mathbb{Z}_2$ quotient of $(x_1^2 + \delta_{4,c}x_4^c = 0) \subset \mathbb{P}(3,2,1,4)$, where $\delta_{4,c}$ is the Kronecker's delta symbol. In any event, this is irreducible.

Then the invariant and diagram is exactly the same as the $l = 1$ in Case 1. For reference, we have

$$E^3 = \frac{2}{4}, \quad F^3 = \frac{4^2}{3l}, \quad T(f,g) = \frac{1}{4}(-2 + \frac{1}{3}) < 0.$$

We summarize the result into following diagram.

$$\begin{array}{ccc} Z & \overset{-\,-\rightarrow}{\longrightarrow} & Z' \\ {\scriptstyle \frac{1}{4}}\Big\downarrow{\scriptstyle wt=w_2} & & {\scriptstyle \frac{1}{2}}\Big\downarrow{\scriptstyle wt=w_2'} \\ Q_2 \in Y & & Y' \ni Q_4' \\ {\scriptstyle \frac{2}{2}}\Big\downarrow{\scriptstyle wt=w_1} & & {\scriptstyle \frac{1}{2}}\Big\downarrow{\scriptstyle wt=w_1'} \\ X & \overset{=}{\longrightarrow} & X \end{array}$$

Where

$$\begin{array}{ll} w_1 = (2,2,1,1), & w_1' = \frac{1}{2}(3,1,1,2) \\ w_2 = \frac{1}{4}(4,1,1,3), & w_2' = \frac{1}{2}(1,3,1,2). \end{array}$$

Note that f', g' are the weighted blowup of type v_1 as in [4, §4].

Case 3. The case of Theorem 1.1.(ii) in [5].

The equation is given as

$$\begin{cases} \varphi_1 : x_1^2 + x_4x_5 + r(x_3)x_2 + p(x_3,x_4) = 0 \\ \varphi_2 : x_2^2 + s(x_3,x_4)x_1x_3 + q(x_3,x_4) - x_5 = 0 \end{cases}$$

with $v = \frac{1}{2}(1,1,1,0,0)$. The map $f : Y \to X$ is given by weighted blowup with vector $v_1 = (l+1, l, 1, 1, 2l+1)$. We take $Z \to Y$ to be the extraction over the quotient singularity Q_5, which is a quotient singularity of type $\frac{1}{4l+2}(3l+2, l, 1)$.

We can write $p(x_3,x_4) = p_0(x_3) + x_4p_1(x_3,x_4)$. By replacing x_5 with $x_5 - p_1(x_3,x_4)$, we may and so assume that $p = p(x_3)$.

We need to distinguish into two subcases according to the parity of l.

Subcase 3.1 l is odd.

In this situation, we need to use the fact that either $x_3^{2l+2} \in \varphi_1$ or

$x_2x_3^{l+2} \in \varphi_1$ (cf. [5, Theorem 1.1.ii.b,c]). By this fact, one sees that the compatible weighted blowup is given by $w_2 = \frac{2l}{4l+2}(3l+2, l, 1, 2l+2, 2l)$.

We now pick $D_{0,X} = (x_3 = 0) \in |-K_X|$ again and it is easy to see that $D_0 \cap E$ is $\mathbb{Z}_2$ quotient of $(x_1^2 = x_2^2 + a_{0,2l}x_4^{2l} = 0) \subset \mathbb{P}(l+1, l, 1, 1, 2l+1)$, where $a_{0,2l}$ is the coefficient. In any event, this is irreducible.

We have

$$\begin{array}{l} E^3 = \frac{4}{4l+2}, \quad F^3 = \frac{(4l+2)^2}{l(3l+2)}, \\ T(f,g) = \frac{1}{4l+2}(-4 + \frac{2l}{l(3l+2)}) < 0. \end{array}$$

We summarize the result into following diagram.

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[d, "\frac{1}{4}"', "wt=w_2"] & & Z' \arrow[d, "\frac{1}{2}"', "wt=w_2'"] \\ Q_5 \in Y \arrow[d, "\frac{2}{2}"', "wt=w_1"] & & Y' \ni Q_4' \arrow[d, "\frac{1}{2}"', "wt=w_1'"] \\ X \arrow[rr, "="] & & X \end{tikzcd}$$

Where

$$\begin{array}{ll} w_1 = (l+1, l, 1, 1, 2l+1), & w_1' = \frac{1}{2}(l+2, l, 1, 2, 2l) \\ w_2 = \frac{1}{4l+2}(3l+2, l, 1, 2l+2, 2l), & w_2' = \frac{1}{2}(l, l, 1, 2, 2l-1). \end{array}$$

Subcase 3.2 l is even.

In this situation, we need to use the fact that either $x_3^{2l} \in \varphi_2$ or $x_1x_3^{l-1} \in \varphi_2$ (cf. [5, Theorem 1.1.ii.a]). Then the compatible weighted blowup is given by $w_2 = \frac{1}{4l+2}(l+1, 3l+1, 1, 2l+2, 2l)$.

We pick $D_{0,X} = (x_3 = 0) \in |-K_X|$ again such that $D_0 \cap E$ is irreducible similarly. We have

$$\begin{array}{l} E^3 = \frac{4}{4l+2}, \quad F^3 = \frac{(4l+2)^2}{(l+1)(3l+1)}, \\ T(f,g) = \frac{1}{4l+2}(-4 + \frac{2l}{(l+1)(3l+1)}) < 0. \end{array}$$

We summarize the result into following diagram.

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[d, "\frac{1}{4l+2}"', "wt=w_2"] & & Z' \arrow[d, "\frac{1}{2}"', "wt=w_2'"] \\ Q_5 \in Y \arrow[d, "\frac{2}{2}"', "wt=w_1"] & & Y' \ni Q_4' \arrow[d, "\frac{1}{2}"', "wt=w_1'"] \\ X \arrow[rr, "="] & & X \end{tikzcd}$$

Where

$$\begin{array}{ll} w_1 = (l+1, l, 1, 1, 2l+1), & w_1' = \frac{1}{2}(l+1, l+1, 1, 2, 2l) \\ w_2 = \frac{1}{4l+2}(l+1, 3l+1, 1, 2l+2, 2l), & w_2' = \frac{1}{2}(l+1, l-1, 1, 2, 2l+2). \end{array}$$

3.5. **discrepancy a/n to a cA/n point.** This case is described in [7, Theorem 1.1.i], the local equation is given by

$$\varphi : x_1x_2 + g(x_3^r, x_4) = 0 \subset \mathbb{C}^4/v,$$

where $v = \frac{1}{n}(1, -1, b, 0)$.

The map f is given by weighted blowup with weight $v_1 = \frac{1}{n}(r_1, r_2, a, r)$. We may write $r_1 + r_2 = dan$ for some $d > 0$ with the term $x_3^{dn} \in \varphi$. We also have that $s_1 := \frac{a-br_1}{n}$ is relatively prime to r_1 and $s_2 := \frac{a+br_2}{n}$ is relatively prime to r_2 (cf. [7, Lemma6.6]). We thus have the following:

$$\begin{cases} a = br_1 + ns_1, \\ 1 = q_1r_1 + s_1^*s_1, \\ a = -br_2 + ns_2, \\ 1 = q_2r_2 + s_2^*s_2, \end{cases}$$

for some $0 \le s_i^* < r_i$ and some q_i.

We set

$$\delta_1 := -nq_1 + bs_1^*, \quad \delta_2 := -nq_2 - bs_2^*.$$

One sees easily that

$$\begin{cases} \delta_1r_1 + n = as_1^*, \\ \delta_2r_2 + n = as_2^*. \end{cases}$$

Claim 1. $a > \delta_i \neq 0$ for $i = 1, 2$.
To see this, first notice that if $\delta_1 = 0$, then $s_1^* = tn, q_1 = tb$ for some integer t. It follows that $1 = ta$, which contradicts to $a > 1$. Hence $\delta_1 \neq 0$ and similarly $\delta_2 \neq 0$.

Note that $\delta_ir_i = as_i^* - n < as_i^* < ar_i$. Hence we have $\delta_i < a$ for $i = 1, 2$. This completes the proof of the Claim 1.

Moreover, we need the following:
Claim 2. $\delta_i > 0$ for some i.
If $\delta_i < 0$, then $n = -\delta_ir_i + as_i^* \ge r_i$. In fact, the equality holds only when $s_i^* = 0$, which implies in particular that $r_i = 1$. We can not have the equalities simultaneously for $i = 1, 2$ otherwise, $r_1 = r_2 = 1$ yields $2 = r_1 + r_2 = adn \ge 2n \ge 4$. Therefore

$$2n > r_1 + r_2 = adn \ge 2n,$$

which is absurd. This completes the proof of the Claim.

Remark 3.1. Suppose that both $\delta_1, \delta_2 > 0$ and $(a, r_1) = 1$, then we have $\delta_1+\delta_2 = a$. To see this, note that $as_2^* = n+\delta_2r_2 = n+\delta_2(adn-r_1)$. Therefore,

$$a(s_2^* - \delta_2dn) = n + (-\delta_2)r_1.$$

By $(a, r_1) = 1$ and comparing it with $as_1^* = n + (\delta_1)r_1$, we have $\delta_1 = -\delta_2 + ta$ for some $t \in \mathbb{Z}$. Since $0 < \delta_1 + \delta_2 < 2a$, it follows that $\delta_1 + \delta_2 = a$.

Subcase 1. Suppose that $\delta_1 > 0$.
Notice that $r_1 = 1$ implies that $s_1^* = 1, q_1 = 1$ and hence $\delta_1 = -n$. Therefore, we must have $r_1 > 1$. Let $g : Z \to Y$ be Kawamata blowup over Q_1, which is a quotient singularity of type $\frac{1}{r_1}(r_1 - s_1^*, 1, s_1^*)$. We take $w_2 = \frac{1}{r_1}(r_1 - s_1^*, dr, 1, s_1^*)$ which is a compatible weighted blowup.

We pick $D_{0,X} = (x_4 = 0)$ then $E \cap D_0$ is defined by $x_1x_2 + x_3^{dn} = 0$ which is clearly irreducible. We have $c_0 = n, q_0 = s_1^*$ and hence $c_0 - aq_0 = -\delta_1 r_1 < 0$. Also

$$\begin{array}{l} E^3 = \frac{dr^2}{r_1r_2}, \quad F^3 = \frac{(r_1)^2}{s_1^*(r_1-s_1^*)}, \\ T(f,g,D_0) = \frac{1}{r_1}(-\frac{adn}{r_2} + 1) < 0. \end{array}$$

Hence Hypothesis $\flat$ holds

We summarize the result into following diagram.

$$\begin{array}{ccc} Z & \stackrel{--\to}{\longrightarrow} & Z' \\ \frac{1}{r_1}\Big\downarrow wt=w_2 & & \frac{\delta_1}{n}\Big\downarrow wt=w_2' \\ Q_1 \in Y & & Y' \ni Q_4' \\ \frac{a}{n}\Big\downarrow wt=w_1 & & \frac{a-\delta_1}{n}\Big\downarrow wt=w_1' \\ X & \stackrel{=}{\longrightarrow} & X \end{array}$$

Where

$$\begin{array}{ll} w_1 = \frac{1}{n}(r_1, r_2, a, n), & w_1' = \frac{1}{n}(r_1 - s_1^*, r_2 - \delta_1 dn + s_1^*, a - \delta_1, n) \\ w_2 = \frac{1}{r_1}(r_1 - s_1^*, dn, 1, s_1^*), & w_2' = \frac{1}{n}(s_1^*, \delta_1 dn - s_1^*, \delta_1, n). \end{array}$$

Note that $0 < a' := a - \delta_1 < a$ and both f', g' are extremal contractions with discrepancies $< \frac{a}{r}$.

Subcase 2. Suppose that $\delta_2 > 0$.
Again, $r_2 > 1$ under this assumption. Let $g : Z \to Y$ be Kawamata blowup over Q_2, which is a quotient singularity of type $\frac{1}{r_2}(r_2 - s_2^*, 1, s_2^*)$. We take $w_2 = \frac{1}{r_2}(dr, r_2 - s_2^*, 1, s_2^*)$ which is a compatible weighted blowup.

We pick $D_{0,X} = (x_4 = 0)$ again which is irreducible. We have $c_0 = n, q_0 = s_2^*$ and hence $c_0 - aq_0 = -\delta_2 r_2 < 0$. Also

$$\begin{array}{l} E^3 = \frac{dr^2}{r_1r_2}, \quad F^3 = \frac{(r_2)^2}{s_2^*(r_2-s_2^*)}, \\ T(f,g,D_0) = \frac{1}{r_2}(-\frac{adn}{r_1} + 1) < 0. \end{array}$$

Hence Hypothesis $\flat$ holds

We summarize the result into following diagram.

$$\begin{array}{ccc}
Z & \xrightarrow{\dashrightarrow} & Z' \\
{\scriptstyle \frac{1}{r_1}}\downarrow{\scriptstyle wt=w_2} & & {\scriptstyle \frac{\delta_2}{n}}\downarrow{\scriptstyle wt=w_2'} \\
Q_2 \in Y & & Y' \ni Q_4' \\
{\scriptstyle \frac{a}{n}}\downarrow{\scriptstyle wt=w_1} & & {\scriptstyle \frac{a-\delta_2}{n}}\downarrow{\scriptstyle wt=w_1'} \\
X & \xrightarrow{=} & X
\end{array}$$

Where

$w_1 = \frac{1}{n}(r_1, r_2, a, n)$, $\quad w_1' = \frac{1}{n}(r_1 + s_2^* - \delta_2 dn, r_2 - s_2^*, a - \delta_2, n)$
$w_2 = \frac{1}{r_2}(dn, r_2 - s_2^*, 1, s_2^*)$, $\quad w_2' = \frac{1}{n}(\delta_2 dn - s_2^*, s_2^*, \delta_2, n)$.

It is easy to see that if $r_1 \geq r_2$, then $\delta_1 > 0$. Hence extracting over Q_1 provides the desired factorization. Similar argument holds if $r_2 \geq r_1$. Therefore, one can conclude that Theorems holds by extracting over the point of highest index.

4. FURTHER REMARKS

It is easy to see that our method also work for any divisorial contraction to a point of index 1 which is a weighted blowup. Let us take $f : Y \to X$ the weighted blowup with weight $(1, a, b)$ for example, where $a < b$ are relatively prime. Write $ap = bq + 1$. Then by our method, one sees easily that $g : Z \to Y$ is weighted blowup with weight $\frac{1}{b}(p, 1, b-p)$ over Q_3. After 2-ray game, we have that g' is the weighted blowup with weight $(1, q, p)$ over Q_1' and f' is the weighted blowup with weight $(1, a-q, b-p)$. Also $Z \dashrightarrow Z'$ is a toric flip. All the other known examples fit into our framework nicely as well.

We would like to raise the following

Problem 1. Can every 3-fold divisorial contraction to a point be realized as a weighted blowup?

Assuming the affirmative answer, then by the method we provided in this article, it is reasonable to expect, as in Corollary 1.3, that for any 3-fold divisorial contraction $Y \to X$ to a singular point $P \in X$ of index $r = 1$ with discrepancy $a > 1$, there exists a sequence of birational maps

$$Y =: X_n \dashrightarrow \ldots \dashrightarrow X_0 =: X$$

such that each map $X_{i+1} \dashrightarrow X_i$ is one of the following:

(1) a divisorial extraction over a singular point of index $r_i \geq 1$ with discrepancy $\frac{1}{r_i}$.
(2) a divisorial contraction to a singular point of index $r_i \geq 1$ with discrepancy $\frac{1}{r_i}$.
(3) a flip or flop.

Together with the factorization result of [1], we have the following:

Conjecture 4.1. Let $Y \dashrightarrow X$ be a birational map which is flip, a divisorial contraction to a point, or a divisorial contraction to a curve. There exists a sequence of birational maps

$$Y =: X_n \dashrightarrow \ldots \dashrightarrow X_0 =: X$$

such that each map $X_{i+1} \dashrightarrow X_i$ is one of the following:

(1) a divisorial extraction or contraction over a point with minimal discrepancy,
(2) a blowup of a lci curve.
(3) a flop.

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