

ON 6-CANONICAL MAP OF IRREGULAR THREEFOLDS OF GENERAL TYPE

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ABSTRACT. We prove that, for any nonsingular projective irregular 3-fold of general type, the 6-canonical map is birational onto its image.

1. Introduction

Given a nonsingular projective variety V of general type, by definition, the pluricanonical map φ_m is birational for all sufficiently large integer m . It is natural and interesting to find an effective bound for m . By the result of Hacon-McKernan [7], Takayama [10] and Tsuji (cf. [11]), one knows that there exists a positive integer r_n depending only on $n = \dim(V)$ such that φ_m is birational for all $m \geq r_n$. In the case of threefolds, the previous work of the first two authors (cf. [3, 4]) shows that $r_3 \leq 73$.

In this note we study irregular threefolds (i.e. $q(V) > 0$) of general type. Recent developments on the technique inspired by the Fourier-Mukai transform show that the geometry of irregular threefolds is very similar to that of general fibers of the Albanese map. Noting that the 5-canonical map of a general type surface is birational, one may expect that φ_5 is birational too for those threefolds which admit a fibration over (a subvariety of) an abelian variety. Indeed, given a nonsingular projective irregular threefold of general type, it has been proved by Chen and Hacon [5, Theorem 2.8, Proposition 2.9] that φ_m is birational for all $m \geq 7$ and, moreover, that φ_5 is birational if $\chi(\omega_X) > 0$.

The aim of this paper is to prove the following:

Theorem 1.1. *Let V be a nonsingular projective irregular 3-fold of general type. Then φ_6 is birational.*

2. Proof of the main theorem

2.1. Reductions. In order to prove Theorem 1.1, we have the following reduction to special cases:

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- (1) Let V be a nonsingular projective 3-fold of general type. Take any birational projective model W of V so that W has at worst canonical singularities. Then V and W share the same birational invariants and $\Phi_{mK_W} \approx \Phi_{mK_V}$. Therefore it is sufficient to prove the statement of Theorem 1.1 just replacing V with any suitable birational model W .
- (2) By Chen and Hacon [5, Proposition 2.9], one only needs to consider the following situation (since, otherwise, $|6K|$ gives a birational map):
 - (†) The Albanese map of V induces the fibration $a_V : V \rightarrow C$ onto an elliptic curve C , of which the general fiber is a $(1, 2)$ surface S , i.e. $(K_{S_0}^2, p_g(S)) = (1, 2)$, where S_0 is assumed to be the minimal model of S .
- (3) Also due to Chen and Hacon [5, Theorem 1.1], we may assume that $\chi(\mathcal{O}_V) \geq 0$ (since, otherwise, $|5K|$ gives a birational map).
- (4) By running the minimal model program, one gets a relative minimal model $X \rightarrow C$ of a_V where X has \mathbb{Q} -factorial terminal singularities. Then $K_{X/C}$ is nef (see, for instance, Ohno [8, Theorem 1.4]), which means that X is minimal since K_C is trivial. In the proof of Theorem 1.1, we may and do replace V by a minimal model X (i.e. K_X nef) which has at worst \mathbb{Q} -factorial terminal singularities.

Corollary 2.1. *Suppose V (or X) satisfies 2.1(2) and 2.1(3). Then $q(X) = 1$, $p_g(X) = h^2(\mathcal{O}_X) \leq 2$ and thus $\chi(\mathcal{O}_X) = 0$.*

Proof. Clearly one has $q(V) = 1$. Since $q(S) = 0$, we see $h^2(\mathcal{O}_V) = h^1(a_*\omega_V)$. So one has $\chi(\mathcal{O}_V) = h^2(\mathcal{O}_V) - p_g(V) = h^1(a_*\omega_V) - h^0(a_*\omega_V) = -\deg(a_*\omega_{V/C}) \leq 0$ by the semi-positivity theorem of Fujita [6]. Thus $\chi(\mathcal{O}_V) = 0$ and $p_g(V) = h^2(\mathcal{O}_V)$. Also by the semi-positivity of $a_*\omega_V = a_*\omega_{V/C}$, $p_g(V) = h^2(\mathcal{O}_V) = h^1(a_*\omega_V) \leq \text{rk}(a_*\omega_V) = 2$. By Reid's R-R formula in [9], one can see $P_2(V) > 0$ and $P_{m+1}(V) > P_m(V)$ for all $m \geq 2$. \square

2.2. Definitions and lemmas. Before proving the main result, we would like to recall some notion and results in Chen and Hacon [5].

Definition 2.2. For any vector bundle E on an elliptic curve, we write $E = \oplus E_i$, where each E_i is indecomposable. We define $\nu(E) := \min\{\mu(E_i)\}$, where $\mu(E_i) = \frac{\deg(E_i)}{\text{rk}(E_i)}$ is the slope of E_i .

Definition 2.3. A coherent sheaf \mathcal{F} on an abelian variety A is said to be IT^0 if $H^i(A, \mathcal{F} \otimes P) = 0$ for all $i > 0$ and all $P \in \text{Pic}^0(A)$.

Lemma 2.4. ([5, Lemma 4.8]) *Let E_1, E_2 be vector bundles on an elliptic curve.*

- (1) *If E_1, E_2 are indecomposable and $\text{Hom}(E_1, E_2) \neq 0$, then $\mu(E_2) \geq \mu(E_1)$.*

(2) *If there exists a surjective map $E_1 \rightarrow E_2$, then $\nu(E_2) \geq \nu(E_1)$.*

Lemma 2.5. ([5, Lemma 4.10]) *Let E be an IT^0 vector bundle on an elliptic curve which admits a short exact sequence*

$$0 \rightarrow F \rightarrow E \rightarrow Q \rightarrow 0$$

of coherent sheaves such that Q has generic rank = 0 (resp. ≤ 1). Then $\nu(E) \geq \nu(F)$ (resp. $\nu(E) \geq \min\{1, \nu(F)\}$).

2.3. Multiplication maps $\varphi_{m,n}$ and $\psi_{m,n}$. Consider the fibration $a : X \rightarrow C$ as in 2.2. Let F be a general fiber F of a . Let $R_m := H^0(F, \omega_F^m)$ and $E_m := a_*\omega_X^m$. By Chen and Hacon [5, Lemma 4.1], E_m is an IT^0 vector bundle of rank $P_m(F)$ for all $m \geq 2$. We also remark that $\nu(E_m) \geq 0$ by the semi-positivity theorem (see, for instance, Viehweg [12]) and Atiyah's description of vector bundles over elliptic curves (cf. [1]). We consider the multiplication map of pluricanonical systems on the fiber F , say

$$\varphi_{m,n} : R_m \otimes R_n \rightarrow R_{m+n}.$$

This naturally induces a map between vector bundles

$$\psi_{m,n} : E_m \otimes E_n \rightarrow E_{m+n}$$

where $m, n > 0$. Clearly if cokernel of $\varphi_{m,n}$ has dimension $\leq r$, then cokernel of $\psi_{m,n}$ has rank $\leq r$.

2.4. Proof of Theorem 1.1. First of all, we recall that the linear system $|6K_V|$ separates two general points on two distinct general fibers of the Albanese map a_V (see [5, Theorem 2.8 (2)]). Hence we just need to show that $|6K_V|$ separates two general points on a general fiber of a_V to conclude the proof of Theorem 1.1.

We now take the birational model $a : X \rightarrow C$ of V as in 2.1(1)~(4).

Step 1. We construct a relative canonical model $W \rightarrow C$ of a .

We may take an integer $m \gg 0$ and pick a very ample divisor L on C so that

- i. for the general fiber F of a , $|mK_F|$ is base point free and $\Phi_{|mK_F|}(F)$ is the canonical model of F ;
- ii. $|a^*L + mK_X|$ is free;
- iii. $a_*\omega_X^m \otimes \mathcal{O}_C(L)$ is generated by global sections and then the restriction map $H^0(X, a^*L + mK_X) \rightarrow H^0(F, mK_F)$ is surjective for general F ;
- iv. $a_*\omega_X^2 \otimes \mathcal{O}_C(L)$ is generated by global sections and then the restriction map $H^0(X, a^*L + 2K_X) \rightarrow H^0(F, 2K_F)$ is surjective for general F .

The linear system $|a^*L + mK_X|$ defines a morphism $X \rightarrow \mathbb{P}^N$ over C and let W be its image. Then we get a relative canonical model $g : W \rightarrow C$. Clearly, by definition, a factors through g . Denote by G

the general fiber of g . Then $W|_G$ is exactly the canonical model of F for general F .

Step 2. The relative bicanonical map $h : Y \rightarrow C$ of g .

It is known (cf. Catanese [2, 1.3 Example]) that the canonical model G of any (1,2) surface is a degree 10 weighted hypersurface, with at worst rational double points, in $\mathbb{P}(1, 1, 2, 5)$. Namely, if x, y, z, u are coordinates of $\mathbb{P}(1, 1, 2, 5)$, then G is given by the homogeneous equation $u^2 - f_{10}(x, y, z)$ for some homogeneous polynomial $f_{10}(x, y, z)$ of degree 10 in x, y, z . Furthermore the bicanonical map φ_2 of G is a double covering onto $\mathbb{P}(1, 1, 2)$ branched along a reduced divisor $B_0 = \text{div}(f_{10}) \subset \mathbb{P}(1, 1, 2)$ of degree 10.

By the choice of m , we may assume that the rational map

$$\Phi_{|a^*(L)+2K_X|} : X \dashrightarrow Y$$

factors through W where Y is assumed to be the closure of the image. Notice also that $a : X \rightarrow C$ factors through Y . Moreover, there is a natural injection $Y \hookrightarrow \mathbb{P}(a_*\omega^2) = \mathbb{P}(E_2)$, where $\mathbb{P}(E_2)$ is a \mathbb{P}^3 -bundle over C . We have a new fibration $h : Y \rightarrow C$ which is induced from the bicanonical map of g .

Let H be the general fiber of $h : Y \rightarrow C$. Over a general point of C , we have morphisms $F \rightarrow G \rightarrow H$ where F is a minimal (1,2) surface, G is the degree 10 hypersurface in $\mathbb{P}(1, 1, 2, 5)$ with RDPs and $H \cong \mathbb{P}(1, 1, 2)$. We have seen that both $X \dashrightarrow Y$ and $W \dashrightarrow Y$ are well-defined over general points of C . Replacing both X and W with suitable birational models \hat{X} and \hat{W} by a necessary birational modification to those indeterminacies, we have the following commutative diagram:

$$\begin{array}{ccccccc} \hat{X} & \xrightarrow{\sigma} & \hat{W} & \xrightarrow{\tau} & Y & \longrightarrow & \mathbb{P}(E_2) \\ \hat{a} \downarrow & & \hat{g} \downarrow & & h \downarrow & & \downarrow p \\ C & \xrightarrow{=} & C & \xrightarrow{=} & C & \xrightarrow{=} & C. \end{array}$$

where \hat{X} (resp. \hat{W}) coincides with X (resp. W) over a Zariski open subset U of C and \hat{a} (resp. \hat{g}) factors through a (resp. g).

Step 3. The decomposition of E_m by the double covering construction.

Shrinking U , if necessary, so that $\tau : W_U = \hat{W}_U \rightarrow Y_U$ is a double covering branched along an even reduced divisor $B_U \subset Y_U$. Let B_1 be the closure of B_U in Y . Then

$$\mathcal{O}_Y(B_1) = \mathcal{O}_{\mathbb{P}(E_2)}(10) \otimes p^*\mathcal{M}|_Y$$

for some line bundle \mathcal{M} on C . Set $B = B_1$ (resp. $B = B_1 + H_0$) if $\deg(\mathcal{M})$ is even (resp. odd), then $\mathcal{O}_Y(B) = \mathcal{L}^{\otimes 2}$, where $\mathcal{L} = (\mathcal{O}_{\mathbb{P}(E_2)}(5) \otimes \pi^*\mathcal{M}')|_Y$ for some \mathcal{M}' .

Let $\mu : \tilde{Y} \rightarrow Y$ be the log resolution of (Y, B) and let $\tilde{B} := \mu^*B - 2\lfloor \frac{\mu^*B}{2} \rfloor$ and $\tilde{\mathcal{L}} = \mu^*\mathcal{L} \otimes \mathcal{O}(-\lfloor \frac{\mu^*B}{2} \rfloor)$. Clearly \tilde{B} is a reduced SNC divisor and $\mathcal{O}(\tilde{B}) = \tilde{\mathcal{L}}^{\otimes 2}$. Let $\tilde{\pi} : \tilde{X} \rightarrow \tilde{Y}$ be the double cover over \tilde{Y} branched along \tilde{B} . One sees that \tilde{X} has at worst canonical singularities by local consideration. We thus have

$$\tilde{\pi}_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}) = \mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^m \oplus \mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^{m-1}$$

for all $m > 0$. Now if we take a common birational modification to both \hat{X} and \tilde{X} and take push-forwards in two directions respectively, we shall get the following decomposition

$$E_m := E_{m,0} \oplus E_{m,1},$$

where

$$\begin{aligned} E_m &:= a_* \mathcal{O}_X(mK_X); \\ E_{m,0} &:= h_* \mu_* (\mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^m); \\ E_{m,1} &:= h_* \mu_* (\mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^{m-1}). \end{aligned}$$

Step 4. Calculating $\nu(E_{6,i})$.

It is rather easy to check that $\text{rk}(E_{m,0}) = h^0(H, \mathcal{O}(m))$ and $\text{rk}(E_{m,1}) = h^0(H, \mathcal{O}(m-5))$ for a general fiber H of h . Indeed, for $t \in U$,

$$\begin{aligned} E_m \otimes k(t) &\cong H^0(F_t, \mathcal{O}_{F_t}(mK_X)) \cong H^0(G_t, \mathcal{O}_{G_t}(mK_W)) \\ &\cong H^0(\mathbb{P}(1, 1, 2, 5), \mathcal{O}(m)), \end{aligned}$$

$$\begin{aligned} E_{m,0} \otimes k(t) &\cong H^0(H_t, \mathcal{O}_{H_t}(mK_Y + mL)) \cong H^0(\mathbb{P}(1, 1, 2), \mathcal{O}(m)) \text{ and} \\ E_{m,1} \otimes k(t) &\cong H^0(H_t, \mathcal{O}_{H_t}(mK_Y + (m-1)L)) \cong H^0(\mathbb{P}(1, 1, 2), \mathcal{O}(m-5)). \end{aligned}$$

It follows that $\psi_{m,n}$ induces a map

$$E_{m,0} \otimes E_{n,0} \rightarrow E_{m+n,0}.$$

Since $E_{m,0} = E_m$ for $m \leq 4$. One sees that

$$\psi_{4,2} : E_4 \otimes E_2 \cong E_{4,0} \otimes E_{2,0} \rightarrow E_{6,0}$$

is generically surjective. Since E_2 is a non-zero IT^0 sheaf, we have $h^0(E_2) \geq 1$. Hence $\nu(E_2) \geq \frac{1}{4}$. Since $\psi_{2,2}$ is generically surjective, we have $\nu(E_4) \geq \nu(E_{4,0}) \geq \frac{1}{2}$ by Lemma 2.5. Similarly, $\nu(E_{6,0}) \geq \frac{3}{4}$. Moreover, $E_{6,1}$ is IT^0 of rank 2 by Lemma 2.4, hence $\nu(E_{6,1}) \geq \frac{1}{2}$.

Step 5. Birationality of φ_6 .

We need the following:

Lemma 2.6. *Let \mathcal{F} be a coherent sheaf on X and $\mathcal{E} := a_* \mathcal{F}$ on C . Suppose that \mathcal{E} is an IT^0 vector bundle. Then for any general fiber X_t , the image of the restriction map $H^0(C, \mathcal{E}) \cong H^0(X, \mathcal{F}) \xrightarrow{\text{res}} H^0(X_t, \mathcal{F}|_{X_t})$ has dimension $\geq \text{rk}(\mathcal{E}) \cdot \min\{\nu(\mathcal{E}), 1\}$.*

Proof. Take the decomposition of $\mathcal{E} = \bigoplus \mathcal{E}_i$ into indecomposable bundles. For each i , there is an induced exact sequence

$$0 \rightarrow \mathcal{E}_i \otimes \mathcal{O}_C(-t) \rightarrow \mathcal{E}_i \rightarrow \mathcal{E}_i \otimes k(t) \rightarrow 0.$$

Let $d_i = \deg(\mathcal{E}_i)$ and $r_i = \text{rk}(\mathcal{E}_i)$, then $\mathcal{E}_i \otimes \mathcal{O}_C(-t)$ has rank r_i and degree $d_i - r_i$. If $d_i = r_i$, then $\mathcal{E}_i \otimes \mathcal{O}_C(-t)$ is a indecomposable rank r_i vector bundle of degree 0. Hence $\mathcal{E}_i \otimes \mathcal{O}_C(-t) \cong U_{r_i} \otimes P$ for some $P \in \text{Pic}^0(C)$ and U_{r_i} is a unipotent vector bundle (cf. [1]). Whenever $P = \mathcal{O}$, we pick $t' \neq t$ and consider $\mathcal{E}_i \otimes \mathcal{O}_C(-t') \cong U_{r_i} \otimes \mathcal{O}(-t' + t)$ instead so that it has no global section. Hence we may and do assume that $H^0(\mathcal{E}_i \otimes \mathcal{O}_C(-t)) = 0$ for general $t \in C$ if $d_i = r_i$.

It now follows that $h^0(\mathcal{E}_i \otimes \mathcal{O}_C(-t)) = \max\{0, d_i - r_i\}$ for general t . Hence the image of $H^0(\mathcal{E}_i) \rightarrow H^0(\mathcal{E}_i \otimes k(t))$ has dimension d_i (resp. r_i) if $d_i < r_i$ (resp. $d_i \geq r_i$). The statement now follows by simply taking the sum. \square

Let $V_{m,i}$ ($i = 0, 1$) be the image of the following map

$$H^0(C, E_{m,i}) \hookrightarrow H^0(C, E_m) \xrightarrow{res} H^0(F_t, \mathcal{O}(mK)|_{F_t})$$

for a general point $t \in C$. Then we have $\dim V_{6,0} \geq 12$ and $\dim V_{6,1} \geq 1$ by Lemma 2.6.

Claim. *The subsystem given by the vector space*

$$V_{6,0} + V_{6,1} \subset H^0(G_t, \mathcal{O}(6))$$

gives a birational map on G_t for all general $t \in C$.

We consider the local sections explicitly. Let x, y, z, u be all the 4 coordinates of $\mathbb{P}(1, 1, 2, 5)$ with weights 1, 1, 2, 5. Then $E_{m,0} \otimes k(t)$ is generated by sections in $\{x^i y^j z^k \mid i + j + 2k = m\}$ and $E_{m,1} \otimes k(t)$ is generated by sections in $\{x^i y^j z^k u \mid i + j + 2k = m - 5\}$. In a word, either xu or yu extends to global sections in $H^0(X, 6K_X)$. Furthermore, at least 12 linearly independent sections in $E_{m,0} \otimes k(t)$ can be extended to global sections in $H^0(X, 6K_X)$.

To prove the claim, we put $H = H_t$ and let $\Sigma_0 \subset H^0(H, \mathcal{O}_H(6))$ (resp. $\Sigma_1 \subset H^0(H, \mathcal{O}_H(6))$) be the subspace spanned by $\{x^6, \dots, y^6\}$ (resp. by $\{x^4 z, x^3 y z, \dots, y^4 z\}$). We see that $\dim \Sigma_0 = 7$ and $\dim \Sigma_1 = 5$. By dimensional considerations, one has $\dim V_{6,0} \cap \Sigma_0 \geq 3$ and $\dim V_{6,0} \cap \Sigma_1 \geq 1$. Pick linearly independent elements $\sigma_{0,1}, \sigma_{0,2}, \sigma_{0,3} \in V_{6,0} \cap \Sigma_0$ and $z\sigma_1 \in V_{6,0} \cap \Sigma_1$. We consider the map $\tilde{\varphi} : H \dashrightarrow \mathbb{P}^3$ defined by these 4 sections. It is easy to see that $\tilde{\varphi}$ has image of dimension 2. Indeed, consider the map $\varphi : H \dashrightarrow \mathbb{P}^{11}$ given by $V_{6,0}$ with image H' . Since φ factors through $\tilde{\varphi}$, one sees that H' is a surface and clearly $\deg(H') \geq 10$. Since

$$\deg(\varphi) \cdot \deg(H') \leq (\mathcal{O}_H(6) \cdot \mathcal{O}_H(6))_H = 18,$$

it follows that φ has degree 1, hence is birational.

Since $G_t \cong X_{10} \rightarrow \mathbb{P}(1, 1, 2) \cong H$ is a $2 : 1$ map and u can separate points on general fibers of this double covering. Hence the sections in $V_{6,1}$ separate points on general fibers of this double covering.

The Claim now follows and hence this completes the proof of Theorem 1.1. \square

Example 2.7. Suppose that there exists a minimal irregular threefold X with a fibration $f : X \rightarrow C$ fibered by $(1, 2)$ surfaces. Suppose that $K_X^3 = \frac{1}{2}$ and $B(X) = \{3 \times (1, 2)\}$. By Reid's R-R formula, one has $P_2(X) = 1$, $P_3(X) = 2$, $P_4(X) = 5$, $P_5(X) = 9$ and $P_6(X) = 16$. We show that $|5K_X|$ may be non-birational.

Note that $\text{rk}(E_{5,0}) = 12$ and $\text{rk}(E_{5,1}) = 1$. Assume $h^0(E_{5,0}) = 8$ and $h^0(E_{5,1}) = 1$.

Now $H^0(F_t, 5K|_{F_t})$ is generated by

$$\{x^5, \dots, y^5, x^3z, x^2yz, xy^2z, y^3z, xz^2, yz^2, u\}.$$

If $V_{5,1}$ is generated by $\{x^5, \dots, y^5, xz^2, yz^2\}$ and $V_{5,2}$ is generated by u , then these sections can not distinguish points (x_0, y_0, z_0, u_0) from $(x_0, y_0, -z_0, u_0)$. In other words, it may only give a $2 : 1$ map on F_t instead of a birational map.

However, we do not know whether this kind of examples really exists or not.

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