THE NOETHER INEQUALITY FOR GORENSTEIN MINIMAL 3-FOLDS

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Abstract. We prove the Conjecture of Catenese–Chen–Zhang: the inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ holds for all projective Gorenstein minimal 3-folds X of general type.

1. **Introduction**

In the classification theory of algebraic varieties, the Noether inequality, which asserts that $K^2 \geq 2p_g - 4$ for minimal surfaces of general type, plays a pivotal role. It is thus natural and important to explore the higher dimensional analogue.

There are several attempts toward this direction. A naive guess is that, for minimal variety X of general type, $K_X^{\dim X} \geq 2(p_g(X) - \dim X)$, which holds in dimension 1 and 2. However, Kobayashi [6] constructed examples of canonically polarized threefolds with $p_g(X) = 3k+4$ and $K_X^3 = 4k+2$ for $k \geq 1$. Hence the inequality $K_X^3 \geq 2p_g(X) - 6$ fails in dimension 3 and one can only expect that $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$.

The aim of this paper is to confirm the conjecture ([5, Conj. 4.4], in 2006) of Catanese–Chen–Zhang and to prove the following:

Theorem 1.1. *The inequality*

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$$

holds for all projective Gorenstein minimal 3-folds X of general type.

Theorem 1.1 was proved by the second author [2] when X is canonically polarized and by Catanese–Chen–Zhang [5] while X is smooth minimal. We refer to the relevant work [6, 2, 3, 5] for more details of the history of this topic.

The main obstacle in proving the above theorem is the existence of Gorenstein terminal singularities in the base locus of the canonical linear system $|K_X|$, while X is canonically fibred by a family of curves of genus 2. By using certain conceived and explicit resolution of Gorenstein terminal singularities, which we call *feasible Goresntein*

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resolution, we are able to resolve the base locus and prove the statement.

Throughout we work over the complex number field $\mathbb{C}$.

2. Special resolutions to Gorenstein terminal singularities (X, P), pairs (X, D) and linear systems $(X, |M|)$

First of all, we recall the following result of the first author:

Theorem 2.1. *(*[1, Theorem 1.3]*) Let X be an algebraic 3-fold with at worst terminal singularities. For any terminal singularity $P \in X$, there exists a sequence of birational morphisms:*

$$\tau_P : Y = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X,$$

such that Y is smooth on $\tau_P^{-1}(P)$ and, for all i, the morphism $\pi_i : X_{i+1} \to X_i$ is a divisorial contraction to a singular point $P_i \in X_i$ of index $r_i \geq 1$ with discrepancy $1/r_i$.

Indeed, given a Gorenstein terminal singularity $(P \in X)$, the resolution can be constructed in explicit as follows.

(1) Take a divisorial contraction $\pi_1 : X_1 \to X$ contracting E_1 to the point P with discrepancy 1, i.e. $K_{X_1} = \pi_1^*(K_X) + E_1$.
(2) If there are some higher index points on $E_1 \subset X_1$, there exists a Gorenstein partial resolution

$$X_{n_1} \to X_{n_1 - 1} \to \ldots \to X_2 \to X_1$$

such that,
  - for any $j > 0$, the birational morphism $\pi_{j+1} : X_{j+1} \to X_j$ is a divisorial contraction to a point $P_j \in X_j$ of index $r_j > 1$ with discrepancy $\frac{1}{r_j}$;
  - X_{n_1} has only Gorenstein terminal singularities of which each one is "milder" than $P \in X$.

(3) Inductively, we have a sequence of birational morphisms

$$\tau_P \colon Y = X_{n_l} \to X_{n_{l-1}} \to \ldots \to X_{n_1} \to X,$$

such that the birational morphism $\tau_{j+1} : X_{n_{j+1}} \to X_{n_j}$ is constructed parallel to those in Steps (1) and (2), $X_{n_{j+1}}$ has only Gorenstein terminal singularities and Y is non-singular on $\tau_P^{-1}(P)$.

Definition 2.2. Given a Gorenstein terminal singularity $P \in X$, the birational map $X_{n_1} \to X \ni P$ constructed as in Steps (1) and (2) is called a *feasible Gorenstein partial resolution of* $P \in X$, or *fG partial resolution* for short. The birational morphism $\tau_P : Y \to X \ni P$ constructed as in Step (3) is called a *feasible resolution of* $P \in X$. Clearly, $X_{n_{j+1}}$ is a fG partial resolution of X_{n_j} for any $j > 0$.

Now given a Gorenstein projective 3-fold X with terminal singularities. Let $P \in X$ be a singular point and D be an effective Cartier divisor on X with $P \in D$. We may consider a fG partial resolution of $P \in X$, say

$$Z := X_n \to \ldots \to X_1 \to X_0 = X, \tag{2.1}$$

so that the birational morphism $\pi_P : Z \to X$ is composed of a sequence of divisorial contractions $X_{i+1} \to X_i$ to points $P_i \in X_i$ of index $r_i > 1$ with discrepancy $1/r_i$ for all $i > 1$ together with a divisorial contraction $X_1 \to X$ to $P \in X$ with discrepancy 1. Clearly, Z is still a projective Gorenstein 3-fold with at worst terminal singularities.

For any $i > 0$, let D_i be the proper transform of D in X_i and write $D_{Z/X} := \pi_P^*(D) - D_n$. Similarly, let K_i be the canonical divisor of X_i and write $K_{Z/X} := K_Z - \pi_P^*(K_X)$. Also let E_i be the exceptional divisor of the contraction morphism $X_i \to X_{i-1}$ and E_{i,X_j} denote the proper transform of E_i on X_j.

Theorem 2.3. *Given a projective Gorenstein 3-fold X with terminal singularities. Let $P \in X$ be a singular point and D be an effective Cartier divisor on X with $P \in D$. Let $\pi_P : Z \to X$ be the fG partial resolution as in (2.1). Then $D_{Z/X} \geq K_{Z/X}$.*

Proof. First of all, we have $K_{X_1/X} = E_1$ and $D_{X_1/X} = b_1 E_1$, where $b_1 = \mathrm{mult}_P D \in \mathbb{Z}_{>0}$ is the multiplicity. Clearly, we have $D_{X_1/X} \geq K_{X_1/X}$.

Suppose we have $D_{X_i/X} \geq K_{X_i/X}$. Write $K_{X_i/X} = \sum_{j=1}^{i} a_j E_j$, and $D_{X_i/X} = \sum_{j=1}^{i} b_j E_j$ with $b_j \geq a_j \in \mathbb{Z}$ for all j. Since $\pi_i : X_{i+1} \to X_i$ is a divisorial contraction to a point P_i of index $r > 1$ with discrepancy $1/r$. Let

$$\begin{aligned} \pi_i^*(E_{j,X_i}) &= E_{j,X_{i+1}} + \frac{\alpha_{i,j}}{r} E_{i+1}; \\ \pi_i^*(D_i) &= D_{i+1} + \frac{\beta_i}{r} E_{i+1} \end{aligned} \tag{2.2}$$

where $\alpha_{i,j} \geq 0$ for each j and $\beta_i \geq 0$. It follows that

$$\begin{aligned} K_{X_{i+1}/X} &= \sum_{j=1}^{i} a_j E_j + \left(\frac{\sum_{j=1}^{i} a_j \alpha_{i,j}}{r} + \frac{1}{r}\right) E_{i+1}; \\ D_{X_{i+1}/X} &= \sum_{j=1}^{i} b_j E_j + \left(\frac{\sum_{j=1}^{i} b_j \alpha_{i,j}}{r} + \frac{\beta_i}{r}\right) E_{i+1}. \end{aligned} \tag{2.3}$$

Since (X, P) is Gorenstein, both $\frac{\sum_{j=1}^{i} a_j \alpha_j}{r} + \frac{1}{r}$ and $\frac{\sum_{j=1}^{i} b_j \alpha_j}{r} + \frac{\beta_i}{r}$ are positive integers. Hence

$$\frac{\sum_{j=1}^{i} a_j \alpha_{i,j} + 1}{r} = \lceil \frac{\sum_{j=1}^{i} a_j \alpha_{i,j}}{r} \rceil \leq \lceil \frac{\sum_{j=1}^{i} b_j \alpha_{i,j}}{r} \rceil \leq \frac{\sum_{j=1}^{i} b_j \alpha_{i,j} + \beta_i}{r}.$$

Therefore, $D_{X_{i+1}/X} \geq K_{X_{i+1}/X}$. We are done by induction. $\square$

Now, for the given terminal Gorenstein singularity $P \in X$, the feasible resolution τ_P as in the above Step (3) can be rephrased as:

$$\tau_P : Z_l \to Z_{l-1} \to \cdots \to Z_1 \to X \ni P \tag{2.4}$$

by setting $Z_j := X_{n_j}$, where Z_l is smooth on $\tau_P^{-1}(P)$ and each birational morphism $Z_i \to Z_{i-1}$ is a fG partial resolution for all i. Therefore Theorem 2.3 and simple induction directly imply the following:

Corollary 2.4. *For the feasible resolution (2.4), we have* $D_{Z_j/X} \geq K_{Z_j/X}$ *for* $1 \leq j \leq l$.

In the last part of this section, we focus on moving linear systems. Suppose that $|M|$ is a moving linear system (i.e. without fixed part) on the given projective Gorenstein terminal 3-fold X with $\mathrm{Bs}|M| \neq \emptyset$. Similar to usual resolution of indeterminancies, we can have a *Gorenstein resolution of indeterminancies* as follows:

(i) If $|M|$ is free out of singularities, i.e. $\mathrm{Bs}|M| \cap \mathrm{Sing}(X) = \emptyset$, then we do nothing.

(ii) If there is a point $P \in \mathrm{Bs}|M| \cap \mathrm{Sing}(X)$, we take a fG-partial resolution $Z_1 \to X \ni P$ and consider the linear system $|M_1|$, where M_1 is the proper transform of M on Z_1.

(iii) Inductively, we will end up with a chain of fG partial resolutions $Z_n \to \ldots \to Z_1 \to X$ so that $|M_n|$ is free out of singularities of Z_n (see (2.4)), since 3-dimensional terminal singularities are isolated.

(iv) If $|M_n|$ is base point free on Z_n, then we stop. Note that Z_n is a Gorenstein terminal 3-fold.

(v) If $|M_n|$ has base points, then $\mathrm{Bs}|M_n|$ consists of smooth points of Z_n by our construction. We then consider the usual resolution of indeterminancies over $\mathrm{Bs}|M_n|$, say $Z_k \to \ldots \to Z_n$, which is composed of a sequence of blow-ups along smooth points or curves by Hironaka's big theorem.

(vi) Thus we may end up with a 3-fold Z_k so that $|M_k|$ is base point free. We call

$$\mu\colon Z_k \xrightarrow{\tau_k} \ldots \xrightarrow{\tau_{n+1}} Z_n \xrightarrow{\tau_n} \ldots \xrightarrow{\tau_1} X \tag{2.5}$$

a Gorenstein resolution of indeterminancies of $|M|$. Note that Z_k is a Gorenstein terminal 3-fold in general.

Theorem 2.5. *Let $|M|$ be a moving linear system on a projective Gorenstein terminal 3-fold X and $D \in |M|$ be a general member. Let $\mu : Z_k \to X$ be the Gorenstein resolution of indeterminancies as in (2.5). Then $2D_{Z_k/X} \geq K_{Z_k/X}$.*

Proof. We keep the notation as in above Steps (i)$\sim$ (vi). For each $i < n$, we have $D_{Z_{i+1}/Z_i} \geq K_{Z_{i+1}/Z_i}$ by Theorem 2.3. For each $i \geq n$, τ_{i+1} is a blowup along a smooth curve or a smooth point, contained in D_{Z_i}. Let E_{i+1} be the exceptional divisor. Then $2D_{Z_{i+1}/Z_i} \geq 2E_{i+1} \geq K_{Z_{i+1}/Z_i}$.

Since $D_{Z_{i+1}/X} = \tau_{i+1}^* D_{Z_i/X} + D_{Z_{i+1}/Z_i}$ and $K_{Z_{i+1}/X} = \tau_{i+1}^* K_{Z_i/X} + K_{Z_{i+1}/Z_i}$. The statement now follows easily by induction. □

3. **The canonical family of curves of genus 2**

Let X be a projective Gorenstein minimal 3-fold of general type. The fact that K_X^3 being even allows us to assume $p_g(X) \geq 5$ in order to prove Theorem 1.1. Thus we may always consider the non-trivial canonical map φ_1. Set $d := \dim \overline{\varphi_1(X)}$.

The following inequalities are already known:

I. If $d \neq 2$, then

$$K_X^3 \geq \min\{2p_g(X) - 6, \ \frac{7}{5}p_g(X) - 2\}$$

by [3, Theorem 5 (1)] and Catanese–Chen–Zhang [5, Theorem 4.1].

II. If $d = 2$ and X is canonically fibred by curves C of genus $g(C) \geq 3$, then $K_X^3 \geq 2p_g(X) - 4$ by [3, Theorem 4.1(ii)].

Theorem 3.1. *Let X be a projective minimal smooth 3-fold of general type. Suppose that $d = 2$ and X is canonically fibred by curves of genus 2. Then*

$$K_X^3 \geq \frac{1}{3}(4p_g(X) - 10).$$

The inequality is sharp.

Proof. Write $|K_X| = |M| + F$, where $|M|$ is the moving part and F is the fixed part. Let

$$\mu : X' = Z_k \to \ldots \to Z_1 \to X$$

be the Gorenstein resolution of indeterminancies as (2.5). Let $g = \varphi_1 \circ \mu$ and take the Stein factorization, we have the induced fibration $f \colon X' \longrightarrow W$.

A general fiber of f is a smooth curve of genus 2 by assumption of the theorem. Let D be a general member of $|M|$ and $S := D_{X'}$ be the general member of the moving part of $|\mu^* M|$. Then we have

$$\mu^* K_X = \mu^* M + \mu^* F = S + D_{X'/X} + \mu^* F.$$

Set $E' := D_{X'/X} + \mu^* F$.

On the surface S, set $L := \mu^*(K_X)|_S$. We also have $S|_S \equiv aC$ where $a \geq p_g(X) - 2$ and C is a general fiber of the restricted fibration $f|_S : S \longrightarrow f(S)$. Note that the above C lies in the same numerical class as that of a general fiber of f. One has

$$(\mu^* K_X^2 \cdot S) \geq (\mu^* K_X \cdot_S S) \geq a(L \cdot C) \geq (L \cdot C)(p_g(X) - 2).$$

If $(L \cdot C) \geq 2$, then we have already $K_X^3 \geq (\mu^* K_X^2 \cdot S) \geq 2p_g(X) - 4$. It remains to consider the case $(L \cdot C) = 1$. Note that, in this situation,

$|M|$ must have base points. Otherwise, $\mu =$ identity and

$$(L \cdot C) = (K_X|_S \cdot C) = ((K_X + S)|_S \cdot C) = (K_S \cdot C) = 2,$$

a contradiction.

Denote $E'|_S := E'_V + E'_H$, where E'_V is the vertical part and E'_H is the horizontal part with respect to $f|_S$. Since $(E'_H \cdot C) = (E'|_S \cdot C) = (L \cdot C) = 1$, E'_H is an irreducible curve and is a section of the restricted fibration $f|_S$.

Denote $K_{X'/X}|_S := E_V + E_H$, where E_V is the vertical part and E_H is the horizontal part. From $(K_S \cdot C) = 2$, one sees that $(K_{X'/X}|_S \cdot C) = 1$ and hence $(E_H \cdot C) = 1$. This also means that E_H is an irreducible curve and we may assume that $E_H = E_0|_S$ for some μ-exceptional divisor E_0. Notice that $2E' \geq K_{X'/X}$ by Theorem 2.5. In particular E_0 is contained in E'. Therefore, $E'_H = E_H$ and $2E'_V \geq E_V$.

Let $G := E_H = E'_H$. Since $2E'_V - E_V$ is effective and vertical, we see that $2(E'_V \cdot G) \geq (E_V \cdot G)$. On the surface S, we have

$$\begin{aligned} & (2\mu^* K_X|_S + E'_V) \cdot G \\ = \ & (\mu^* K_X|_S + S|_S + 2E'_V + E'_H) \cdot G \\ \geq \ & (\mu^* K_X|_S + S|_S + E_V + E_H) \cdot G \\ = \ & (\mu^* K_X + K_{X'/X}|_S + S|_S) \cdot G \\ = \ & (K_S \cdot G) \geq -2 - G^2 \end{aligned}$$

We also have

$$\begin{aligned} (\mu^* K_X|_S - E'_V) \cdot G &= (S|_S \cdot G) + (E'_H \cdot G) \\ &= a(C \cdot G) + G^2 \\ &\geq p_g(X) - 2 + G^2. \end{aligned}$$

Combining these, we get $3(\mu^*(K_X)|_S \cdot G) \geq p_g(X) - 4$ and therefore

$$(\mu^*(K_X)|_S \cdot E'|_S) \geq (\mu^*(K_X)|_S \cdot G) \geq \frac{1}{3}(p_g(X) - 4).$$

Finally we have

$$\begin{aligned} K_X^3 &= \mu^*(K_X)^3 \geq (\mu^*(K_X)^2 \cdot S) \\ &= (\mu^*(K_X)|_S \cdot S|_S) + (\mu^*(K_X)|_S \cdot E'|_S) \\ &\geq (p_g(X) - 2) + \tfrac{1}{3}(p_g(X) - 4) = \tfrac{2}{3}(2p_g(X) - 5). \end{aligned}$$

The inequality is sharp by virtue of Kobayashi's example [6]. □

Theorem 1.1 follows directly from known results I, II and Theorem 3.1.

We would like to ask the following:

Open problem 3.2. *Is the inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ true for any projective minimal 3-fold X of general type?*

Some known results includes: if $p_g(X) \geq 3$, then $K_X^3 \geq 1$ and if $p_g(X) \geq 4$, then $K_X^3 \geq 2$ (cf. [4, Theorem 1.5]).

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