INCREASING STABILITY OF THE INVERSE BOUNDARY VALUE PROBLEM FOR THE SCHRÖDINGER EQUATION

V ISAKOV, S NAGAYASU, G UHLMANN, AND J-N WANG

Abstract. In this work we study the phenomenon of increasing stability in the inverse boundary value problem for the Schrödinger equation. This problem was previously considered by Isakov in which he discussed the phenomenon in different ranges of the wave number (or energy). The main contribution of this work is to provide a unified and easier approach to the same problem based on the complex geometrical optics solutions.

1. Introduction

Most of inverse problems are known to be severely ill-posed. This weakness makes it extremely difficult to design reliable reconstruction algorithms in practice. However, in some cases, it has been observed numerically that the stability increases with respect to some parameter such as the wave number (or energy) (see, for example, [6] for the inverse obstacle scattering problem). Several rigorous justifications of the increasing stability phenomena in different settings were obtained by Isakov *et al* [9, 11, 12, 3, 4]. In particular, in [12], Isakov considered the Helmholtz equation with a potential

$$\big(\Delta + k^2 + q(x)\big)u(x) = 0 \text{ in } \Omega \subset \mathbb{R}^n \tag{1.1}$$

with $n \geq 3$. He obtained stability estimates of determining q by the Dirichlet-to-Neumann map for different ranges of k, which demonstrate the increasing stability phenomena in k. The purpose of this work is to provide a more straightforward way to derive a similar estimate for the inverse boundary value for (1.1). In [12], Isakov used *real* geometrical optics solutions for the large wave number k. In this work, by more careful choice of an additional large parameter and a priori constraints we are able to use *complex* geometrical optics (CGO) solutions introduced by Calderón [5] and Sylvester-Uhlmann [17] for all $k \geq 1$. This will simplify the proof in [12]. Recently similar results were obtained by Isaev and Novikov [10] by using less explicit and more complicated methods of scattering theory.

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In this work, instead of considering the Dirichlet-to-Neumann map, we define the boundary measurements to be the Cauchy data corresponding to (1.1)

$$\mathcal{C}_q = \left\{ \left(u|_{\partial\Omega}, \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} \right), \text{ where } u \text{ is a solution to (1.1)} \right\}.$$

Hereafter, ν is the unit outer normal vector of $\partial\Omega$. Assume that $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$ are two Cauchy data associated with refraction indices q_1 and q_2, respectively. To measure the distance between two Cauchy data, we define

$$\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) = \max\left\{ \max_{(f,g)\in\mathcal{C}_{q_1}} \min_{(\widetilde{f},\widetilde{g})\in\mathcal{C}_{q_2}} \frac{\|(f,g)-(\widetilde{f},\widetilde{g})\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}}, \right.$$
$$\left. \max_{(f,g)\in\mathcal{C}_{q_2}} \min_{(\widetilde{f},\widetilde{g})\in\mathcal{C}_{q_1}} \frac{\|(f,g)-(\widetilde{f},\widetilde{g})\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}} \right\},$$

where

$$\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}} = \left(\|f\|^2_{H^{1/2}(\partial\Omega)} + \|g\|^2_{H^{-1/2}(\partial\Omega)} \right)^{1/2}.$$

Note that $\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$ is a case of distance between two subspaces in a Hilbert space. This notion has a long tradition and still attracts attentions, for instance, see the book [1, Ch. III] and the recent literature [14].

Our main theorem is stated as follows.

Theorem 1.1. *Let $n \geq 3$. Assume $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$ are Cauchy data corresponding to $q_1(x)$ and $q_2(x)$, respectively. Let $s > n/2$ and $M > 0$. Assume $\|q_l\|_{H^s(\Omega)} \leq M$ $(l = 1, 2)$ and $\operatorname{supp}(q_1 - q_2) \subset \Omega$. Denote $\widetilde{q}$ the zero extension of $q_1 - q_2$. Then for $k \geq 1$ and $\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq 1/e$ we have the following stability estimate:*

$$\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} \leq Ck^4 \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + C\left(k + \log \frac{1}{\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right)^{-(2s-n)}, \tag{1.2}$$

where $C > 0$ depends only on n, s, Ω, M and $\operatorname{supp}(q_1 - q_2)$.

From estimate (1.2), we can see that the influence of the logarithmic part becomes small when k is large. Indeed, in view of the right hand side of (1.2), we can check that

$$k^4\varepsilon \geq \left(k + \log\frac{1}{\varepsilon} \right)^{-(2s-n)}$$

provided $k \geq k(\varepsilon)$, where $k(\varepsilon)$ solves

$$k(\varepsilon)^4\varepsilon = (k(\varepsilon) + \log\frac{1}{\varepsilon})^{-(2s-n)}.$$

We can see that $k(\varepsilon)$ is a decreasing function of ε and $k(\varepsilon) \approx \varepsilon^{-1/(2s-n+4)}$. In other words, given

$$\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq \varepsilon, \tag{1.3}$$

the stability behaves more like Lipschitz when k is sufficient large, precisely, $k \geq k(\varepsilon)$ (see Figure 1). On the other hand, if k is confined in a finite interval, then for small ε, the stability estimate is more or less of a logarithmic type.

We also want to point out that if (1.3) is true for all $k \neq 0$, then by substituting $k = k(\varepsilon)$ into the right hand side of (1.2), we obtain a Hölder type estimate in terms of the Cauchy data $\mathcal{C}_q$ at all wave numbers. To facilitate further discussions, we denote $\mathcal{C}_q$ at wave number k^2 as $\mathcal{C}_{q,k^2}$. It is clear that the information of $\mathcal{C}_{q,k^2}$ is contained in that of the Dirichlet-to-Neumann map $\Lambda_{q,z}$, defined by

$$\Lambda_{q,z} f = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

with u satisfying $(-\Delta - q)u = zu$ in Ω and $u|_{\partial\Omega} = f$, for all $\mathbb{C} \ni z \notin \sigma(-\Delta - q)$, where $\sigma(-\Delta - q)$ is the set of Dirichlet eigenvalues of $-\Delta - q$ on Ω. Note that $\Lambda_{q,z}$ can be extended to a meromorphic function in $\mathbb{C}$ with poles at $\sigma(-\Delta - q)$. $\Lambda_{q,z}$ is closely to the *boundary spectral data* $\{\lambda_k, \partial\phi_k/\partial\nu|_{\partial\Omega}\}_{k=1}^{\infty}$, where $\lambda_k \in \sigma(-\Delta - q)$ and ϕ_k is the corresponding normalized eigenfunction, and the *Hyperbolic Dirichlet-to-Neumann map* given by

$$\Lambda_q^H : f \mapsto \frac{\partial v}{\partial \nu}|_{\partial\Omega\times[0,T]},$$

where v satisfies

$$\begin{cases} \partial_{tt} v - \Delta v - qv = 0 \quad \text{in} \quad \Omega \times [0,T], \\ v = f \quad \text{on} \quad \partial\Omega \times [0,T], \\ v(x,0) = \partial_t v(x,0) = 0 \quad x \in \Omega. \end{cases}$$

In fact, these three data are equivalent (see, for example, [13]). Hölder type stability estimates of recovering q using the boundary spectral data and the hyperbolic Dirichlet-to-Neumann map were first derived by Alessandrini, Sylvester [2] and by Sun [16], respectively.

Finally, we would like to point out that unlike in the acoustic case where the constant associated with the Lipschitz estimate grows exponentially in k [15], the constant here grows only polynomially in k. Similarly, the corresponding constant obtained in [12] (see estimate (8) there) also grows polynomially in k.

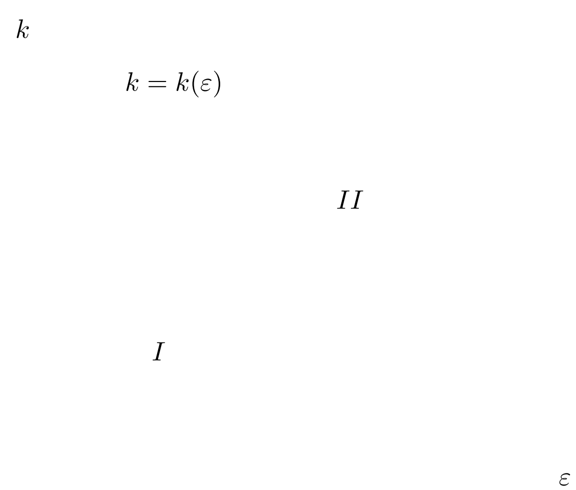

FIGURE 1. The stability estimate is Logarithmic dominated in Region I and is Lipschitz dominated in Region II.

The paper is organized as follows. In Section 2, we will collect some known results about the CGO solutions and an estimate for the difference of potentials, which are essential tools in the proof. In Section 3, we present a detailed proof of Theorem 1.1.

2. Preliminaries

To begin, we state the existence of CGO solutions for (1.1). These special solutions are first constructed by Sylvester and Uhlmann [17]. Another construction based on the Fourier series is given by Hähner [8].

Lemma 2.1. *Let $s > n/2$. Assume that $\zeta = \eta + i\xi$ ($\eta, \xi \in \mathbb{R}^n$) satisfies*

$$|\eta|^2 = k^2 + |\xi|^2 \quad \text{and} \quad \eta \cdot \xi = 0,$$

i.e., $\zeta \cdot \zeta = k^2$. Then there exist constants C_* and $C > 0$, which are independent of k, such that if $|\xi| > C_* \|q\|_{H^s(\Omega)}$ then there exists a solution u to the equation (1.1) *of the form*

$$u(x) = e^{i\zeta \cdot x}\big(1 + \psi(x)\big), \tag{2.1}$$

where ψ has the estimate

$$\|\psi\|_{H^s(\Omega)} \le \frac{C}{|\xi|} \|q\|_{H^s(\Omega)}.$$

Remark 2.2. *Note that the correction term ψ decays in $Im\,\zeta$. This property is crucial in obtaining that the constant associated with the Lipschiz estimate grows only polynomially in k.*

Next inequality follows from the weak formulation of the equation (1.1). We refer to [7] for the proof.

Proposition 2.3. *Let u_l and $\mathcal{C}_{q_l}$ be solution and Cauchy data to the equation* (1.1) *with $q = q_l$, respectively ($l = 1, 2$). Then the following estimate holds:*

$$\left| \int_\Omega (q_2 - q_1) u_1 u_2 \, dx \right|$$
$$\le \left\| \left(u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} \left\| \left(u_2, \frac{\partial u_2}{\partial \nu} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_1}).$$

3. Proof of main theorem

To prove Theorem 1.1, we first derive two lemmas.

Lemma 3.1. *Under the assumptions in Theorem 1.1,*

$$|\mathcal{F}\widetilde{q}(r\omega)| \le Ck^4 e^{Ca} \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + \frac{C}{a} \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} \tag{3.1}$$

holds for $k \ge 1$, $r \ge 0$, $\omega \in \mathbb{R}^n$ with $|\omega| = 1$ and $a > C_* M$ with $k^2 + a^2 > r^2/4$, where $C > 0$ depends only on n, s, M, Ω and $\operatorname{supp}(q_1 - q_2)$ and C_* is the constant given in Lemma 2.1.

Proof. We will use CGO solutions (2.1) with appropriately chosen parameter ζ. Let us denote $\zeta_l = \eta_l + i\xi_l$, $l = 1, 2$. We can choose $\omega^\perp, \widetilde{\omega}^\perp \in \mathbb{R}^n$ satisfying

$$\omega \cdot \omega^\perp = \omega \cdot \widetilde{\omega}^\perp = \omega^\perp \cdot \widetilde{\omega}^\perp = 0 \quad \text{and} \quad |\omega^\perp| = |\widetilde{\omega}^\perp| = 1.$$

Now we set

$$\xi_1 = a\omega^\perp, \quad \eta_1 = -\frac{r}{2}\omega + \sqrt{k^2 + a^2 - \frac{r^2}{4}}\,\widetilde{\omega}^\perp,$$
$$\xi_2 = -\xi_1 \quad \text{and} \quad \eta_2 = -r\omega - \eta_1,$$

and thus

$$\xi_l \cdot \eta_l = 0, \quad |\eta_l|^2 = k^2 + |\xi_l|^2$$

and $|\xi_l| = a \geq C_* M \geq C_* \|q_\ell\|_{H^s(\Omega)}$. From Lemma 2.1, there exist CGO solutions

$$u_l(x) = e^{i\zeta_l x}\big(1 + \psi_l(x)\big)$$

to equation (1.1) with $q = q_l$, where ψ_l satisfies

$$\|\psi_l\|_{H^s(\Omega)} \leq \frac{C}{|\xi_l|}\|q_l\|_{H^s(\Omega)}.$$

Note that ψ_l also satisfies the estimate

$$\|\psi_l\|_{H^s(\Omega)} \leq \frac{C}{|\xi_l|}\|q_l\|_{H^s(\Omega)} \leq \frac{CM}{a} < \frac{CM}{C_* M} = \frac{C}{C_*}. \tag{3.2}$$

Now, by Proposition 2.3 and using the relation $-r\omega = \zeta_1 + \zeta_2$, we have that

$$\begin{aligned} &\left|\int_\Omega \widetilde{q}(x) e^{-ir\omega\cdot x}(1+\psi_1)(1+\psi_2)\,dx\right| = \left|\int_\Omega (q_2 - q_1) u_1 u_2\,dx\right| \\ &\leq \left\|\left(u_1, \frac{\partial u_1}{\partial\nu}\right)\right\|_{H^{1/2}\oplus H^{-1/2}} \left\|\left(u_2, \frac{\partial u_2}{\partial\nu}\right)\right\|_{H^{1/2}\oplus H^{-1/2}} \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_1}). \end{aligned}$$

Subsequently, we obtain

$$\begin{aligned} |\mathcal{F}\widetilde{q}(r\omega)| &= \left|\int_\Omega \widetilde{q}(x) e^{-ir\omega\cdot x}\,dx\right| \\ &\leq \left|\int_\Omega \widetilde{q}(x) e^{-ir\omega\cdot x}(1+\psi_1)(1+\psi_2)\,dx\right| \\ &\qquad + \left|\int_\Omega \widetilde{q}(x) e^{-ir\omega\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\,dx\right| \\ &\leq \left\|\left(u_1, \frac{\partial u_1}{\partial\nu}\right)\right\|_{H^{1/2}\oplus H^{-1/2}} \left\|\left(u_2, \frac{\partial u_2}{\partial\nu}\right)\right\|_{H^{1/2}\oplus H^{-1/2}} \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_1}) \\ &\qquad + \left|\int_\Omega \widetilde{q}(x) e^{-ir\omega\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\,dx\right|. \end{aligned} \tag{3.3}$$

In view of (3.3), we want to estimate $\big\|(u_l, \partial u_l/\partial\nu)\big\|_{H^{1/2}\oplus H^{-1/2}}$. Recall that u_l solves (1.1) with $q = q_l$. Using assumptions $\|q_l\|_{H^s(\Omega)} \leq M$, and $s > n/2$, and $k \geq 1$, we have that

$$\left\|\frac{\partial u_l}{\partial\nu}\right\|_{H^{-1/2}(\partial\Omega)} \leq Ck^2\|u_l\|_{L^2(\Omega)} + C\|\nabla u_l\|_{L^2(\Omega)}$$

and thus

$$\left\|\left(u_l, \frac{\partial u_l}{\partial\nu}\right)\right\|_{H^{1/2}\oplus H^{-1/2}} \leq Ck^2\|u_l\|_{L^2(\Omega)} + C\|\nabla u_l\|_{L^2(\Omega)}.$$

We now choose $R_0 > 0$ large enough such that $\Omega \subset B_{R_0}(0)$. Then we have

$$|u_l(x)| \leq e^{-\xi_l\cdot x}\big(1 + |\psi_l(x)|\big) \leq Ce^{|\xi_l|R_0} = Ce^{aR_0}$$

since

$$|\psi_l(x)| \leq \|\psi_l\|_{L^\infty(\Omega)} \leq C\|\psi_l\|_{H^s(\Omega)} \leq C$$

by $s > n/2$ and (3.2). It follows that

$$\|u_l\|_{L^2(\Omega)} \leq Ce^{aR_0}.$$

On the other hand, in view of $\|\nabla\psi_l\|_{L^2(\Omega)} \le \|\psi_l\|_{H^s(\Omega)} \le C$ $(s > n/2 \ge 3/2 > 1)$ and (3.2), we can estimate

$$\begin{aligned}\|\nabla u_l\|_{L^2(\Omega)} &= \left\|iu_l\zeta_l + e^{i\zeta_l\cdot\bullet}\nabla\psi_l\right\|_{L^2(\Omega)} \le |\zeta_l|\|u_l\|_{L^2(\Omega)} + e^{|\xi_l|R_0}\|\nabla\psi_l\|_{L^2(\Omega)}\\ &\le C\big(k+|\xi_l|\big)e^{aR_0} + Ce^{|\xi_l|R_0} = C(k+a)e^{aR_0} + Ce^{aR_0} \le Cke^{Ca}.\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}\left\|\left(u_l, \frac{\partial u_l}{\partial\nu}\right)\right\|_{H^{1/2}\oplus H^{-1/2}} &\le Ck^2\|u_l\|_{L^2(\Omega)} + C\|\nabla u_l\|_{L^2(\Omega)}\\ &\le Ck^2e^{Ca} + Cke^{Ca} \le Ck^2e^{Ca}.\end{aligned}\tag{3.4}$$

Note that here C depends on n, s, M, and the diameter of Ω.

Let $\chi \in C_0^\infty(\Omega)$ be a cut-off function satisfying $\chi \equiv 1$ near $\operatorname{supp}(q_1 - q_2)$, then we have

$$\begin{aligned}&\left|\int_\Omega \widetilde{q}(x)e^{-ir\omega\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\,dx\right|\\ &\quad= \left|\int_\Omega \widetilde{q}(x)\chi(x)e^{-ir\omega\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\,dx\right|\\ &\quad\le \int_\Omega |\widetilde{q}(x)||\chi(\psi_1+\psi_2+\psi_1\psi_2)|\,dx\\ &\quad\le \|\widetilde{q}\|_{H^{-s}(\Omega)}\|\chi(\psi_1+\psi_2+\psi_1\psi_2)\|_{H^s(\Omega)}.\end{aligned}\tag{3.5}$$

Since $s > n/2$ and (3.2), we can estimate

$$\begin{aligned}&\|\chi(\psi_1+\psi_2+\psi_1\psi_2)\|_{H^s(\Omega)}\\ &\le \|\chi\|_{H^s(\Omega)}\big(\|\psi_1\|_{H^s(\Omega)} + \|\psi_2\|_{H^s(\Omega)} + \|\psi_1\|_{H^s(\Omega)}\|\psi_2\|_{H^s(\Omega)}\big)\\ &\le \|\chi\|_{H^s(\Omega)}\left(\frac{CM}{a} + \frac{CM}{a} + \frac{C}{C_*}\cdot\frac{CM}{a}\right) \le \frac{C}{a}.\end{aligned}\tag{3.6}$$

Finally, (3.1) follows from (3.3), (3.4), (3.5), and (3.6). □

The following lemma is an easy corollary of Lemma 3.1.

Lemma 3.2. *Suppose that the assumptions in Theorem 1.1 hold. Let $R > C_*M$ with C_* being the constant given in Lemma 2.1. Then for $k \ge 1$, $r \ge 0$ and $\omega \in \mathbb{R}^n$ with $|\omega| = 1$, the following estimates hold true: if $0 \le r \le k+R$ then*

$$|\mathcal{F}\widetilde{q}(r\omega)| \le Ck^4e^{CR}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) + \frac{C}{R}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)};\tag{3.7}$$

if $r \ge k+R$ then

$$|\mathcal{F}\widetilde{q}(r\omega)| \le Ck^4e^{Cr}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) + \frac{C}{r}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}.\tag{3.8}$$

Proof. It is enough to take $a = R$ when $0 \le r \le k+R$, and take $a = r$ when $r \ge k+R$ in Lemma 3.1. □

Now we prove our main theorem.

Proof of Theorem 1.1. Written in polar coordinates, we have that

$$
\begin{aligned}
\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} &= C\int_0^\infty \int_{|\omega|=1} |\mathcal{F}\widetilde{q}(r\omega)|^2(1+r^2)^{-s}r^{n-1}\,d\omega dr \\
&= C\Big(\int_0^{k+R}\int_{|\omega|=1} |\mathcal{F}\widetilde{q}(r\omega)|^2(1+r^2)^{-s}r^{n-1}\,d\omega dr \\
&\quad + \int_{k+R}^{T}\int_{|\omega|=1} |\mathcal{F}\widetilde{q}(r\omega)|^2(1+r^2)^{-s}r^{n-1}\,d\omega dr \\
&\quad + \int_{T}^{\infty}\int_{|\omega|=1} |\mathcal{F}\widetilde{q}(r\omega)|^2(1+r^2)^{-s}r^{n-1}\,d\omega dr\Big) \\
&=: C(I_1+I_2+I_3),
\end{aligned}
\tag{3.9}
$$

where $R > C_*M$ and $T \geq k+R$ are parameters which will be chosen later.

Our task now is to estimate each integral separately. We begin with I_3. Since $|\mathcal{F}\widetilde{q}(r\omega)| \leq C\|q_1-q_2\|_{L^2(\Omega)}$, $q_1-q_2 \in H^s_0(\Omega)$ and $s > n/2$, we get

$$
\begin{aligned}
I_3 &\leq C\int_T^\infty \|q_1-q_2\|^2_{L^2(\Omega)}(1+r^2)^{-s}r^{n-1}\,dr \leq CT^{-m}\|q_1-q_2\|^2_{L^2(\Omega)} \\
&\leq CT^{-m}\left(\varepsilon\|q_1-q_2\|^2_{H^{-s}(\Omega)} + \frac{1}{\varepsilon}\|q_1-q_2\|^2_{H^s(\Omega)}\right) \\
&\leq CT^{-m}\left(\varepsilon\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{1}{\varepsilon}\right)
\end{aligned}
\tag{3.10}
$$

for $\varepsilon > 0$, where $m := 2s-n$.

On the other hand, by estimate (3.7), we can obtain

$$
\begin{aligned}
I_1 &\leq \int_0^{k+R}\left(Ck^4e^{CR}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) + \frac{C}{R}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}\right)^2(1+r^2)^{-s}r^{n-1}\,dr \\
&\leq C\left(k^8e^{CR}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \frac{1}{R^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\right)\int_0^\infty(1+r^2)^{-s}r^{n-1}\,dr \\
&= C\left(k^8e^{CR}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \frac{1}{R^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\right).
\end{aligned}
\tag{3.11}
$$

In the same way, using estimate (3.8), we have

$$
\begin{aligned}
I_2 &\leq C\int_{k+R}^{T}\left(Ck^4e^{Cr}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) + \frac{C}{r}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}\right)^2(1+r^2)^{-s}r^{n-1}\,dr \\
&\leq Ck^8\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2\int_{k+R}^{T}e^{Cr}(1+r^2)^{-s}r^{n-1}\,dr \\
&\qquad\qquad + C\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\int_{k+R}^{T}(1+r^2)^{-s}r^{n-1}\,dr \\
&\leq C\left(k^8e^{CT}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \frac{1}{R^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\right),
\end{aligned}
\tag{3.12}
$$

where we have used

$$\int_{k+R}^{T} e^{Cr}(1+r^2)^{-s}r^{n-1}\,dr \le e^{CT}\int_{k+R}^{T}(1+r^2)^{-s}r^{n-1}\,dr$$
$$\le e^{Ct}\int_0^\infty (1+r^2)^{-s}r^{n-1}\,dr = Ce^{CT},$$

$$\int_{k+R}^{T}(1+r^2)^{-s}r^{n-1}\,dr \le \int_{k+R}^{T} r^{-2s+n-1}\,dr$$
$$\le \frac{1}{2s-n+2}\frac{1}{(k+R)^{2s-n+2}} \le \frac{C}{(k+R)^2} \le \frac{C}{R^2},$$

and $s > n/2$, $k \ge 1$. Combining (3.9)–(3.12) gives

$$\begin{aligned}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} &\le C(I_1+I_2+I_3)\\ &\le C\left(k^8e^{CR}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \frac{1}{R^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\right)\\ &\quad + C\left(k^8e^{CT}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \frac{1}{R^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\right)\\ &\quad + CT^{-m}\left(\varepsilon\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{1}{\varepsilon}\right)\\ &\le C\left(\frac{2}{R^2}+\varepsilon T^{-m}\right)\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + Ck^8e^{CR}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2\\ &\quad + Ck^8e^{CT}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \frac{CT^{-m}}{\varepsilon}.\end{aligned} \tag{3.13}$$

To continue, we consider the following two cases:

$$\text{(i) } k+R \le p\log\frac{1}{A} \quad \text{and} \quad \text{(ii) } k+R \ge p\log\frac{1}{A},$$

where $R > C_*M$ and $p > 0$ are constants which will be determined later. We begin with the first case (i). Taking

$$R > 2\sqrt{C} \tag{3.14}$$

and $\varepsilon = cT^m$ ($c \ll 1$), we deduce that

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \le Ck^8A + Ck^8e^{CT}A + CT^{-2m} \tag{3.15}$$

for any $T \ge k+R$ by (3.13), where $A = \operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2$.

Now we choose $T = p\log(1/A)$, which is greater than or equal to $k+R$ by the condition (i). Our current aim is to show that there exists $C_1 > 0$ such that

$$k^8e^{CT}A \le C_1\left(k+\log\frac{1}{A}\right)^{-2m} \tag{3.16}$$

and

$$T^{-2m} \le C_1\left(k+\log\frac{1}{A}\right)^{-2m}. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15) clearly implies (1.2). We remark that (3.17) is equivalent to

$$C_1^{-1/2m}\left(k+\log\frac{1}{A}\right)\le p\log\frac{1}{A}. \tag{3.18}$$

Since we have

$$k+\log\frac{1}{A}\le(k+R)+\log\frac{1}{A}\le(p+1)\log\frac{1}{A}$$

by (i), condition (3.18) (i.e. (3.17)) holds whenever

$$C_1^{-1/2m}\le\frac{p}{p+1}. \tag{3.19}$$

On the other hand, condition (3.16) is equivalent to

$$8\log k+(Cp-1)\log\frac{1}{A}+2m\log\left(k+\log\frac{1}{A}\right)\le\log C_1. \tag{3.20}$$

Using (i), we can bound the left-hand side of (3.20) by

$$(\text{LHS of (3.20)})\le 8\log p+2m\log(p+1)+(Cp-1)\log\frac{1}{A}+2(m+4)\log\log\frac{1}{A}.$$

Choosing

$$p\le\frac{1}{2C}, \tag{3.21}$$

we can see that

$$\begin{aligned}&(\text{LHS of (3.20)})\\&\le 8\log\frac{1}{2C}+2m\log\left(\frac{1}{2C}+1\right)-\frac{1}{2}\log\frac{1}{A}+2(m+4)\log\log\frac{1}{A}\\&\le 8\log\frac{1}{2C}+2m\log\left(\frac{1}{2C}+1\right)+\max_{z\ge 2}\left(-\frac{1}{2}z+2(m+4)\log z\right)\\&= 8\log\frac{1}{2C}+2m\log\left(\frac{1}{2C}+1\right)+2(m+4)\big(\log(4m+16)-1\big).\end{aligned}$$

Therefore, condition (3.20) (i.e. (3.16)) is satisfied provided

$$8\log\frac{1}{2C}+2m\log\left(\frac{1}{2C}+1\right)+2(m+4)\big(\log(4m+16)-1\big)\le\log C_1. \tag{3.22}$$

Next we consider case (ii). We choose $T=k+R$ and observe that the term I_2 in (3.9) does not appear in this case. Hence, instead of (3.13), we have

$$\begin{aligned}&\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\\&\le C\left(\frac{1}{R^2}+\varepsilon T^{-m}\right)\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}+Ck^8e^{CR}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2+\frac{CT^{-m}}{\varepsilon}\end{aligned}$$

Setting $\varepsilon=T^m/R^2$ implies that

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\le\frac{2C}{R^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}+Ck^8e^{CR}A+CR^2(k+R)^{-2m}.$$

Now we choose

$$R>2\sqrt{C} \tag{3.23}$$

and obtain that

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\le Ck^8A+C(k+R)^{-2m},$$

which implies the desired estimate (1.2) since from condition (ii) we have

$$k+R \geq \frac{k}{2}+\frac{k+R}{2} \geq \frac{k}{2}+\frac{p}{2}\log\frac{1}{A} \geq \frac{\min\{p,1\}}{2}\left(k+\log\frac{1}{A}\right).$$

As the last step, we choose appropriate R, p, and C_1 to complete the proof. We first pick $R > C_* M$ sufficiently large satisfying (3.14) and (3.23) and then choose p small enough satisfying (3.21). Finally, we take C_1 large enough satisfying (3.19) and (3.22). □

4. Conclusion

We think that increasing stability is an important feature of the inverse boundary problem for the Schrödinger potential which should lead to higher resolution of numerical algorithms. It is important to collect numerical evidence of this phenomenon. Our method is based on the CGO solutions constructed in [8] where the constants in Lemma 2.1 are explicit. So most likely one can give explicit constants in Theorem 1.1 at least for particular domains Ω like balls. Contrary to the acoustic case [15], the constants in the estimate (1.2) depend only polynomially on k. It is an important and challenging question to determine whether the exponential dependence on k of the estimates in [15] is indeed generic if there are no assumptions on rays.

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