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A β-expectation tolerance interval for general balanced mixed linear models

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Abstract

A β-expectation tolerance interval procedure is derived from the concept of generalized pivotal quantity, which has been frequently used to obtain confidence intervals in situations where standard procedures do not lead to useful solutions. The proposed procedure can be applied to general balanced mixed linear models. Some practical examples are given to illustrate the proposed procedure. In addition, detailed simulation studies are conducted to evaluate its performance, showing that it can be recommended for use in practical applications.

Keywords: Chi-squared approximation; Generalized p-value; Generalized confidence interval; Variance component

1. Introduction

Statistical tolerance intervals are useful in life-testing, process reliability studies, pharmaceutical engineering, and many other areas. Two basic types of tolerance intervals have received considerable attention: (i) β-content tolerance intervals and (ii) β-expectation tolerance intervals. Most papers concerning construction of these two types of tolerance intervals are restricted to the case of a simple random sample (SRS) from $N\left(\mu, \sigma^2\right)$ or balanced

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one-way random effects models. Recently, Liao and Iyer (2004) have developed a β-content tolerance interval procedure applicable to all balanced mixed linear models. Earlier work related to the construction of β-content tolerance intervals refers to Lemon (1977), Beckman and Tietjen (1989), Mee and Owen (1983), Mee (1984), Vangel (1992), Wang and Iyer (1994), Liao and Iyer (2001), and Fernholz and Gillespie (2001) among others.

In this study, our main interest lies in β-expectation tolerance intervals which are also called *prediction intervals* for a single future observation or *mean-coverage* tolerance intervals. A β-expectation tolerance interval for a SRS from $N\left(\mu, \sigma^2\right)$ is first presented in Wilks (1941). Paulson (1943) proves that the problem of finding such a tolerance interval is exactly equivalent to finding a confidence interval for one single future observation. Fraser and Guttman (1956) establish the relationship between β-expectation tolerance intervals and hypothesis tests. Guttman (1970) deals with the problem from a Bayesian viewpoint. Mee (1984) considers the β-expectation tolerance intervals for balanced one-way random effects models and proposes an approximate method based on the result of Wilks (1941). Wang (1988) also provides an iterative algorithm to obtain an approximate β-expectation tolerance interval for balanced one-way random effects models. To the best of our knowledge, no β-expectation tolerance interval for more complex models has been considered in the literature.

We develop a β-expectation tolerance interval procedure for all mixed linear models provided balanced data is available. Our method is based on the concept of *generalized pivotal quantity*, presented in Weerahandi (1993), which has been proven successful in constructing β-content tolerance intervals by Liao and Iyer (2004). In the next section, we review the definition of β-expectation tolerance interval and the concept of generalized pivotal quantity. The derivation of the proposed β-expectation tolerance interval is presented in Section 3. Two practical examples are given in Section 4 to illustrate the proposed procedure. Section 5 contains a simulation study to evaluate the performance of the proposed method. A comparison between the proposed method and Mee's (1984) method is provided in Section 6.

2. Preliminaries

We first review the definition of β-expectation tolerance interval for a random variable and the concept of generalized pivotal quantity.

2.1. β-expectation tolerance interval

Let F denote the cumulative distribution for a random variable. An interval $[L(\boldsymbol{Y}), U(\boldsymbol{Y})]$ based on the data vector $\boldsymbol{Y}$ is said to be a two-sided β-expectation tolerance interval for F if $E\{F[U(\boldsymbol{Y})] - F[L(\boldsymbol{Y})]\} = \beta$. Thus, one can state that a proportion β of the population modeled by F is contained in the interval $[L(\boldsymbol{Y}), U(\boldsymbol{Y})]$ *on average*. Additionally, one can also state with confidence coefficient β that a future sample of size one from the underlying distribution is contained in the interval $[L(\boldsymbol{Y}), U(\boldsymbol{Y})]$. Similarly, one-sided β-expectation tolerance limits for F can be defined.

2.2. Generalized pivotal quantity

Let $\boldsymbol{y}$ be the realized value of the data vector $\boldsymbol{Y}$ and $\boldsymbol{\xi}$ be the vector of model parameters. Moreover, let ψ be a function of $\boldsymbol{\xi}$ for which a confidence interval is sought. According to Weerahandi (1993), a function $R = r(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi})$ of $\boldsymbol{Y}$, $\boldsymbol{y}$ and $\boldsymbol{\xi}$ is called a generalized pivotal quantity for ψ if it satisfies the following two conditions:

(i) R has a probability distribution that is free of unknown parameters.
(ii) The observed value of R, namely $r(\boldsymbol{y}; \boldsymbol{y}, \boldsymbol{\xi})$, depends on $\boldsymbol{\xi}$ only through ψ.

The percentiles of R can be analytically evaluated in simple problems but more conveniently estimated using Monte-Carlo algorithms in complex problems.

3. Derivation of β-expectation tolerance intervals

The problem of interest can be formulated as follows. It is desired to construct β-expectation tolerance intervals for distribution $N\left(\theta, \tau^2\right)$, where $\tau^2 = \sum_{i=1}^q h_i \sigma_i^2$. Here each σ_i^2 denotes a linear combination of the variance components in the model of interest and h_i are known constants. It is assumed that mutually independent statistics T, $S_1^2, S_2^2, \ldots, S_q^2$ are available such that

(i) $T \sim N\left(\theta, \sigma^2\right)$ with $\sigma^2 = \sum_{i=1}^q c_i \sigma_i^2$ and c_i being known constants.
(ii) $U_i = n_i S_i^2 / \sigma_i^2 \sim \chi^2_{n_i}$, for $i = 1, 2, \ldots, q$.

Since the construction for one-sided β-expectation tolerance limit is similar to the two-sided case, we now consider the two-sided case in detail. The following lemma due to Paulson (1943) describes the relationship between β-expectation tolerance intervals and confidence intervals for some function of the observations in a future independent sample.

Lemma 3.1 (*Paulson, 1943*). *If confidence limits $V_1(\boldsymbol{Y})$ and $V_2(\boldsymbol{Y})$ of level β are determined for V, a function of a future sample of new independent observations from the population, and if*

$$P = \int_{V_1}^{V_2} \mathrm{d}G(v),$$

where G is the distribution function of V, then $E[P] = \beta$.

We thus directly apply Lemma 3.1 to construct the tolerance interval of interest. Let $V_1(\boldsymbol{Y}) = T + D_L\left(S_1^2, S_2^2, \ldots, S_q^2\right)$ and $V_2(\boldsymbol{Y}) = T + D_U\left(S_1^2, S_2^2, \ldots, S_q^2\right)$. Also let $W \sim N\left(\theta, \tau^2\right)$ denote a new random observation independent of the data vector $\boldsymbol{Y}$. We now need to seek a 100β percent confidence interval $[V_1(\boldsymbol{Y}), V_2(\boldsymbol{Y})]$ for W. Namely, we need to determine the margin errors $D_L\left(S_1^2, S_2^2, \ldots, S_q^2\right)$ and $D_U\left(S_1^2, S_2^2, \ldots, S_q^2\right)$ such that

$$\begin{aligned}\beta &= \Pr\left(T + D_L < W < T + D_U\right)\\ &= \Pr\left(\frac{D_L}{\sqrt{\tau^2+\sigma^2}} < \frac{W-T}{\sqrt{\tau^2+\sigma^2}} < \frac{D_U}{\sqrt{\tau^2+\sigma^2}}\right)\\ &= \Pr\left(\frac{D_L}{\sqrt{\tau^2+\sigma^2}} < Z < \frac{D_U}{\sqrt{\tau^2+\sigma^2}}\right)\\ &= \Pr\left(D_L < \delta < D_U\right),\end{aligned}$$

where $\delta = Z\sqrt{\tau^2+\sigma^2}$. We thus use

$$\left[T + \delta_{\beta_1}, T + \delta_{\beta_2}\right] \tag{3.1}$$

as the required β-expectation tolerance interval, where δ_{β_1} and δ_{β_2} denote the $\left(100\beta_1\right)$th and $\left(100\beta_2\right)$th percentiles of δ with $\beta_2 - \beta_1 = \beta$.

Obviously, δ involves standardized normal variate Z and parameters $\sigma_1^2, \sigma_2^2, \ldots, \sigma_q^2$ and its percentiles are generally unavailable. We thus substitute the following generalized pivotal quantities for τ^2 and σ^2 in δ so as to obtain the required percentiles

$$R_{\tau^2} = \sum_{i=1}^{q} \frac{h_i n_i s_i^2}{U_i} = \sum_{i=1}^{q} \frac{h_i \sigma_i^2 s_i^2}{S_i^2} \tag{3.2}$$

and

$$R_{\sigma^2} = \sum_{i=1}^{q} \frac{c_i n_i s_i^2}{U_i} = \sum_{i=1}^{q} \frac{c_i \sigma_i^2 s_i^2}{S_i^2}, \tag{3.3}$$

where $s_1^2, s_2^2, \ldots, s_q^2$ denote the observed values of $S_1^2, S_2^2, \ldots, S_q^2$. From the first expressions of (3.2) and (3.3), R_{τ^2} and R_{σ^2} have distributions that are free of model parameters. When $s_1^2, s_2^2, \ldots, s_q^2$ are substituted for the observable random variables $S_1^2, S_2^2, \ldots, S_q^2$ in the second expressions of (3.2) and (3.3), R_{τ^2} and R_{σ^2} become τ^2 and σ^2, respectively. Therefore, R_{τ^2} and R_{σ^2} satisfy the requirements for being generalized pivotal quantities for τ^2 and σ^2, respectively. We now define

$$R_\delta = Z\sqrt{\max\{0, R_{\tau^2} + R_{\sigma^2}\}}. \tag{3.4}$$

Hence, δ_{β_1} and δ_{β_2} in (3.1) can be estimated by the corresponding percentiles of R_δ, denoted by R_{δ,β_1} and R_{δ,β_2}, respectively. For practical convenience, we may simply set $\beta_1 = (1-\beta)/2$ and $\beta_2 = (1+\beta)/2$, resulting in an equal tailed interval. The required percentiles may be estimated with the following Monte-Carlo algorithm.

Step 1: Choose a large simulation sample size, say $M = 10,000$. For i equal to 1–M, carry out the following two steps.

Step 2: Generate a standardized normal deviate Z_i and chi-squared random deviates $U_{1,i}, U_{2,i}, \ldots, U_{q,i}$ with $n_1, n_2, \ldots, n_q$ degrees of freedom (df), respectively. These variates must be independent.

Step 3: Compute $R_{\delta,i}$ using Eq. (3.4) for R_δ.

Table 1
Breaking strength (pounds tension) of nine batches of cement briquettes

1	2	3	4	5	6	7	8	9
553	553	510	520	543	492	542	581	578
550	599	580	559	500	530	550	550	531
568	579	529	539	562	528	580	529	562
541	545	535	510	540	510	545	570	525
537	540	537	540	535	571	520	524	549

R_{δ,β_1} and R_{δ,β_2} are just the $(100\beta_1)$th and $(100\beta_2)$th sample percentiles of the collection of values $R_{\delta,1}, R_{\delta,2}, \ldots, R_{\delta,M}$.

4. Illustrative examples

The following practical examples are given to illustrate the proposed procedure.

Example 4.1. Mee (1984), who cited Bowker and Lieberman (1972, p. 439) as the original source, uses a cement briquettes experiment to illustrate his method. Five specimens from each of nine batches of cement briquettes are analyzed for breaking strength. The data is given in Table 1.

The experimental data can be fitted by the one-way random effects model

$$Y_{ij} = \mu + B_i + e_{ij},$$

for $i = 1, 2, \ldots, a,\ j = 1, 2, \ldots, b$, where μ denotes the constant term, B_i the batch effects and e_{ij} the measurement errors. B_i and e_{ij} are random effects normally distributed with 0 mean and variances equal to σ_B^2 and σ_e^2, respectively.

Let $\sigma_1^2 = \sigma_e^2$ and $\sigma_2^2 = b\sigma_B^2 + \sigma_e^2$. We are interested in finding a two-sided β-expectation tolerance interval for the distribution of measured values, namely $N\left(\theta, \tau^2\right)$ with $\theta = \mu$ and $\tau^2 = \sigma_B^2 + \sigma_e^2 = (1 - 1/b)\sigma_1^2 + (1/b)\sigma_2^2$. Moreover, we have sufficient statistics $T = \overline{Y} \sim N\left(\theta, \sigma^2\right)$, where $\sigma^2 = (1/a)\sigma_B^2 + (1/ab)\sigma_e^2 = (1/ab)\sigma_2^2$; $S_1^2 = MSE$ (error mean square) with $a(b-1)S_1^2/\sigma_1^2 \sim \chi^2_{a(b-1)}$ and $S_2^2 = MSB$ (mean square between batches) with $(a-1)S_2^2/\sigma_2^2 \sim \chi^2_{(a-1)}$.

From Table 1, $a = 9$, $b = 5$, $\bar{y} = 543.8$, $s_1^2 = 526$, $s_2^2 = 630$. The proposed method yields a two-sided ($\beta = 0.90$)-expectation tolerance interval [503, 585].

Example 4.2. Liao and Iyer (2001) describe a gauge study for comparing the quality between a newly developed glucose monitoring meter for in-home use by patients with diabetes (called test meter) and a marked one (called reference meter). Let X denote a measurement using a test meter and Y denote a measurement using a reference meter. Then X and Y are modeled as follows:

$$X_{ijkl} = \mu_{\mathrm{T}} + M_i + B_j + L_k + e_{ijkl},$$

for $i=1,2,\ldots,m$, $j=1,2,\ldots,B$, $k=1,2,\ldots,L$ and $l=1,2,\ldots,E$, where μ_{T} denotes the expected reading when using a test meter, M_i the effect of test meter i, B_j the effect of the jth blood sample, L_k the effect of the kth strip-lot and e_{ijkl} measurement error. Likewise,

$$Y_{ijkl} = \mu_{\mathrm{R}} + M'_i + B_j + L_k + e'_{ijkl}$$

for $i=1,2,\ldots,n$, $j=1,2,\ldots,B$, $k=1,2,\ldots,L$ and $l=1,2,\ldots,E$, where μ_{R} denotes the expected reading when using a reference meter, M'_i the effect of reference meter i, B_j the effect of the jth blood sample, L_k the effect of the kth strip-lot and e'_{ijkl} measurement error. The effects M_i, M'_i, B_j, L_k, e_{ijkl}, e'_{ijkl} are random effects, normally distributed with zero mean and standard deviations equal to σ_{T}, σ_{R}, σ_B, σ_L, σ_e and σ_e, respectively (the variances of e_{ijkl} and e'_{ijkl} are assumed to be equal).

The theoretical mean for the ith test meter when using blood sample j and strip-lot k equals $\mu_{\mathrm{T}} + M_i + B_j + L_k$. The theoretical mean reading, averaging over *all* reference meters, for the same blood sample and strip-lot equals $\mu_{\mathrm{R}} + B_j + L_k$. This theoretical mean reading is used as the reference value against which the readings from individual test meters will be compared to assess their accuracy. The deviation of the reading obtained using a single test meter from the mean over all reference meters is thus equal to $D_i = \mu_{\mathrm{T}} - \mu_{\mathrm{R}} + M_i$. It is the distribution of D_i that is of interest. Instead of β-content tolerance intervals applied in Liao and Iyer (2001, 2004), we use β-expectation tolerance intervals to measure the quality of a batch of test meters. A batch of test meters is deemed to have met the quality requirements if its two-sided ($\beta=0.95$)-expectation tolerance interval for the distribution of D_i completely falls into the threshold interval $[-5, 5]$.

We now apply the β-expectation tolerance interval given in Section 3 to this problem. Let

$$\overline{X} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{B}\sum_{k=1}^{L}\sum_{l=1}^{E} X_{ijkl}}{mBLE},$$

$$MS_{\mathrm{T}} = \frac{BLE\sum_{i=1}^{m}\left(\overline{X}_{i\ldots} - \overline{X}\right)^2}{m-1},$$

$$MSE_{\mathrm{T}} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{B}\sum_{k=1}^{L}\sum_{l=1}^{E}\left(X_{ijkl} - \overline{X}_{i\ldots} - \overline{X}_{.j..} - \overline{X}_{..k.} + 2\overline{X}\right)^2}{mBLE - m - B - L + 2}.$$

Similarly, let $\overline{Y}$, MS_{R} and MSE_{R} denote the corresponding sample mean, mean square for the reference meter effect and error mean square for the model fitted to the reference meters data. Then the statistics $\overline{X}$, $\overline{Y}$, MS_{T}, MSE_{T}, MS_{R} and MSE_{R} are mutually independent. Let $\sigma_1^2 = \sigma_{\mathrm{T}}^2 + \sigma_e^2/k_0$, $n_1 = m-1$, $S_1^2 = MS_{\mathrm{T}}/k_0$; and $\sigma_2^2 = \sigma_{\mathrm{R}}^2 + \sigma_e^2/k_0$, $n_2 = n-1$, $S_2^2 = MS_{\mathrm{R}}/k_0$, where $k_0 = BLE$. It follows that $n_1 S_1^2/\sigma_1^2 \sim \chi^2_{n_1}$, $n_2 S_2^2/\sigma_2^2 \sim \chi^2_{n_2}$. Also $v_1 MSE_{\mathrm{T}}/\sigma_e^2 \sim \chi^2_{v_1}$ and $v_2 MSE_{\mathrm{R}}/\sigma_e^2 \sim \chi^2_{v_2}$, where $v_1 = mk_0 - m - B - L + 2$ and $v_2 = nk_0 - n - B - L + 2$.

MSE_{T} and MSE_{R} are pooled to get $MSE = (v_1 MSE_{\mathrm{T}} + v_2 MSE_{\mathrm{R}}) / (v_1 + v_2)$, so we have $n_3 S_3^2/\sigma_3^2 \sim \chi^2_{n_3}$, where $\sigma_3^2 = \sigma_e^2/k_0$, $n_3 = v_1 + v_2$ and $S_3^2 = MSE/k_0$. We are interested in a two-sided β-expectation tolerance interval for $N\left(\theta, \tau^2\right)$, where $\theta = \mu_{\mathrm{T}} - \mu_{\mathrm{R}}$ and $\tau^2 = \sigma_{\mathrm{T}}^2 = \sigma_1^2 - \sigma_3^2$. Also observe that $T = \overline{X} - \overline{Y} \sim N\left(\theta, \sigma^2\right)$, where $\sigma^2 = \sigma_1^2/m + \sigma_2^2/n$.

For the data provided in Liao and Iyer (2001), $m = 44$, $n = 10$, $B = L = E = 3$, $\bar{x} - \bar{y} = -1.13654$, $s_1^2 = 0.61928$, $s_2^2 = 0.63132$ and $s_3^2 = 0.19052$. A two-sided ($\beta = 0.95$)-expectation tolerance interval for the distribution of D_i is obtained as $[-2.5900, 0.3278]$ which is completely contained in the threshold interval $[-5, 5]$. Therefore, one concludes that the batch of test meters has satisfied the quality requirement.

5. Simulation study

To evaluate the performance of the proposed procedure, the following simulation study is carried out based on the glucose monitoring meter experiment of Example 4.2. We specify the values of β=0.025, 0.05, 0.1, 0.9, 0.95 and 0.975. Moreover, without loss of generality, we may assume that $\theta = 0$ and $\sigma_{\mathrm{R}} = 1$. For fixed $B = L = E = 3$ and specified values of $m = 10, 30, 60$; $n = 5, 20, 50$; $\sigma_{\mathrm{T}} = 0.5, 1.0, 2.0, 4.0$ and $\sigma_e = 0.5, 1.0, 2.0, 4.0$, we generate a normal random deviate T from $N\left(0, \sigma^2\right)$ and three chi-squared random deviates U_1, U_2 and U_3 with n_1, n_2, n_3 df, using functions *rnorm* and *rchisq*, respectively, in the statistical package *S*-plus. The corresponding sample statistics $S_1^2 = U_1\sigma_1^2/n_1$, $S_2^2 = U_2\sigma_2^2/n_2$, $S_3^2 = U_3\sigma_3^2/n_3$ are then generated.

Furthermore, we compute the quantities of $R_{\delta,\beta}$ using the Monte-Carlo algorithm described in Section 3. Also let $p = F\left(T + R_{\delta,\beta}\right)$, where F is the cumulative distribution function of $N\left(0, \tau^2\right)$. The procedure is repeated 10,000 times for each parameter combination and the average of p is computed. The results are displayed in Figs. 1–6. Each panel in figures plots the simulated confidence coefficient for a specific combination of m and n. The plotting symbols "1", "2", "3" and "4" are designated for cases with $\sigma_e = 0.5$, 1, 2 and 4, respectively. For most parameter combinations, the constructed tolerance limits are successful in maintaining the confidence level close to the nominal values of β, particularly when σ_{T} is larger than σ_{R}. Nonetheless, the results indicate that when σ_{T} is smaller than σ_{R} and n is small ($n = 5$), the simulated value of β is slightly smaller or larger than the nominal one according as $\beta \leqslant 0.1$ or $\beta \geqslant 0.9$. Fortunately, in most practical situations, σ_{T} is usually larger than σ_{R} because the reference meters tend to have much higher precision than the test meters.

6. A comparison with Mee's method

Mee (1984) applies Satterthwaite (1946) approximation to determine the df of Student t-distribution in Wilks' (1941) method for the balanced one-way random effects model. It may be of interest to compare the performance of this approximate method with ours. Based on the balanced one-way random effects model described in Example 4.1, some simulation results, using the simulation procedure of Section 5, are reported in Table 2.

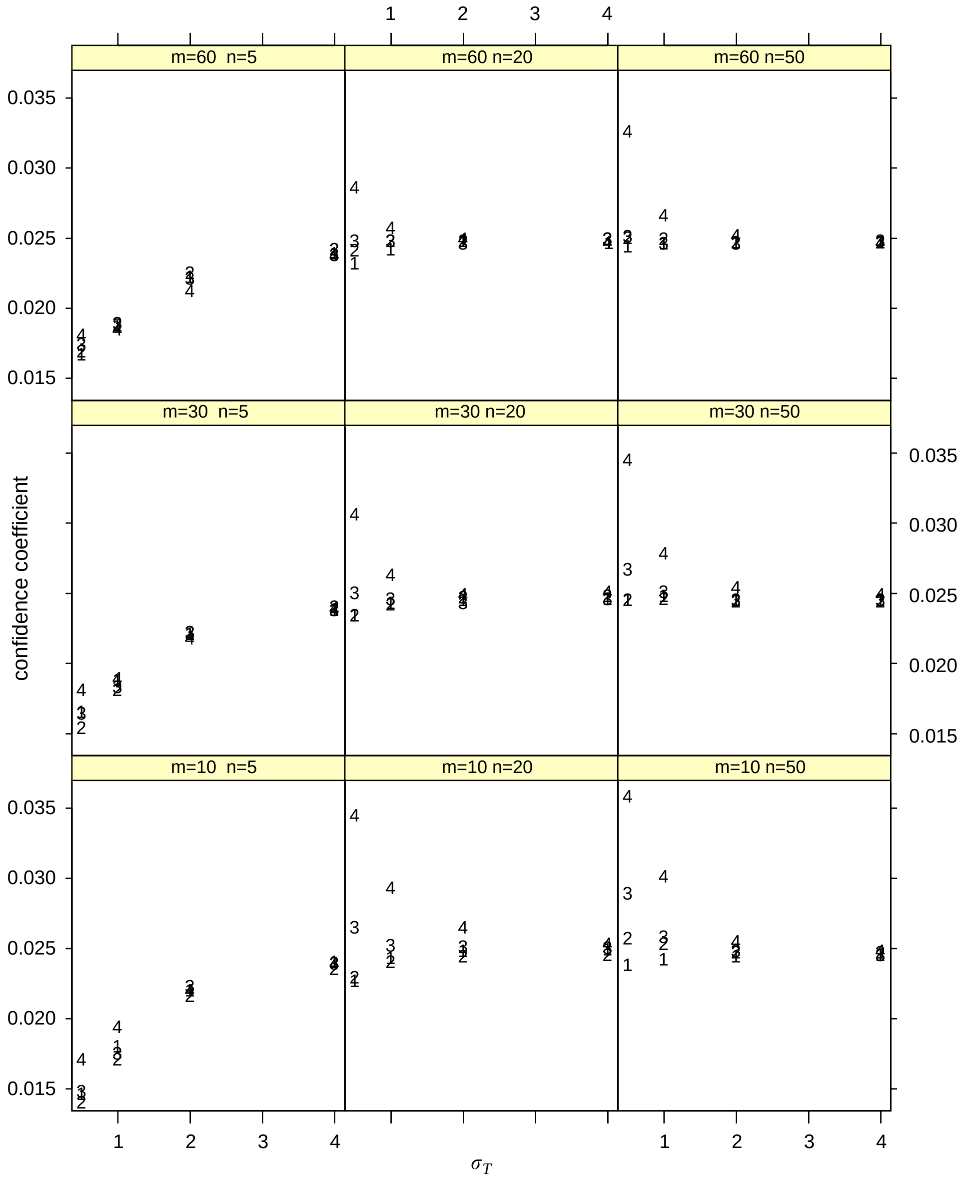

Fig. 1. Simulated confidence coefficients for one-sided ($\beta = 0.025$)-expectation tolerance limits, based on the glucose monitoring meter experiment for $B = L = E = 3$ and $\sigma_R = 1$.

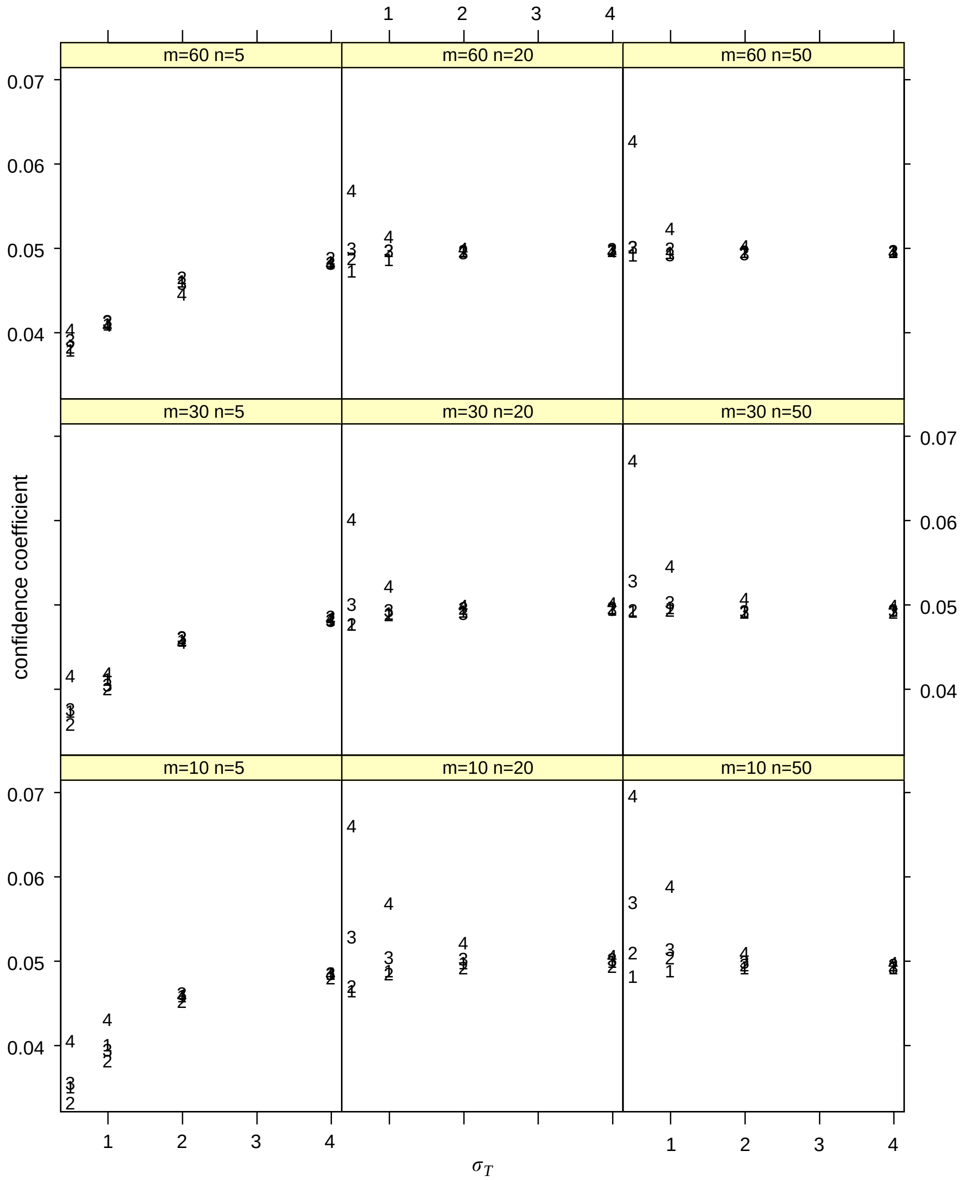

Fig. 2. Simulated confidence coefficients for one-sided ($\beta = 0.05$)-expectation tolerance limits, based on the glucose monitoring meter experiment for $B = L = E = 3$ and $\sigma_{\mathrm{R}} = 1$.

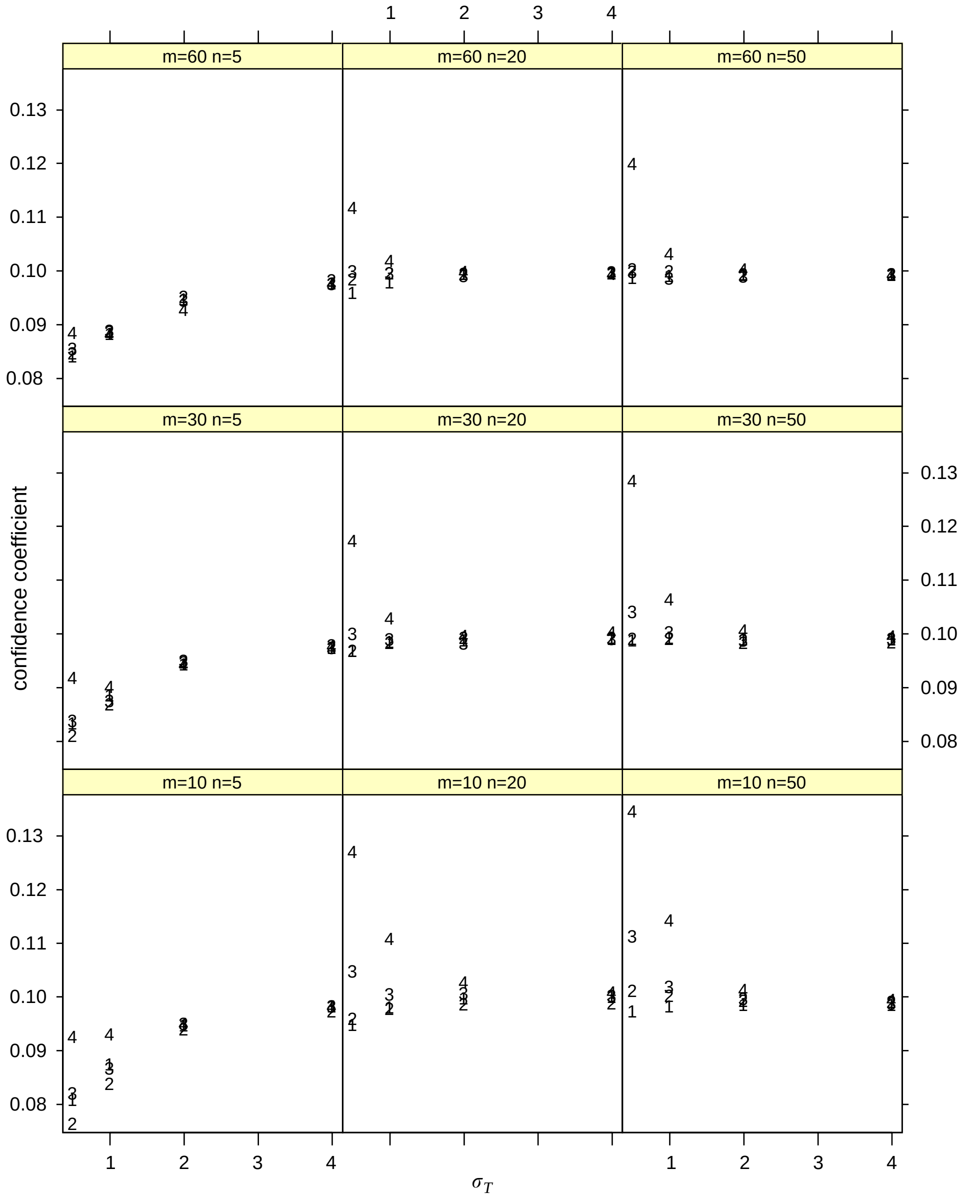

Fig. 3. Simulated confidence coefficients for one-sided ($\beta = 0.10$)-expectation tolerance limits, based on the glucose monitoring meter experiment for $B = L = E = 3$ and $\sigma_R = 1$.

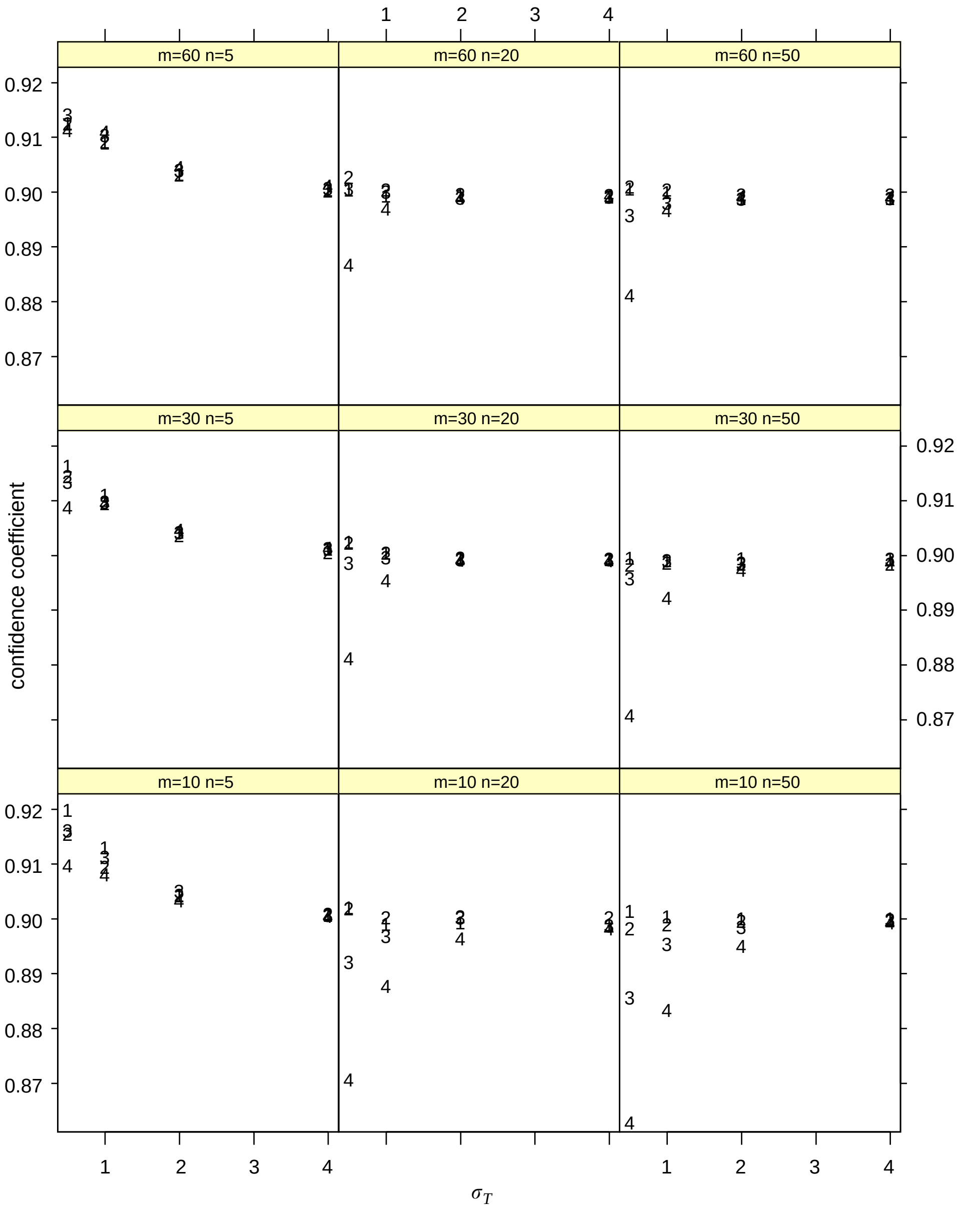

Fig. 4. Simulated confidence coefficients for one-sided ($\beta = 0.90$)-expectation tolerance limits, based on the glucose monitoring meter experiment for $B = L = E = 3$ and $\sigma_R = 1$.

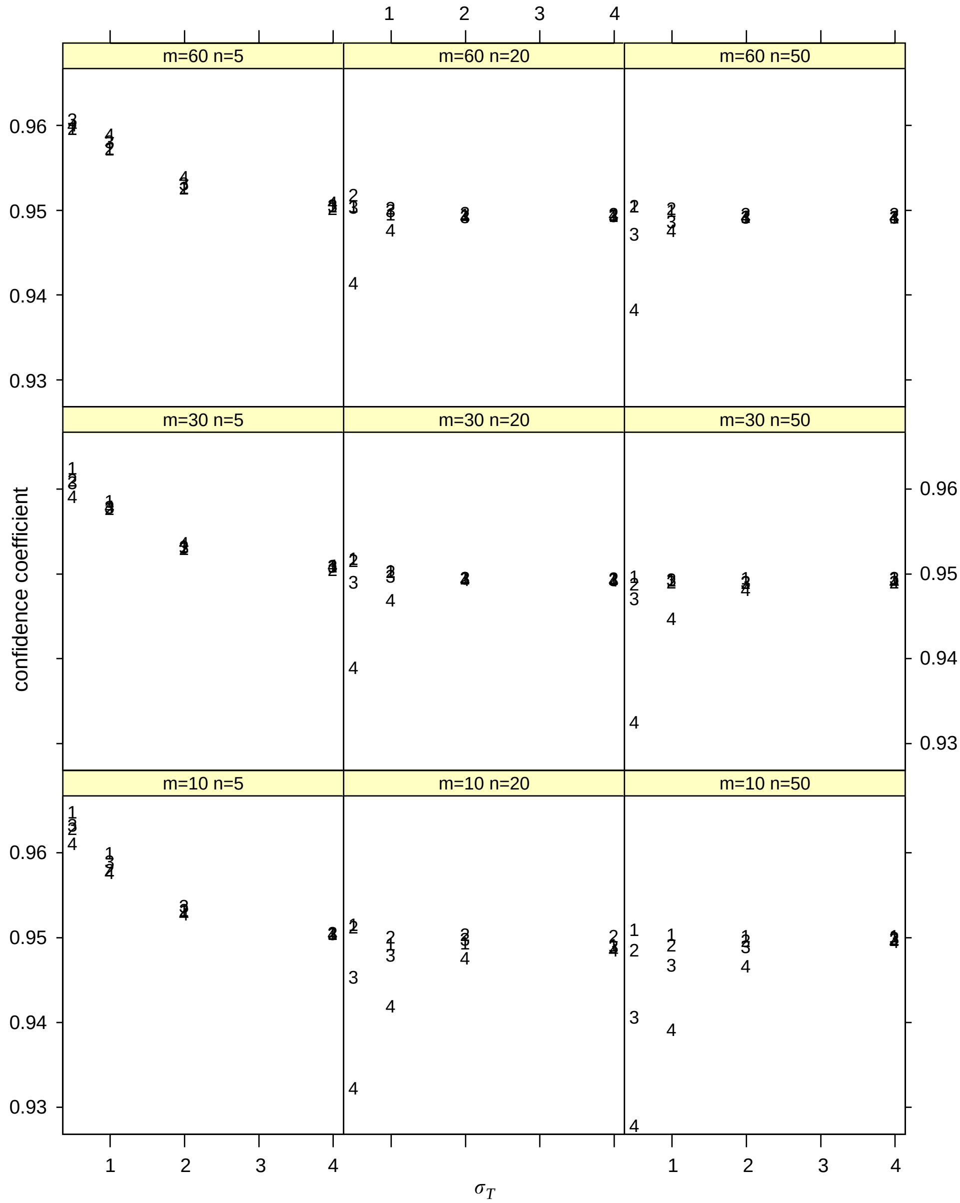

Fig. 5. Simulated confidence coefficients for one-sided ($\beta = 0.95$)-expectation tolerance limits, based on the glucose monitoring meter experiment for $B = L = E = 3$ and $\sigma_R = 1$.

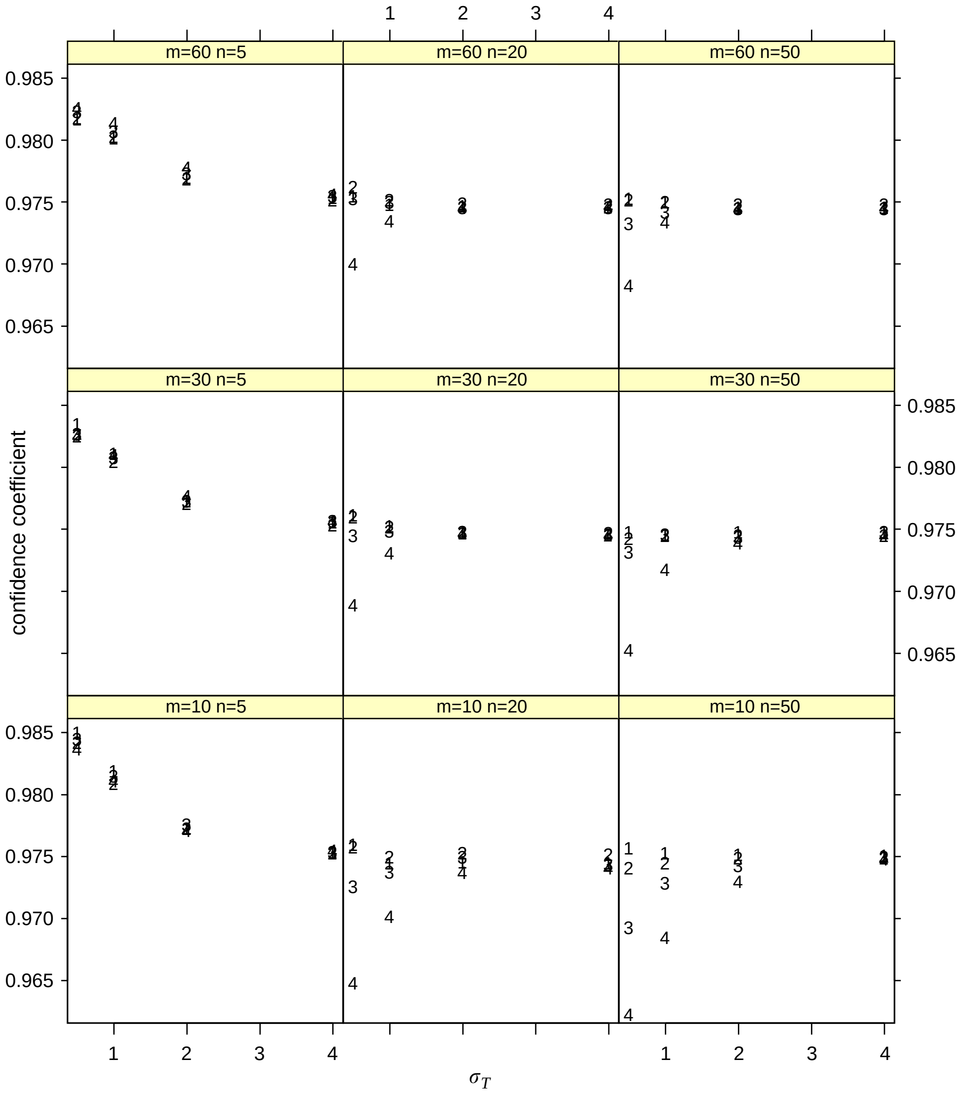

Fig. 6. Simulated confidence coefficients for one-sided ($\beta = 0.975$)-expectation tolerance limits, based on the glucose monitoring meter experiment for $B = L = E = 3$ and $\sigma_R = 1$.

For all the cases considered in the simulation study, our method yields not only the simulated confidence coefficients closer to the nominal values, but also shorter expected lengths. Our method obviously outperforms Mee's.

Table 2
Simulated confidence coefficients for two-sided β-expectation tolerance intervals using Mee's method and the proposed method. The simulated expected lengths are given in parentheses

a	b	σ_B^2/σ_e^2	$\beta=0.90$		$\beta=0.95$		$\beta=0.99$	
5	2	0.1	0.9440	(14.9970)[a]	0.9811	(19.4384)	0.9989	(32.1741)
			0.9342	(14.3661)[b]	0.9738	(18.2212)	0.9970	(28.2176)
		0.5	0.9392	(7.8600)	0.9780	(10.1612)	0.9985	(16.8696)
			0.9302	(7.5847)	0.9702	(9.6506)	0.9965	(15.1550)
		1.0	0.9337	(6.3448)	0.9753	(8.3069)	0.9980	(13.6968)
			0.9258	(6.2299)	0.9681	(7.9492)	0.9956	(12.5011)
		2.0	0.9255	(5.4818)	0.9713	(7.0941)	0.9975	(11.7616)
			0.9207	(5.3727)	0.9642	(6.9550)	0.9944	(11.1111)
		10.0	0.9100	(4.6446)	0.9586	(6.0114)	0.9939	(10.0628)
			0.9082	(4.5953)	0.9541	(5.9321)	0.9915	(9.8143)
5	5	0.1	0.9463	(13.7587)	0.9815	(17.0871)	0.9992	(24.8831)
			0.9236	(12.6582)	0.9668	(15.4683)	0.9952	(22.2444)
		0.5	0.9502	(7.6478)	0.9830	(9.6085)	0.9995	(15.4989)
			0.9246	(6.9646)	0.9672	(8.5815)	0.9952	(12.9347)
		1.0	0.9446	(6.1633)	0.9826	(7.9453)	0.9995	(12.6896)
			0.9220	(5.8472)	0.9653	(7.2858)	0.9945	(11.3365)
		2.0	0.9370	(5.4486)	0.9771	(7.0458)	0.9989	(11.4945)
			0.9186	(5.1638)	0.9618	(6.5868)	0.9932	(10.4172)
		10.0	0.9132	(4.6366)	0.9624	(6.0594)	0.9956	(9.9538)
			0.9057	(4.5705)	0.9534	(5.8872)	0.9910	(9.7340)
5	9	0.1	0.9361	(12.6565)	0.9811	(16.2675)	0.9989	(23.4475)
			0.9186	(12.1046)	0.9632	(14.7047)	0.9947	(20.5924)
		0.5	0.9552	(7.3393)	0.9880	(9.4094)	0.9995	(13.7992)
			0.9239	(6.7732)	0.9660	(8.3869)	0.9949	(12.3833)
		1.0	0.9547	(6.1623)	0.9866	(7.8584)	0.9997	(12.6949)
			0.9217	(5.7231)	0.9642	(7.1666)	0.9943	(10.9801)
		2.0	0.9498	(5.4196)	0.9834	(6.9372)	0.9994	(11.1791)
			0.9178	(5.1436)	0.9602	(6.4531)	0.9934	(10.2497)
		10.0	0.9298	(4.6396)	0.9707	(5.9900)	0.9970	(9.9202)
			0.9072	(4.5573)	0.9537	(5.8881)	0.9913	(9.6690)
9	5	0.1	0.9180	(11.8076)	0.9658	(14.4961)	0.9957	(20.0072)
			0.9113	(11.6894)	0.9578	(14.0728)	0.9925	(18.9589)
		0.5	0.9279	(6.4346)	0.9710	(7.9070)	0.9967	(10.9858)
			0.9119	(6.2508)	0.9593	(7.5993)	0.9928	(10.4076)
		1.0	0.9292	(5.3374)	0.9704	(6.5353)	0.9970	(9.2977)
			0.9112	(5.1885)	0.9573	(6.2985)	0.9923	(8.7610)
		2.0	0.9258	(4.6435)	0.9692	(5.7397)	0.9965	(8.2330)
			0.9087	(4.5534)	0.9562	(5.5782)	0.9919	(7.8503)
		10.0	0.9175	(3.9828)	0.9622	(4.9502)	0.9940	(7.1434)
			0.9026	(3.9649)	0.9521	(4.8944)	0.9905	(7.0440)

[a]Tolerance intervals constructed by using Mee's method.
[b]Tolerance intervals constructed by using the proposed method.

7. Concluding remarks

In the present paper, we have obtained a β-expectation tolerance interval for the normal distribution whose mean and variance are functions of parameters, which are estimated using the data from a balanced mixed linear model. The proposed method is mainly based on the concept of generalized pivotal quantity. And it can easily be verified that our method turns out to be that of Wilks (1941) when a SRS from $N\left(\mu, \sigma^2\right)$ is considered. Most importantly, the simulation studies conducted in the study strongly support the proposed method and can be recommended for use in practical applications. The extension to the unbalanced mixed linear models is currently under investigation.

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