CHINESE JOURNAL OF PHYSICS VOL. 29, NO. 3 JUNE 1991

A Variational Calculation Scheme for Dyson Boson Description of Nuclear Collective Motion

Ching-teh Li (李慶德)
Department of Physics, National Taiwan University
Taipei, Taiwan 107, R.O.C.
(Received Ian. 22, 1991)

A variational calculation scheme is proposed to tackle the non-unitarity problem in the Dyson boson description of nuclear collective motion. It is based on a trial-state-optimization variational principle in which one minimizes the deviation of a trial state from being an exact eigenstate, instead of minimizing the energy expectation value in the trial state as is usually done in standard variational calculations. The viability of this variational method is demonstrated through an application to a monopole pairing interaction for multi-j shells.

I. INTRODUCTION

The Dyson Boson Mapping (DBM) of nuclear shell model algebra first introduced by Janssen *et al.*[1] has proved to be a useful tool for studying low-lying nuclear collective states[2] as well as for investigating the microscopic foundation of the interacting boson model (IBM).[3-6] Recently, there has been a growing interest[7-15] in using this boson mapping method to do various calculations for nuclear collective motion. The advantage in using the DBM, which is both exact and finite, is that one can avoid the infinite expansions usually involved in using the Holstein-Primakoff type of boson mappings. The drawback of the DBM, on the other hand, is that the mapping is non-unitary and the resulting boson Hamiltonians are usually non-Hermitian. One thus has to cope with solving the problem of non-Hermitian quantum systems. Moreover, since bi-orthonormal basis states have to be used in the end, there has been some ambiguity in evaluating physical quantities such as the various transition amplitudes. This latter problem has basically been solved.[4,6-8,16] In fact, it was further shown in Ref. [4] that the non-unitarity of the DBM at least in some cases is actually not a drawback but rather a distinct advantage one can exploit together with the finiteness of the mapping. Although so far considerable progress has been made in tackling the non-unitarity problem of the DBM,[6-15] more study still has to be done in order to make the DBM really a powerful method to the description of nuclear collective motion. Another salient problem is the question of how to deal with the spurious *states* in the boson space[6-9,13,15] which have nothing to do with the original fermion system but are often

present in practical calculations. This is actually a problem common to most calculations based on boson mapping methods. As it turns out, the variational calculation scheme to be described below has the merit that it also minimizes the spurious components contained in the various boson states used in nuclear collective-motion calculations.

In applying the DBM method to realistic nuclear systems, in principle one can perform direct diagonalizations of the non-Hermitian Dyson boson Hamiltonians. In practice, however, the dimensionality of the non-Hermitian problem is actually much larger than that of the original fermion problem and can easily become unreasonably large to be handled this way, Thus, one way or another, one has to resort to some kind of variational methods such as Hartree-Bose or Tamm-Dancoff approximation to pin down the most relevant collective degrees of freedom of the problem at hand. The difficulty is that, owing to the non-Hermiticity of the Dyson boson Hamiltonians, the standard variational principle, ***i.e.*** that of Ritz, cannot be applied unless one uses the complicated ***physical boson states*** which are the boson images of the original fermion states. Therefore, it will be very desirable to develop a viable variational calculation scheme for the DBM approach to nuclear collective motion in which one can use relatively simple boson states, instead of the complicated physical boson states. The present paper is devoted to this goal.

This paper is organized in the following manner. In Sect. II we give a brief account of the Dyson boson mapping (DBM) of the shell model algebra. In Sect. III this DBM is applied to a monopole pairing Hamiltonian by keeping only the monopole degrees of freedom. The variational calculation scheme based on a trial state optimization is developed in Sect. IV to tackle the non-unitarity problem of the DBM. Numerical results and comparisons with previous works are presented in Sect. V. Finally, in Sect. VI we give a brief summary and conclusion of the present work.

II. DYSON BOSON MAPPING

In the Dyson representation of nuclear shell model algebra, one uses the following boson mappings for the fermion pairing and multipole operators:

$$a_\alpha^\dagger a_\beta^\dagger \rightarrow P_{\alpha\beta}^\dagger \equiv b_{\alpha\beta}^\dagger - \sum_{\gamma\delta} b_{\alpha\gamma}^\dagger b_{\beta\delta}^\dagger b_{\gamma\delta} \quad , \tag{1a}$$

$$a_\beta a_\alpha \rightarrow b_{\alpha\beta} \quad , \tag{1b}$$

$$a_\alpha^\dagger a_\beta \rightarrow \sum_\gamma b_{\alpha\gamma}^\dagger b_{\beta\gamma} \quad , \tag{1c}$$

where $a_\alpha^\dagger (a_\alpha)$ are fermion creation (annihilation) opeators for single-particle levels $\alpha \equiv (n_\alpha, l_\alpha, j_\alpha, m_\alpha)$ and $b_{\alpha\beta}^\dagger = -b_{\beta\alpha}^\dagger$ are the antisymmetric boson operators satisfying the commutation relations

$$[b_{\alpha\beta}, b_{\gamma\delta}] = [b^{\dagger}_{\alpha\beta}, b^{\dagger}_{\gamma\delta}] = 0 \quad ,$$

$$[b_{\alpha\beta}, b^{\dagger}_{\gamma\delta}] = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} \quad . \tag{2}$$

The Dyson boson images of the pair creation and annihilation operators as given above in (1 a) and (1 b) are clearly non-Hermitian. This in turn implies that the resulting Dyson boson image of the Hamiltonian is usually non-Hermitian.[3,13,15] A salient feature of this mapping is that while the fermion ket states are mapped to the so-called physical boson states by

$$(a^{\dagger}_{\alpha_1} a^{\dagger}_{\beta_1})(a^{\dagger}_{\alpha_2} a^{\dagger}_{\beta_2}) \ldots (a^{\dagger}_{\alpha_n} a^{\dagger}_{\beta_n})| - > \rightarrow P^{\dagger}_{\alpha_1\beta_1} P^{\dagger}_{\alpha_2\beta_2} \ldots P^{\dagger}_{\alpha_n\beta_n} IO > \ , \tag{3}$$

the fermion bra states are mapped to the simple boson states by

$$< - | (a_{\beta_1} a_{\alpha_1})(a_{\beta_2} a_{\alpha_2}) \ldots (a_{\beta_n} a_{\alpha_n}) \rightarrow < 0 | b_{\alpha_1\beta_1} b_{\alpha_2\beta_2} \ldots b_{\alpha_n\beta_n} \quad , \tag{4}$$

where $|-\!>$, and $|0\!>$ are the fermion and boson vacuum states, respectively. It is this latter mapping for the bra states, *i.e.* (4), which makes the relationship between the Dyson boson problem and the corresponding fermion problem very transparent, as has been stressed in Ref. [4]. This special feature of the DBM is also what we are going to exploit below.

III. MONOPOLE PAIRING INTERACTION

For the purpose of illustration, here we consider the following monopole pairing interaction for nondegenerate multi-j shells:

$$H_p = \sum_{jm} h_j a^{\dagger}_{jm} a_{jm} - \frac{G}{2} \sum_{j_1 j_2} [a^{\dagger}_{j_1} a^{\dagger}_{j_1}]_{00} [\tilde{a}_{j_2} \tilde{a}_{j_2}]_{00} \quad , \tag{5}$$

where h_j are single particle energies, G the pairing strength, $\tilde{a}_{jm} = (-1)^{j-m} a_{j,-m}$, and $[\,]_{00}$ denotes angular momentum coupling to total angular momentum $(JM) = (00)$. Employing the DBM given in (1) for fermion bilinear operators and *keeping only the monopole (i.e. $J = 0$)* degrees of freedom, it is straightforward to obtain for the Hamiltonian (5) the following Dyson boson image:

$$\mathcal{H} = 2 \sum_a h_a B^{\dagger}_a B_a - G \sum_{ab} \sqrt{\Omega_a \Omega_b}\, B^{\dagger}_a \left(1 - \frac{B^{\dagger}_a B_a}{\Omega_a}\right) B_b \quad . \tag{6}$$

In (6), whith the notation $a \equiv (n_a, l_a, j_a)$ etc., $\Omega_a = j_a + 1/2$ is the pair degeneracy of the j-shell a, $B^{\dagger}_a$ and B_a a monopole coupling the antisymmetric boson operators $b^{\dagger}_{\alpha\beta}$ and $b_{\alpha\beta}$ to zero angular momentum. These monopole boson operators satisfy the commutation relations

$$[B_a, B_b] = [B_a^\dagger, B_b^\dagger] = 0 \ ,$$

$$[B_a, B_b^\dagger] = \delta_{ab} \ . \tag{7}$$

The Dyson boson Hamiltonian $\mathcal{H}$ as given above in (6) is clearly non-Hermitian, so one has to distinguish between its right-hand-side (r.h.s) and left-hand-side (l.h.s.) eigenstates. For instance, from the simple relationship between the boson bra states and their fermion counterparts as given by (4), it is not difficult to see that the boson condensate bra state

$$<n;\alpha| = <0| \frac{1}{\sqrt{n!}} (\sum_a \alpha_a B_a)^n \tag{8}$$

with suitably chosen coefficients α_a, should be a good approximation to the l.h.s. ground state of the boson Hamiltonian $\mathcal{H}$ for $2n$ particles in the valence i-shells. The corresponding approximation to the r.h.s. ground state, however, is not simply $|n;\alpha>$, the Hermitian conjugate of (8). Instead, it is given by

$$|\overline{n;\alpha}> = \frac{1}{\sqrt{n!}} [\sum_a \alpha_a B_a^\dagger + (1 - \frac{B_a^\dagger B_a}{\Omega_a})]^n |0> \ , \tag{9}$$

where the overbar in $|\overline{n;\alpha}>$ has been used to differentiate it from $|n;\alpha>$. In fact, to obtain (9) from $|n;\alpha>$, one has to replace the boson operators $B_a^\dagger$ in $|n;\alpha>$ by $B_a^\dagger(1 - B_a^\dagger B_a/\Omega_a)$, the actual boson images of the $J = 0$ fermion pair creation operators. While (8) is a simple boson state, (9) is an example of the so-called physical states in the. boson space which obey the Pauli principle exactly. Here we stress that due to the non-Hermiticity of the Dyson boson Hamiltonian, to observe the Pauli principle exactly, one has to use physical boson states such as (9) *only* for the ket states. As for the bra states, the mapping (4) implies that simple boson states such as that given in (8) can always be used.

In the standard variational method, *i.e.* that of Ritz, if one uses (8) as the trial bra state, at the same time one should use (9) as the trial ket state. One can then minimize the energy expectation value $<n;\alpha|\mathcal{H}|\overline{n;\alpha}>/<n;\alpha|\overline{n;\alpha}>$
α_a.

states[6,12]
ket states because Pauli the

spurious and the presence of the

principle leaking, the standard variational applicable. If one still insists to use this variational method anyway, the energy minimization procedure tends to invite more undesired mixture from the spurious components and one often ends up with energies

much lower than that of the true ground state.

To remedy this problem, in what follows, we shall use a different variational method to do approximate calculations for the ground state of the monopole pairing boson Hamiltonian $\mathcal{H}$.

IV. TRIAL-STATE-OPTIMIZATION VARIATIONAL PRINCIPLE

Since the Dyson boson Hamiltonian $\mathcal{H}$ given in (8) is non-Hermitian, one has to distinguish between the r.h.s. and l.h.s. eigenstates. Thus, besides solving the Schrodinger equation

$$(\mathcal{H} - E)\,|\Psi_R\rangle = 0 \tag{10}$$

for the r.h.s. eigenstates $|\Psi_R\rangle$, one also has to solve the Schrodinger equation

$$\langle\Psi_L|\,(\mathcal{H} - E) = 0 \tag{11}$$

for the l.h.s. eigenstates $\langle\Psi_L|$. Here we shall solve both (10) and (11) approximately through a trial-state-optimization (TSO) variational principle which is similar in spirit to the variance-of-local-energy minimization procedure[17] used previously for Hermitian quantum systems.

In the case of solving for the r.h.s. eigenstate variationally with a trial state $|\phi\rangle$, we can write the action of the boson Hamiltonian $\mathcal{H}$ on the trial state as

$$\mathcal{H}\,|\phi\rangle = \epsilon\,|\phi\rangle + (\mathcal{H} - \epsilon)\,|\phi\rangle\,, \tag{12}$$

where ϵ is a constant to be determined. If the second term of the r.h.s of (12) vanishes identically, we have an exact r.h.s. eigenstate of the non-Hermitian Hamiltonian. Since we only have an approximate trial state, $(\mathcal{H}-\epsilon)|\phi\rangle$ is in general nonvanishing. By forming its norm square, which is clearly nonnegative, we have

$$\frac{\langle\phi\,|\,(\mathcal{H} - \epsilon)^{\dagger}\,(\mathcal{H} - \epsilon)\,|\,\phi\rangle}{\langle\phi\,|\phi\rangle} \geqslant 0\ , \tag{13}$$

where the equality holds only for exact r.h.s. eigenstates of $\mathcal{H}$. The nonnegative quantity on the l.h.s. of (13) can be considered as a measure of deviation of the trial state $|\phi\rangle$ from being an exact r.h.s. eigenstate of the Hamiltonian $\mathcal{H}$. Since this measure always has the zero as its lower bound, in the present TSO method we thus minimize this quantity with respect to the parameter ϵ as well as to the trial ket state. It is straightforward to see that the minimization of the l.h.s. of (13) with respect to ϵ simply yields the result $\epsilon = \langle\phi|\mathcal{H}|\phi\rangle / \langle\phi|\phi\rangle$. We are therefore led to the following variational equation for the ket state:

$$\delta\left\{\frac{\langle\phi|\mathcal{H}^{\dagger}\mathcal{H}|\phi\rangle}{\langle\phi|\phi\rangle}-\left|\frac{\langle\phi|\mathcal{H}|\phi\rangle}{\langle\phi|\phi\rangle}\right|^{2}\right\}=0 \quad , \tag{14}$$

where $|\,|^2$ denotes absolute value square and the variations are performed with respect to all parameters in the r.h.s. trial state $|\phi\rangle$, subject to appropriate constraints.

In the case of solving the Schrödinger equation (11) for the l.h.s. eigenstates approximately through the TSO variational principle, in exactly the same fashion one can readily show that the corresponding variational equation for the bra state is given by

$$\delta\left\{\frac{\langle\varphi|\mathcal{H}\mathcal{H}^{\dagger}|\varphi\rangle}{\langle\varphi|\varphi\rangle}-\left|\frac{\langle\varphi|\mathcal{H}|\varphi\rangle}{\langle\varphi|\varphi\rangle}\right|^{2}\right\}=0 \quad , \tag{15}$$

where the variations are performed with respect to all parameters in the l.h.s. trial state $\langle\varphi|$, subject to appropriate constraints.

Note that with the present TSO method, which yields the variational Eqs. (14) and (15), the variational calculations for the r.h.s. and l.h.s. states are done separately. This unique feature allows us to take advantage of the fact that in the Dyson boson representation of the shell model algebra, the bra states are usually simple boson states such as that given in (8). Furthermore, while the standard variational principle is basically a ground state method, the TSO variational principle proposed here is applicable not only to the ground state but also to excited states. In the case of excited states, as long as the trial states are reasonably well chosen, the TSO variational method obviates the need of explicitly orthogonalizing the trial states to all lower-lying exact eigenstates before variations are performed. Even though we are concerned here only with the non-Hermitian boson system represented by the Hamiltonian (6), we emphasize that the present TSO method can also be applied to Hermitian quantum systems[17,18] for which (14) and (15) reduce to a single variational equation.

V. NUMERICAL RESULTS

To illustrate the viability of the above TSO variational principle, we apply it with the non-Hermitian boson Hamiltonian given in (6) to the case of tin isotopes. Following Ref. [6], we take G = 0.187 *MeV* for the pairing strength and $h_a = 0, 0.22, 1.90, 2.20, 2.80$ *MeV* for the single-particle energies of the valence j-shells $d_{5/2}, g_{7/2}, s_{1/2}, d_{3/2}, h_{11/2}$.

Here we take the boson condensate bra state $\langle n;\alpha|$ defined in (8) as the trial state to the l.h.s. ground state of the monopole pairing boson Hamilonian $\mathcal{H}$ for $2n$ particles in the valence i-shells. The parameters α_a are to be determined by using the variational Eq. (15), subject to the constraint

$$\sum_a \alpha_a^2 = 1 \quad , \tag{16}$$

which simply means that the state $<n;\alpha|$ is normalized to unity. We thus obtain the following variational equation

$$\frac{\partial}{\partial\alpha_a}\sigma^2(\mathcal{H}\mathcal{H}^\dagger) = 0 \quad , \tag{17}$$

where

$$\begin{aligned}\sigma^2(\mathcal{H}\mathcal{H}^\dagger) &\equiv <n;\alpha|\,\mathcal{H}\mathcal{H}^\dagger|n;\alpha> - |<n;\alpha|\mathcal{H}|n;\alpha>|^2 \\ &= n[\sum_a t_a^2 - (\sum_a t_a\alpha_a)^2] \\ &\quad + 2n(n-1)[2(\sum_{ab} t_a\alpha_b q_{ab} - \sum_a t_a\alpha_a \sum_{bc} q_{bc}\alpha_b\alpha_c) \\ &\quad + \sum_{ab} q_{ab}^2 - (\sum_{ab} q_{ab}\alpha_a\alpha_b)^2] + 4n(n-1)(n-2)[\sum_{abc} q_{ab}q_{ac}\alpha_b\alpha_c \\ &\quad - (\sum_{ab} q_{ab}\alpha_a\alpha_b)^2]\end{aligned} \tag{18a}$$

with

$$t_a = 2h_a\alpha_a - G\sqrt{\Omega_a}\sum_b\sqrt{\Omega_b}\,\alpha_b \quad , \tag{18b}$$

$$q_{ab} = \frac{1}{2}G(\sqrt{\Omega_b/\Omega_a}\,\alpha_a^2 + \sqrt{\Omega_a/\Omega_b}\,\alpha_b^2) \quad .$$

The quantity $\sigma^2(\mathcal{H}\mathcal{H}^\dagger)$ defined in (18a) can be regarded as a measure of deviation of the trial bra state (8) from being an exact l.h.s. eigenstate of $\mathcal{H}$. Eq. (18) simply means that we are minimizing this quantity with respect to α_a, subject to the constraint (16), in order to make the bra state (8) as close as possible to an exact l.h.s. eigenstate of $\mathcal{H}$. Since $\sigma^2(\mathcal{H}\mathcal{H}^\dagger)$ is proportional to the number of pairs n, it seems to be more meaningful to use its values per pair of particles. In Table I we therefore give the minimum values of $\sigma^2(\mathcal{H}\mathcal{H}^\dagger)/n$,

TABLE I. Minimum values of $\sigma^2(\mathcal{H}\mathcal{H}^\dagger)/n$ and $\sigma^2(\mathcal{H}^\dagger\mathcal{H})/n$ for the l.h.s. and r.h.s. boson condensate trial states, respectively, are shown for tin isotopes, with n being the number of valence pairs.

n	1	2	3	4	5	6	7	8	9
$\frac{\sigma^2(\mathcal{H}\mathcal{H}^\dagger)}{n}$	.0000	.0023	.0045	.0066	.0082	.0089	.0078	.0098	.0377
$\frac{\sigma^2(\mathcal{H}^\dagger\mathcal{H})}{n}$	.0000	.1410	.2942	.4589	.6349	.8225	1.022	1.233	1.456

instead of $\sigma^2(\mathcal{H}\mathcal{H}^\dagger)$, for the tin isotopes. There it can be readily seen that these minimum values are quite small for all the isotopes studied, indicating the boson bra state (8) is indeed a good approximation to the l.h.s. ground state of $\mathcal{H}$. Note also that $\sigma^2(\mathcal{H}\mathcal{H}^\dagger)$ vanishes for $n=1$, meaning that the bra state $<n;\alpha|$ is an exact l.h.s. eigenstate of $\mathcal{H}$ in this particular case.

Once the approximate l.h.s. ground state has been obtained variationally in this fashion, the parameters α_a are known. The physical boson state defined in (9) with the same parameters α_a can be used as the corresponding approximation to the r.h.s. ground state. One can then evaluate the ground state energy approximately by using $<n;\alpha|\mathcal{H}|\overline{n;\alpha}>/<n;\alpha|\overline{n;\alpha}>$. In this way to calculate the ground state energy, the present TSO method actually greatly simplifies the variational calculation because the variational procedure does not involve the complicated physical states $|\overline{n;\alpha}>$ at all. Here, however, instead of this complicated physical state, we shall use the following simple boson ket state

$$In;@> = \frac{1}{\sqrt{n!}}(\sum_a \beta_a B_a^\dagger)^n \; IO> \quad , \tag{19}$$

where the parameters β_a are to be determined variationally by using (14), subject to the normalization condidtion

$$\sum_a \beta_a^2 = 1 \quad . \tag{20}$$

Using (19) as the trial ket state, the present TSO method allows one to obtain the best approximation to the r.h.s. ground state in the form of a boson condensate. Employing the variational Eq. (14), we obtain

$$\frac{\partial}{\partial \beta_a} \; \sigma^2(\mathcal{H}^\dagger \mathcal{H}) = 0, \tag{21}$$

where

$$\sigma^2(\mathcal{H}^\dagger\mathcal{H}) \equiv <n;\beta|\mathcal{H}^\dagger\mathcal{H}|n;\beta> - |<n;\beta|\mathcal{H}|n;\beta>|^2 \quad . \tag{22a}$$

As in the previous case, the quantity $\sigma^2(\mathcal{H}^\dagger\mathcal{H})$ here can be considered as a measure of deviation of the simple boson ket state $n;\beta>$ from being an exact r.h.s. eigenstate of the Hamiltonian $\mathcal{H}$. The explicit formula for $\sigma^2(\mathcal{H}^\dagger\mathcal{H})$ can be readily obtained from (18a) by substituting β_a for α_a, and replacing t_a and q_{ab} respectively with $\tilde{t}_a$ and $\tilde{q}_{ab}$ defined below:

$$\tilde{t}_a = 2h_a\beta_a - G\sqrt{\Omega_a}\sum_b \sqrt{\Omega_b}\,\beta_b \quad , \tag{22b}$$

$$\tilde{q}_{ab} = G\,\delta_{ab}\frac{\beta_a}{\Omega_a}\sum_c \sqrt{\Omega_c}\,\beta_c \quad . \tag{22c}$$

The resultant minimum values of $\sigma^2(\mathcal{H}^\dagger\mathcal{H})/n$ for the tin isotopes are given in the second row of Table I. There one can see that they are not quite small, indicating that (19) is not a very good approximation to the r.h.s. ground state except for $n \approx 1$. Since the bra state (8) is already quite a good approximation to the l.h.s. ground state, it does not really matter much what ket state one actually uses to evaluate the energy. Besides, we are having the best ket state available in the form of (19). For simplicity we thus use $E_0(\alpha,\beta) = \langle n;\alpha|\mathcal{H}|n;\beta\rangle/\langle n;\alpha|n;\beta\rangle$ to calculate the approximate ground state energies for the tin isotopes. Namely,

$$E_0(\alpha,\beta) = 2n\sum_a h_a\alpha_a\tilde{\beta}_a - nG\sum_a\sqrt{\Omega_a}\,\tilde{\beta}_a\left\{\sum_b\sqrt{\Omega_b}\,\alpha_b - (n-1)\sum_b\frac{\alpha_b^2\tilde{\beta}_b}{\sqrt{\Omega_b}}\right\} \tag{23a}$$

with

$$\tilde{\beta}_a = \frac{\beta_a}{\sum_c \alpha_c\beta_c} \quad . \tag{23b}$$

Numerical results for these approximate ground state energies are given in the two columns under the heading TSO in Table II. In the first column the energies are evaluated by using

TABLE II. Approximate ground state energies for tin isotopes in MeV obtained by the present TSO variational method are compared with those obtained by exact diagonalizations and energy-minimization (EM) variational calculations.

n	TSO		Exact	EM	
	$E_0(\alpha,\alpha)$	$E_0(\alpha,\beta)$	E_0	$E_0(\alpha,\alpha)$	$E_0(\alpha,\beta)$
1	-1.534	-1.534	-1.534	-1.534	-1.534
2	-2.626	-2.626	-2.624	-2.630	-2.626
3	-3.263	-3.263	-3.258	-3.291	-3.265
4	-3.426	-3.427	-3.419	-3.517	-3.435
5	-3.076	-3.086	-3.084	-3.310	-3.113
6	-2.13 1	-2.163	-2.209	-2.67 1	-2.26 1
7	-0.412	-0.499	-0.701	-1.600	-0.807
8	2.156	2.25 1	2.161	0.097	1.421
9	5.041	6.062	5.702	1.834	4.612

$\langle n;\alpha|\mathcal{H}|n;\alpha\rangle = E_0(\alpha,\alpha)$ whereas those given in the second column are calculated by using Eq. (23). There one can see that these two sets of the ground state energies are quite close, indicating that the bra boson condensate state $\langle n;\alpha|$ is indeed quite a good approximation to the l.h.s. ground state of $\mathcal{H}$. Nevertheless, by using different bra and ket states, the resulting ground state energies are slightly better. Overall, the present TSO method yields quite accurate ground state energies even up to $n = 9$, i.e. over the middle of the valence

j-shells even though it seems to get worse for larger n. More accurate ground state energies can be obtained if one uses more sophisticated trial ket states than the boson condensate state given in (19), for instance, by including one more collective boson.

For comparison, we also present in the table results from exact diagonalizations as well as those of energy minimization (EM) variational calculations under the headings Exact and EM, respectively. The results of these EM variational calculations are basically obtained by minimizing the approximate ground state energies E_0 given in (23) with respect to both α_a and $\tilde{\beta}_a$, i.e.

$$\frac{\partial}{\partial \alpha_a} E_0(\alpha,\beta) = \frac{\partial}{\partial \tilde{\beta}_a} E_0(\alpha,\beta) = 0 \quad , \tag{24a}$$

subject to the constraints

$$\sum_a \alpha_a \tilde{\beta}_a = 1 \quad . \tag{24b}$$

In the first EM variational calculation, one does not distinguish between the r.h.s. and l.h.s. boson states and thus one sets $\tilde{\beta}_a = \alpha_a$ before the variation. However, in the second EM variational calculation[12] the bra and ket boson condensate states are taken as different and thus $\tilde{\beta}_a \neq \alpha_a$. The first EM calculation yields accurate ground state energies only for small number of pairs. It gives much too low ground state energies when one approaches the middle of the i-shells ($n \approx 8$), indicating severe violations of the Pauli principle there. By using different bra and ket boson condensate states, the second EM calculation has been able to take some effects of the non-Hermiticity of the Dyson boson Hamiltonian into account, as can be seen from the much improved ground state energies obtained over the first EM calculation. In fact, numerically this second EM calculation has exactly the same degree of accuracy as the l-boson approximation[19] to the Holstein-Primakoff boson representation of the monopole pairing model studied here. Since this EM calculation employs the standard variational principle without using the complicated physical ket states (9), near the middle of the valence j-shells it also suffers from yielding much too low ground state energies. By contrast, the present TSO method works well for all tin isotopes studied even when the same type of simple boson states are used. This clearly shows the necessity of usng a viable variational method such as the TSO method proposed here when relatively simple boson states are used in the variational calculations for non-Hermitian Dyson boson systems. Furthermore, by optimizing the trial state so that it is as close as possible to an exact eigenstate of the boson Hamiltonian, the present TSO method also automatically minimizes the spurious components contained in the trial state.

VI. SUMMARY AND CONCLUSION

In this paper we have proposed a variational calculation scheme to tackle the non-

unitarity problem of the Dyson boson mapping (DBM) approach to nuclear collective motion. It is based on a trial-state-optimization (TSO) variational principle in which one minimizes the deviation of a trial state from being an exact eigenstate, instead of minimizing the energy expectation value. The viability of this variational method has been demonstrated through an application to a monopole pairing Hamiltonian in the Dyson boson representation. Since the variational calculations for the bra and ket states are done independent of each other in the present variational method, it allows one to take advantage of the special feature of the DBM: namely, the bra states are often simple boson states while the ket states are complicated physical boson states.

For the simple pairing shell model employed here, of course, there exist more accurate methods such as the number-conserving quasiparticle (NCQP) method.[16] These methods, being purely fermion methods, do not use any boson mappings. Thus, no attempt has been made here to compete with them, in terms of accuracy. Our main concern here is to develop a viable variational calculation scheme to deal with the non-Hermitian problem of the Dyson boson mapping method as well as to minimize the spurious components often present in the various boson states used in calculations for nuclear colletive motion.

The main merit of the present TSO method is that it is applicable not only to the ground states but also to excited states of both Hermitian and non-Hermitian quantum systems. Furthermore, since one minimizes the direct measure of deviation of a trial state from being an exact eigenstate of the Hamiltonian in question, one can explicitly see that this deviation actually gets smaller when better trial states are used.

We expect the present TSO method to be a useful tool for studying nuclear collective motion as well as for investigating other many-body problems. Further developments and applications of this variational method to more realistic nuclear systems thus seem warranted. In particular, this variational method has been generalized to the case of multi-state calculations.[20] This generalization will allow one, for instance, to optimally determine the collective subspace relevant to nuclear collective motion. Actual applications of this new development will be reported elsewhere in the near future.

ACKNOWLEDGEMENT

The author would like to thank Dr. Abraham Klein for a useful discussion and Dr. Che Ming Ko and Dr. Siu A. Chin for valuable comments. This work was supported in part by the NSF of the United States under the grant PHY86-08149 during its early stage and was supported later on by the National Science Council of the Republic of China under the grant NSC80-0208-M002-08.

REFERENCES

1. D. Janssen, F. Döau, S. Frauendorf, and R. V. Jolos, Nucl. Phys. Al 72, 145 (1971).
2. P. Ring and P. Schuck, Phys. Rev. C 16, 801 (1977).
3. H. B. Geyer and S. Y. Lee, Phys. Rev. C 26, 642 (1982).
4. C. T. Li, Phys. Lett. **120B**, 251 (1983).
5. M. R. Zirnbauer, Nucl. Phys. **A419**, 241 (1984).
6. C. T. Li, Phys. Rev. C 29, 2309 (1984).
7. Y. K. Gambhir, J. A. Sheikh, P. Ring, and P. Schuck, Phys. Rev. C 31, 1519 (1985).
8. K. Takada, Nucl. Phys. **A439**, 489 (1985).
   K. Takada, T. Tamura and S. Tazaki, Phys. Rev. C 31, 1948 (1985).
   K. Takada, Phys. Rev. C 34,750 (1986).
9. H. B. Geyer, C. A. Engelbrecht, and F. J. W. Hahne, Phys. Rev. C 33, 1041 (1986).
10. K. Takada and S. Tazaki, Nucl. Phys. A448, 56 (1986).
    K. Takada, and K. Yamada, Nucl. Phys. **A462**, 561 (1987).
    K. Takada, Y. R. Shimizu and H. Thorn, Nucl. Phys. **A485**, 189 (1988).
11. W. Pannert and P. Ring, Nucl. Phys. **A465**, 379 (1987).
12. M. C. Cambiaggio and J. Dukelsky, Phys. Lett. **B197**, 479 (1987).
13. P. Park, Phys. Rev. C 35, 807 (1987).
14. R. Kuchta, Phys. A849, 1 (1988).
15. C. M. Vncent, L. J. Henry, and G.-K. Kim, Phys. Rev. C 39, 1583 (1989).
16. C. T. Li, Nucl. Phys. **A417**, 37 (1984).
17. C. J. Umrigar, K. G. Wilson, and J. W. Wilkins, Phys. Rev. Lett. 60, 1719 (1988).
    R. L. Coldwell, Int. Quant. Chem. **11**, 215 (1977).
    J. H. Barlett, Phys. Rev. 98, 1067 (1955).
18. C. T. Li, to appear in Phys. Rev. C.
19. A. Klein, C. T. Li, T. D. Cohen, and M. Vallieres, Progress in Particle and Nuclear Physics, Vol. 9, ed. D. Wilkinson (Pergamon Press, Oxford, 1983) p. 183.
20. C. T. Li, "*Number-conserving Quasiparticle Method and Boson Expansions*", to appear in Proceedings of the Second International Conference on Medium- and High-Energy Nuclear Physics, Taipei, 1990.