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One Dimensional Planar Classical Heisenberg Model

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We have studied the one dimensional planar classical Heisenberg model in two new perspectives – transfer matrix method and renormalisation group method.

I. INTRODUCTION

Exactly soluble models are intensively studied in statistical mechanics. Usually they occur in the case of unrealistic spatial dimensions – one dimensional and two dimensional models. The one dimensional models are more tractable mathematically. However, the study of one dimensional models suffers one drawback. It had been proven that one dimensional models with short-ranged interactions never exhibit phase transition at finite temperature. Nevertheless, the study of exact results of one dimensional soluble models is aesthetically rewarding. The results are often illuminating, offering demonstration of how exact analysis is carried out. The most famous example is the one dimensional Ising model.' It is the usual textbook example of one dimensional exactly soluble models. Next in complexity is the one dimensional planar classical Heisenberg model (xy model). This is the subject of discussion of our present paper. The planar model is easy to handle mathematically. Unfortunately, in the literature, it is usually discussed as a special case of the isotropic classical Heisenberg model,[2,3] in which the mathematics involved is quite messy. We are going to study the one dimensional planar classical Heisenberg model directly. We shall present new perspectives of the planar model, which we believe to be useful, especially for pedagogical purposes.

In section 2 we shall study the transfer matrix method. Because of a possible Fourier expansion, the results then look quite transparent. We have a knowledge of all the eigenvalues and eigenfunctions of the transfer matrix. We then use perturbation theory to evaluate the partition function in a small magnetic field. Hence we easily obtain the magnetic susceptibility directly. In the literature the magnetic susceptibility is usually obtained indirectly through the fluctuation-dissipation theorem. In section 3, we propose to study the model using renormalisation group transformation. The calculations proceed like the case of the Ising model.' However, there are some subtle differences.

II. TRANSFER MATRIX SOLUTION

First of all, let us present what is meant by a one dimensional planar classical Heisenberg model. The hamiltonian for the planar model is written as

$$H = -J \sum_{i=1}^{N} \cos(\theta_i - \theta_{i+1}) , \tag{1}$$

in which we have a linear chain with classical magnetic interaction in the plane, θ_i means the angle of inclination of the spin on the i^{th} site, and J denotes the coupling strength. In this paper we use the cyclic boundary condition $\theta_{N+1} = \theta_1$ for our convenience, since in the thermodynamical limit boundary conditions do not count.

Our task is to calculate the partition function

$$Z_N = \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} \cdots \int_0^{2\pi} \frac{d\theta_N}{2\pi} \exp\left[\frac{J}{kT} \sum_{i=1}^{N} \cos(\theta_i - \theta_{i+1})\right] , \tag{2}$$

where k is the Boltzmann constant and T is the absolute temperature. We can then define a transfer matrix equation (this is an integral equation because the B's are continuously varying),

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \exp[K\cos(\theta_1 - \theta_2)]\, \phi(\theta_2) = \lambda \phi(\theta_1) , \tag{3}$$

where we set $K = J/kT$. This is an eigenvalue problem for the integral equation. Its solution is readily seen due to the following expansion formula,

$$\exp\ K\cos\theta = \sum_{m=-\infty}^{\infty} I_m(K)\, e^{im\theta} , \tag{4}$$

where m are integers and $I_m(K)$ are the modified Bessel functions. Due to the orthogonality of the functions $e^{im\theta}$, it can be readily seen that the eigenvalues of the integral equation are

$$\lambda_m = I_m(K) , \tag{5}$$

and the normalised eigenfunctions are

$$\phi_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi}} . \tag{6}$$

The largest eigenvalue corresponds to

$$\lambda_{max} = I_0(K) . \tag{7}$$

This is an easy consequence of the Frobenius theorem which states that the largest eigenvalue of a nonnegative matrix is nondegenerate and corresponds to the eigenfunction with no node.

In the thermodynamical limit, we have the well-known result,

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = \log \lambda_{max} = \log I_o(K). \tag{8}$$

This is the exact result for the partition function. Indeed we can get more information from our solution. We have a knowledge of not only the largest eigenvalue of the transfer matrix, but also the whole eigenvalue spectrum and all the eigenfunctions. We can examine the transfer matrix in a magnetic field h

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \exp[K\cos(\theta_1 - \theta_2) + H(\cos\theta_1 + \cos\theta_2)]\,\psi(\theta_2) = \Lambda\psi(\theta_1), \tag{9}$$

where we have put $H = h/(2kT)$.

For h small we can use perturbation method to find out what is the new largest eigenvalue A,,, . Symmetry argument (h $\longleftrightarrow$ h) tells us that we have to work at least up to order h^2. So it is a simple exercise in perturbation theory to obtain

$$\Lambda_{max} = I,(K)\ [1 + H^2 \frac{I_0(K) + I_1(K)}{I_0(K) - I_1(K)}, + \ldots \ . \tag{10}$$

From this we get the magnetic susceptibility directly,

$$\begin{aligned} \chi &= \lim_{N\to\infty} \frac{kT}{N} \left(\frac{\partial^2}{\partial h^2} \log Z_N^h\right)_{h=0} \\ &= \frac{I}{2kT}\ \frac{1+u}{1-u}\ , \end{aligned} \tag{11}$$

where $u(K) = I, (K)/I_0(K)$.
This way of obtaining χ is in contrast to the usual way via the fluctuation-dissipation theorem.

III. RENORMALISATION GROUP METHOD

The basic idea of the renormalisation group method is to integrate over the even site spins to obtain a partition function of similar form as the original one, i.e. we integrate

$$\int_0^{2\pi} \exp[K(\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3))]\,d\theta_2 \tag{12}$$

In fact integral (12) is easy to perform, but the result is not of the form $\exp[K'\cos(\theta_1 - \theta_3)]$ That is the difficult point. Rather we have to write

$$e^{K\cos(\theta_1 - \theta_2)} = \sum_{m=-\infty}^{\infty} I_m(K)\, e^{im(\theta_1 - \theta_2)} \tag{13}$$

The integral (12) can then be written as

$$\sum_{m=-\infty}^{\infty} I_m(K)\, I_m(K)\, e^{im(\theta_1-\theta_3)} \quad . \tag{14}$$

So in this case we do not have one coupling, but we have infinite couplings. Under the renormalisation transformation, for each integer m

$$I_m(K) \rightarrow I_m(K)\, I_m(K) \quad . \tag{15}$$

This is the exact form for the renormalisation transformation. The infinite number of couplings makes it difficult to proceed further calculation. We have to make some "approximations". Since $I_0(K) > I_1(K) > \ldots$, we propose to retain only two of the couplings, supposing at certain point the deleting couplings become irrelevant. This is justified later by the result that this way of calculation agrees with results obtained by other exact methods.

So we propose

$$C\left\{ I_0(K') + 2I_1(K')\cos\theta \right\} = I_0^2(K) + 2I_1^2(K)\cos\theta \quad , \tag{16}$$

where C is an unimportant constant. We get the renormalisation transformation

$$\frac{I_1(K')}{I_0(K')} = \frac{I_1^2(K)}{I_0^2(K)} \quad , \tag{17}$$

or,

$$K' = u^{-1}(u^2(K)) \, . \tag{18}$$

This is the key formula. We now use this relation to calculate the correlation length ξ following the method of Schuster[4] who dealed with the case of the Ising model.

From Eq. (18) we see that the only fixed points are 0 and ∞, confirming that there is no phase transition at finite temperature.

The correlation length ξ obeys the scaling relation

$$\xi(K) = 2\xi(K') \; . \tag{19}$$

This implies,

$$\xi(K) = 2^n\, \xi\left\{ u^{-1}(u^{2n}(K)) \right\} \quad . \tag{20}$$

For $K \gg 1$ the variable n can be chosen such that

$$u^{2n}(K) = \text{constant} \tag{21}$$

This leads to

$$2^n \propto 1/\log(u(K)) \quad , \tag{22}$$

or

$$\xi \propto 1/\log(u(K)) \quad . \tag{23}$$

This is in agreement with exact results.

We can also copy the method of Nauenberg[5] to obtain the free energy of the planar model by relation (18).

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