On a Mechanical Analogue of the Braess Paradox

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The Braess paradox states that adding a branching route to an already congested road might actually worsen traffic jams. An elegant mechanical analogue to this paradox has already been proposed (J. E. Cohen and P. Horowitz, Nature 352, 699-701 (1991)) which seems to defy many people's initial intuition. We examine the motion of this analog mechanical system in an attempt to make the counter-intuitive prediction more comprehensible, while also clarifying a misleading remark contained in this reference.

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To relieve traffic congestion at a certain section on the highway, one's first reaction might be to build a bypass to increase the local traffic flow, in such a way that, when it later again merges with the main traffic, the total time cost to go from the origin to the final destination can be reduced. That this may not always be the case, and indeed that a bypass added to a highly congested area often actually increases the travel time, was discovered by Braess in 1968 [1–4], and later appropriately referred to as the Braess paradox. In 1991 J. E. Cohen and P. Horowitz suggested a simple mechanical analogue (among other physical examples) of the Braess paradox, to illustrate how one's intuition might fail even in a seemingly trivial experimental setup [5]. Though this "paradox" might appear rather trivial to a physicist once (s)he has seen it, it might still present some mystery to beginning mechanics students. One purpose of this short note is, therefore, to give an elementary account of the motion of this system, so that one can feel more at ease with this least expected outcome. Another point motivating our present work is to clarify a misleading remark contained in Ref. [5]. (See the end of the next paragraph for details.)

The model proposed by Cohen and Horowitz is shown in Fig. 1(a); it consists of a weight M suspended under the pivot P by two identical (idealized massless) springs which are connected by a taut massless string $\overline{AB}$. From the pivot P one also hangs a slack string (again, of no mass) which is connected to the top B of the lower spring, as a kind of "relief wire." Similarly, the lower end A of the upper spring and the mass M is also linked by a limp string $\widetilde{AQ}$, that is in every way identical to the string $\widetilde{PB}$. When the system is in equilibrium, one measures the height of M. Then one cuts the connecting string $\overline{AB}$ and asks if M will have a lower height when things are back in static equilibrium again, as illustrated in Fig. 1(b). The counter-intuitive part of the story is that in some suitable parameter regimes the suspended weight actually rises to a higher elevation. Not only that, Cohen and Horowitz also remarked that, at least in their experimental setup, the weight

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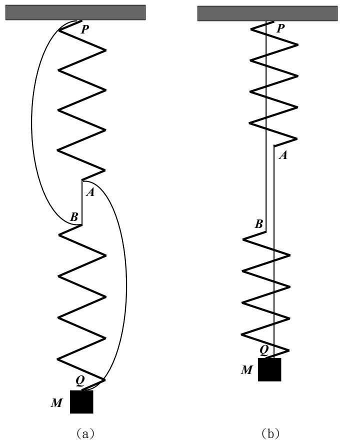


FIG. 1: A weight M is supported by two springs in series, with a connecting string $\overline{AB}$ holding the springs together. Two limp relief wires $\widetilde{PB}$ and $\widetilde{QA}$ are also attached to the system. (b) After one cuts $\overline{AB}$, the relief wires and the springs will act together to support the weight when a new equilibrium configuration is reached.

can even rise in the initial transient, right after one cuts the connecting string! Taken at its face value, this observation really challenges one's physics intuition: How can the weight *not* fall right after one cuts the connecting string? As we will show below, indeed, the weight necessarily falls by a short distance at an early stage of the time evolution. Beyond that brief moment, however, it can be so quickly pulled up by the relief wires that it may be virtually impossible to observe the fall under ordinary experimental settings.

To understand how this is possible, we will consider a slightly more realistic model in the following. (See Fig. 2.) First, we assume that each of the two ends A and B of the two idealized springs actually has a small mass m attached to it. This serves as a reminder of the fact that the springs in reality cannot be massless [6]. Next, we suppose that the otherwise limp relief wires behave like a very strong spring with a spring constant K when their lengths are stretched beyond a threshold length L. Letting s and k denote the natural length of the springs and their associated spring constant, and denoting the vertical displacement of the masses A, B, and M by z_1, z_2, and z_3, respectively, we may write down the equations of motion for the system *after the connecting string* $\overline{AB}$ *is cut* as

$$\begin{aligned}
m\frac{d^2z_1}{dt^2} &= -k(z_1 - s) + f(z_3 - z_1 - L) + mg - \gamma'\frac{dz_1}{dt}\,, \\
m\frac{d^2z_2}{dt^2} &= -f(z_2 - L) + k(z_3 - z_2 - s) + mg - \gamma'\frac{dz_2}{dt}\,, \\
M\frac{d^2z_3}{dt^2} &= -k(z_3 - z_2 - s) - f(z_3 - z_1 - L) - \gamma\frac{dz_3}{dt} + Mg\,,
\end{aligned} \tag{1}$$

where f describes the constraining force coming from the relief wire and satisfies

$$f(\zeta) \equiv \begin{cases} K\zeta & (\zeta \geq 0) \\ 0 & (\zeta < 0) \end{cases}\,,$$

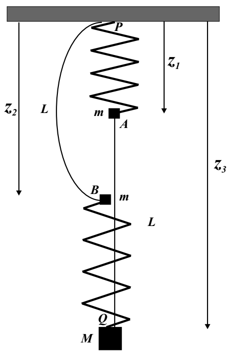


FIG. 2: The modified model we are considering here has one additional small mass m attached to one end of each of the two idealized springs $\overline{AP}$ and $\overline{BQ}$. Also shown are the various variables pertinent to the problem. (Note: This figure does *not* depict the initial configuration of the system, but rather a state, after one cuts the connecting wire $\overline{AB}$, such that the relief wire $\widetilde{QA}$ has become taut.)

and g is the gravitational acceleration. In the above, we have also introduced a very small damping γ and γ' on the motion of the weight M and the mass points A and B, to guarantee that the system will after a long time eventually come to a new static equilibrium, although in all our later analysis we will also assume that the characteristic damping times, M/γ and m/γ' are so large compared to all the other time scales of the problem, that we may safely ignore them as far as the short-term time variation is concerned. Before the connecting string of length l is cut, however, the static equilibrium configuration $(z_1, z_2, z_3) = (z_{10}, z_{20}, z_{30})$ satisfies

$$\begin{aligned} 0 &= -k(z_{10} - s) + mg + T\,, \\ 0 &= k(z_{30} - z_{20} - s) + mg - T\,, \\ 0 &= -k(z_{30} - z_{20} - s) + Mg\,, \end{aligned} \tag{2}$$

with T standing for the tension on the taut string $\overline{AB}$ and z_{20} given by

$$z_{20} = z_{10} + l$$

as a constraint.

The solution for Eq. (2) is trivially

$$\begin{aligned} T &= (m + M)g\,, \\ z_{10} &= s + \frac{(2m + M)g}{k}\,, \\ z_{20} &= s + \frac{(2m + M)g}{k} + l\,, \\ z_{30} &= 2s + \frac{2(m + M)g}{k} + l\,; \end{aligned} \tag{3}$$

whereas the final equilibrium configuration after $\overline{AB}$ is cut can be solved from Eq. (1); it is

$$\begin{aligned} z_1 &= s + \frac{(k+K)M + (3k+K)m}{2k(k+K)} g \,, \\ z_2 &= L + \frac{(k+K)M + (k+3K)m}{2K(k+K)} g \,, \\ z_3 &= L + s + \frac{k+K}{2kK}(m+M)g \,. \end{aligned} \tag{4}$$

The condition that the weight M rises in height after the cut is thus

$$L + s + \frac{k+K}{2kK}(m+M)g < 2s + \frac{2(m+M)g}{k} + l \,, \tag{5}$$

which certainly can be fulfilled if we play around with the many parameters involved. In particular, the situation is much simplified when $m \to 0$ and $K \to \infty$, because the solutions of z_3 in both Eqs. (3) and (4) reduce to the familiar results for springs placed in series and in parallel, respectively. In this case, the criterion for having a rising weight becomes

$$L + s + \frac{Mg}{2k} < 2s + \frac{2Mg}{k} + l \,.$$

However, this is not what interests us most here. Instead, we would like to know how the system evolves into the final state, especially at the initial stage of its evolution, when it seems natural to assume that the weight M must drop a little bit at the very least due to gravity.

To this end, we first note that the initial time evolution is governed only by

$$\begin{aligned} m\frac{d^2 z_1}{dt^2} &= -k(z_1 - s) + mg \,, \\ m\frac{d^2 z_2}{dt^2} &= k(z_3 - z_2 - s) + mg \,, \\ M\frac{d^2 z_3}{dt^2} &= -k(z_3 - z_2 - s) + Mg \,, \end{aligned} \tag{6}$$

that is, the relief wires remain slack, because no infinite acceleration is involved here due to our very assumption that the springs are massive. Clearly, the motion of mass A decouples from that of mass B and the weight M, as it should. Hence it is easy to solve for the motion of mass A; the answer is

$$z_1 = s + \frac{mg}{k} + \left(z_{10} - s - \frac{mg}{k}\right)\cos\sqrt{\frac{k}{m}}t \tag{7}$$

$$= s + \frac{mg}{k} + \frac{(m+M)g}{k}\cos\sqrt{\frac{k}{m}}t \,. \tag{8}$$

The time dependence of the center of mass of B and M can be easily calculated; it reads

$$z_{cm} \equiv \frac{mz_2 + Mz_3}{m+M} = \frac{mz_{20} + Mz_{30}}{m+M} + \frac{1}{2}gt^2 \equiv z_{cm0} + \frac{1}{2}gt^2 \,. \tag{9}$$

Then the equation for z_3 becomes

$$\mu \frac{d^2 z_3}{dt^2} = -k z_3 + \frac{k}{2} g t^2 + k z_{cm0} + \mu \left(g + \frac{ks}{M} \right) ,$$

which has the solution

$$z_3 = z_{cm0} + \frac{\mu s}{M} + \frac{1}{2} g t^2 + \left(z_{30} - z_{cm0} - \frac{\mu s}{M} \right) \cos \sqrt{\frac{k}{\mu}} t \, , \tag{10}$$

with

$$\mu \equiv \frac{mM}{m+M}$$

being the reduced mass. Eq. (10) can be further simplified to read

$$z_3 = z_{30} + \frac{1}{2} g t^2 - \frac{\mu g}{k} \left(1 - \cos \sqrt{\frac{k}{\mu}} t \right) , \tag{11}$$

using the fact that

$$k \left(z_{30} - z_{cm0} - \frac{\mu s}{M} \right) = \frac{m}{m+M} (z_{30} - z_{20} - s) = \mu g \, .$$

Hence, the initial time evolution of the weight M is

$$z_3 \approx z_{30} + \frac{1}{4!} \frac{kg}{\mu} t^4 \, ,$$

showing that *the mass M necessarily drops in height at the initial stage of time evolution, although its fall due to gravity is significantly compensated for by the upward pull of the spring, in such a way as to make the fall proportional to the fourth power of time, instead of the ordinary square dependence.* If we think about it, this result should not come as any surprise at all, because the net acceleration of the weight at $t = 0$ is exactly zero, due to the precise balance between the upward pull of the spring on the weight and the downward pull of the gravity. Also, because the spring is already in a stretched state at $t = 0$, subsequent evolution right after the connecting string is cut must proceed in such a way so as to reduce the length of the spring, which effectively reduces the pull from the spring and permits gravity to eventually win out in this "tug of war." From Eqs. (9) and (11) the solution for z_2 can also be computed to be

$$z_2 = z_{20} + \frac{1}{2} g t^2 + \frac{M}{m} \frac{\mu g}{k} \left(1 - \cos \sqrt{\frac{k}{\mu}} t \right) . \tag{12}$$

Next, we need to investigate when the relief wires will become taut and exert a force on the system. Clearly, this can happen only when either $z_3 \geq z_1 + L$ or $z_2 \geq L$. For the former to happen the time t must satisfy

$$z_{30} + \frac{1}{2} g t^2 - \frac{\mu g}{k} \left(1 - \cos \sqrt{\frac{k}{\mu}} t \right) \geq L + s + \frac{mg}{k} + \left(z_{10} - s - \frac{mg}{k} \right) \cos \sqrt{\frac{k}{m}} t \, ,$$

or

$$\frac{1}{2}gt^2 - \frac{\mu g}{k}\left(1-\cos\sqrt{\frac{k}{\mu}}t\right) + \frac{(m+M)g}{k}\left(1-\cos\sqrt{\frac{k}{m}}t\right) \geq (L - z_{30} + z_{10})\,.$$

The equality sign holds when t reaches the threshold time τ_1, above which the relief wire $\widetilde{AQ}$ becomes taut. Unfortunately, τ_1 satisfies a transcendental equation, so that we can obtain a simple analytical form for it only if certain approximations are made. For example, we expect τ_1 to be small if $L - z_{30} + z_{10}$ is small (the length of the relief wire barely exceeds the initial separation between A and M). In this case we may expand

$$\frac{1}{2}g\tau_1^2 - \frac{\mu g}{k}\left(1-\cos\sqrt{\frac{k}{\mu}}\tau_1\right) + \frac{(m+M)g}{k}\left(1-\cos\sqrt{\frac{k}{m}}\tau_1\right) = (L - z_{30} + z_{10}) \quad (13)$$

up to the second order in τ_1 to yield

$$\tau_1 \approx \sqrt{\frac{2m}{m+M}\frac{(L - z_{30} + z_{10})}{g}}\,. \quad (14)$$

We can similarly compute the time τ_2 when the slack relief wire $\widetilde{BP}$ first becomes taut. The condition $z_2(t=\tau_2) = L$ gives

$$\frac{1}{2}g\tau_2^2 + \frac{M}{m}\frac{\mu g}{k}\left(1-\cos\sqrt{\frac{k}{\mu}}\tau_2\right) = L - z_{20}\,, \quad (15)$$

which has

$$\tau_2 \approx \sqrt{\frac{2m}{m+M}\frac{(L - z_{20})}{g}} \quad (16)$$

as the approximate solution when $L - z_{20}$ is small. Since it is obviously true that

$$z_{20} - (z_{30} - z_{10}) = \frac{2mg}{k} > 0\,,$$

we therefore conclude that $\tau_2 < \tau_1$, implying that the left relief wire $\widetilde{BP}$ will first become operative in trying to prevent the system from further falling off before the wire $\widetilde{AQ}$ does. During this period, the weight M has dropped a vertical distance of

$$z_3 - z_{30} \approx \frac{1}{4!}\frac{kg}{\mu}\tau_2^4 \approx \frac{1}{3}\frac{m}{M}\frac{k(L - z_{20})^2}{(m+M)g}\,,$$

which is very small when m is small.

Another physically interesting case in which we may perform an approximate analysis to show that $\tau_2 < \tau_1$ is when $m \ll M$. To avoid unnecessary digression, however, we have chosen to relegate the details of the calculation to the Appendix.

Once the relief wire connecting mass point B is activated when the other relief wire is still slack, the equations of motion for z_2 and z_3 become

$$\begin{aligned} m\frac{d^2 z_2}{dt^2} &= -K(z_2 - L) + k(z_3 - z_2 - s) + mg\,, \\ M\frac{d^2 z_3}{dt^2} &= -k(z_3 - z_2 - s) + Mg\,; \end{aligned} \tag{17}$$

and the "initial conditions" at $t = \tau_2$ are

$$\begin{aligned} z_2 &= L\,, \\ z_3 &\approx z_{30} + \frac{1}{24}\frac{kg}{\mu}\tau_2^4\,, \\ \frac{dz_2}{dt} &\approx \left(1 + \frac{M}{m}\right) g\tau_2 \approx \frac{Mg}{m}\tau_2\,, \\ \frac{dz_3}{dt} &\approx \frac{1}{6}\frac{kg}{\mu}\tau_2^3\,; \end{aligned} \tag{18}$$

correct to the first nontrivial order. Once again, the exact analytical expression for the solution of Eq. (17) is messy and tends to obscure the underlying physics, so it is advisable to resort to approximate solutions. This time, we invoke the assumptions that, (1) the typical acceleration experienced by mass point B is dominated by the restoring force of the spring rather than the local gravitational pull, and (2) K is very much greater than k. These assumptions greatly simplify Eq. (17); it now reads

$$\begin{aligned} m\frac{d^2 z_2}{dt^2} &\approx -K(z_2 - L)\,, \\ M\frac{d^2 z_3}{dt^2} &= -k(z_3 - z_2 - s) + Mg\,, \end{aligned} \tag{19}$$

which has the following as its solution for z_2

$$z_2 \approx L + \frac{Mg}{\sqrt{mK}}\tau_2 \sin\sqrt{\frac{K}{m}}(t - \tau_2)\,.$$

This solution is valid for only a very short period of time τ_{pull1} described by

$$\tau_{pull1} = \pi\sqrt{\frac{m}{K}}\,,$$

because the wire will immediately become limp again after this brief interaction time. Although z_2 immediately regains its starting value (of L), the velocity dz_2/dt of the mass point B is reversed during this period. *The most noticeable effect of this relief wire is then seen to be to quickly reverse the velocity during this period and make itself slack again. Its effect on z_3, however, is barely noticeable, in view of Eq. (19) and the brevity of the interaction.* Of course, once the direction of the motion of the mass point B is reversed, we do expect the spring $\overline{BQ}$ to quickly stretch out again, to exert an upward force on the weight M.

Exploiting Eqs. (11) and (12), the stretching or the compressional state of the spring $\overline{BQ}$ during this period of time can be checked from the sign of

$$z_3 - z_2 - s = z_{30} - z_{20} - s - \frac{Mg}{k}\left(1 - \cos\sqrt{\frac{k}{\mu}}t\right) = \frac{Mg}{k}\cos\sqrt{\frac{k}{\mu}}t\,,$$

as long as

$$\sqrt{\frac{k}{\mu}}\tau_2 < \frac{\pi}{2}\,, \tag{20}$$

or equivalently

$$\sqrt{\frac{2k(L - z_{20})}{Mg}} < \frac{\pi}{2}\,. \tag{21}$$

If the relief wires are barely slack, then the spring is always in a stretched state. If Eq. (21) is satisfied then $\overline{BQ}$ will again begin to stretch more when the relief wire $\widetilde{BP}$ first becomes taut. This suggests that the spring $\overline{BQ}$ will remain a major force, continuously counteracting the gravitational pull on the weight.

Do we expect Eq. (21) to hold for the case we are dealing with here? Suppose we are interested in arrangements that will eventually lead to a rise in the weight M, then Eq. (5) must be satisfied. In the limit when K is very large this becomes

$$L - s - l < \frac{3}{2}\frac{(m+M)g}{k}\,. \tag{22}$$

By Eqs. (3) and (22) we know

$$\sqrt{\frac{2k(L - z_{20})}{Mg}} = \sqrt{\frac{2k(L - s - l - \frac{2m+M}{k}g)}{Mg}} < \sqrt{1 - \frac{m}{M}}\,,$$

which certainly is less than $\pi/2$, as required by Eq. (21). This therefore gives us a strong indication that indeed the spring on the left ($\overline{BQ}$) is usually in a stretched condition and ready to (partially) support the weight from falling. Although the above was only derived for the limiting case when the relief wires are barely slack, we should also mention that Eq. (20) still holds if it is the mass ratio m/M which is small instead. The argument leading to this assertion is given in the Appendix.

The effect of the relief wire $\widetilde{AQ}$ on the mass point A and weight M can be analogously analyzed. Thus, by Eq. (1)

$$m\frac{d^2 z_1}{dt^2} = -k(z_1 - s) + f(z_3 - z_1 - L) + mg \approx K(z_3 - z_1 - L)\,,$$

$$M\frac{d^2z_3}{dt^2} = -k(z_3 - z_2 - s) - f(z_3 - z_1 - L) - \gamma\frac{dz_3}{dt} + Mg \approx -K(z_3 - z_1 - L)\,,$$

we obtain

$$\frac{d^2(z_3 - z_1)}{dt^2} = -\frac{K}{\mu}(z_3 - z_1 - L)\,,$$

and

$$\frac{d^2(mz_1 + Mz_3)}{dt^2} = 0\,.$$

These equations can be easily solved to yield

$$\begin{aligned} z_3 - z_1 &= L + \sqrt{\frac{\mu}{K}}v_0 \sin\sqrt{\frac{K}{\mu}}(t - t_0)\,, \\ m\frac{dz_1}{dt} + M\frac{dz_3}{dt} &= (m + M)v_{cm} = \text{constant}\,, \end{aligned} \tag{23}$$

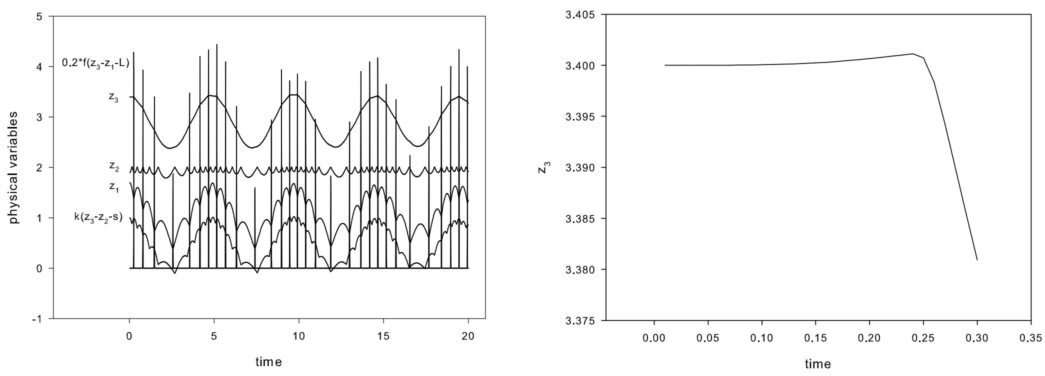
where v_{cm} and v_0 are, respectively, the velocity of their center of mass and the initial relative velocity between M and A when the wire begins to tense up. By our assumption the relative velocity is necessarily greater than zero, Eq. (23) tells us that

$$M\frac{dz_3}{dt} = Mv_{cm} + \mu v_0 \cos\sqrt{\frac{K}{\mu}}(t - t_0)\,,$$

so that at the end of the brief interaction time, $\tau_{pull2} \equiv \pi\sqrt{\mu/K}$, the mass point M necessarily gains a net momentum of the amount

$$-2\mu v_0\,.$$

In other words, *the relief wire $\widetilde{AQ}$ is the one responsible for giving M a prominent sporadic upward pull whenever it gets tensed.* Although we remarked before that a force which tends to continuously hold the weight up is supplied by the spring $\overline{BP}$, because it is usually in a stretched state, the wire $\widetilde{AQ}$ actually plays an equally important role in upsetting the gravity (to conspire the "paradox" we are investigating here). This is illustrated in Fig. 3(a) using the following parameters: $k = M = g = 1$, $\gamma = 0.001$, $\gamma' = 0.01$, $m = 0.1$, $K = 1000$, $l = 1$, $L = 2$, $s = 0.2$. In this figure, one readily notices that the force $k(z_3 - z_2 - s)$ exerted on the weight by the spring $\overline{BP}$ is almost always positive, confirming our observation that it is constantly in a stretched condition; yet its magnitude never exceeds the weight Mg (=1 in this illustration), simultaneously suggesting that the tension $f(z_3 - z_1 - L)$ on the wire $\widetilde{AQ}$ is just as responsible for supporting the weight. In fact, this figure shows that both forces are acting in phase to comprise a strong force that quickly pulls the weight up after the connecting string is cut or when the weight is in a lower position.



(a) (b)

FIG. 3: (a) Time evolution of the various variables of the system. Note that the restoring force $k(z_3 - z_2 - s)$ falls below zero only sporadically, indicating that the spring $\overline{PB}$ is usually in a stretched state. The delta function-like pulls of the relief wire $\widetilde{QA}$ (represented by the function $f(z_3 - z_1 - L)$) is equally important for raising the weight up. (b) A blowup of the initial portion of the time evolution, to show that z_3 indeed increases (meaning the weight M falls) for a brief moment right after one cuts the connecting wire $\overline{AB}$.

In summary, we notice that several time scales are present in this exercise even when we assume $m \ll M$: the damping times $T_{d1} = M/\gamma$ and $T_{d2} = m/\gamma'$, the relief wire interaction time $T_{rw} = \sqrt{m/K}$, the natural period of the massive spring $T_m = \sqrt{m/k}$, and the natural period of the weight $T_M = \sqrt{M/k}$. The assumptions that $T_{d1}, T_{d2} \gg T_M \gg T_m \gg T_{rw}$ greatly simplify the analysis: The left-most inequality allows us to deal with a quasi-conservative system whose time evolution can be more simply expressed, and the right-most inequality reduces the response time of the relief wire so that we can focus entirely on the motion of the small masses A, B and the weight M. The middle inequality then allows us to treat the influence of the springs and the relief wires on the weight in a time-averaged manner, from which one can derive a simple picture for the motion of the weight. Under these grossly simplified (but still physically interesting) assumptions, we learned that the weight must necessarily drop in height immediately after the connecting string is cut. However, because the drop is so little and so brief in duration when $m \ll M$, one may not be able to readily verify it in a less well-controlled experiment. This then explains the counter-intuitive observation of the *initial* rise of the weight reported in Ref. [5]. Indeed, a casual glance at Fig. 3(a) also gives one the false impression that the weight M rises initially, as z_3 appears to be decreasing right after $t = 0$. It is only by exploding the initial portion of the time evolution (Fig. 3(b)) that one can see clearly that M actually falls a very short distance before it is lifted upward.

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APPENDIX A: APPENDIX

In this appendix we show why $\tau_2 < \tau_1$ for the case $m \ll M$. Defining

$$\xi_1 \equiv \sqrt{\frac{k}{\mu}}\tau_1 \text{ and } \alpha \equiv \frac{m}{M}\,,$$

we then need to solve

$$\frac{\alpha}{2(1+\alpha)}\xi_1^2 - \frac{\alpha}{(1+\alpha)}(1-\cos\xi_1) + (1+\alpha)\left(1-\cos(\xi_1(1+\alpha)^{-1/2})\right) = \frac{k(L-z_{30}+z_{10})}{Mg}$$

in view of Eq. (13). Expanding the expression up to the first order in α, we obtain

$$\frac{\alpha}{2}\xi_1^2 - \frac{\alpha}{2}\xi_1\sin\xi_1 + (1-\cos\xi_1) = \frac{k(L-z_{30}+z_{10})}{Mg}\,.$$

The lowest order solution ξ_{10} to this equation can be easily obtained by inverting

$$1-\cos\xi_1^{(0)} = \frac{k(L-z_{30}+z_{10})}{Mg}\,, \tag{A1}$$

while its next order solution $\xi_1^{(1)}$ is simply given by

$$1-\cos\xi_1^{(1)} = \frac{k(L-z_{30}+z_{10})}{Mg} - \frac{\alpha}{2}\xi_1^{(0)2} + \frac{\alpha}{2}\xi_1^{(0)}\sin\xi_1^{(0)}\,. \tag{A2}$$

We are interested in arrangements that will lead to a rise in the equilibrium height of the weight when the relief wires have a very large K; the condition of Eq. (5) for this case becomes

$$L-s-l < \frac{3}{2}\frac{(m+M)g}{k}\,. \tag{A3}$$

Employing this result, we see that the right-hand side of Eq. (A1) satisfies

$$\frac{k(L-z_{30}+z_{10})}{Mg} = \frac{k(L-s-l)}{Mg} - 1 < \frac{1}{2} + \frac{3\alpha}{2}\,,$$

which in turn implies that

$$\xi_1^{(0)} < \frac{\pi}{3} + O(\alpha)\,.$$

Next, we try to obtain a solution to Eq. (15) correct to the same order in α. In terms of the new variable

$$\xi_2 \equiv \sqrt{\frac{k}{\mu}}\tau_2 \,,$$

this equation becomes

$$\frac{\alpha}{2}\xi_2^2 + \frac{1}{(1+\alpha)}\left(1-\cos\xi_2\right) = \frac{k(L-z_{20})}{Mg} \,.$$

This equation has the lowest order solution $\xi_2^{(0)}$ satisfying

$$1-\cos\xi_2^{(0)} = \frac{k(L-z_{20})}{Mg} \,, \tag{A4}$$

and the next order solution satisfying

$$1-\cos\xi_2^{(1)} = \frac{k(L-z_{20})}{Mg} - \frac{\alpha}{2}\xi_2^{(0)2} + \alpha\left(1-\cos\xi_2^{(0)}\right) \,. \tag{A5}$$

Again, the right-hand side of Eq. (A4) can be easily shown to satisfy

$$\frac{k(L-z_{20})}{Mg} = \frac{k(L-s-l)}{Mg} - (1+2\alpha) < \frac{1}{2} - \frac{\alpha}{2}$$

by Eq. (A3) so that

$$\xi_2^{(0)} < \frac{\pi}{3} - O(\alpha) \,.$$

Incidentally, we note in passing that this immediately tells us that Eq. (20) in the main text is indeed satisfied.

To compare the magnitudes of $\xi_1^{(1)}$ and $\xi_2^{(1)}$, we first note that the right-hand side of Eq. (A4) is less than the right-hand side of Eq. (A1):

$$\frac{k(L-z_{30}+z_{10})}{Mg} - \frac{k(L-z_{20})}{Mg} = 2\alpha > 0 \,.$$

This simply says that $\xi_2^{(0)} < \xi_1^{(0)}$. However, the fact that $\xi_1^{(0)} - \xi_2^{(0)}$ is only of order $O(\alpha)$ also means that we simply have to compute ξ_1 and ξ_2 to the next order of accuracy to say for sure that indeed $\xi_2 < \xi_1$. To proceed to show that $\xi_2^{(1)} < \xi_1^{(1)}$, all we need to do is prove that the right-hand side of Eq. (A2) is greater than that of Eq. (A5). However this is obvious, because

$$\left(\frac{k(L-z_{30}+z_{10})}{Mg} - \frac{\alpha}{2}\xi_1^{(0)2} + \frac{\alpha}{2}\xi_1^{(0)}\sin\xi_1^{(0)}\right) - \left(\frac{k(L-z_{20})}{Mg} - \frac{\alpha}{2}\xi_2^{(0)2} + \alpha\left(1-\cos\xi_2^{(0)}\right)\right)$$
$$\approx 2\alpha + \frac{\alpha}{2}\xi_1^{(0)}\sin\xi_1^{(0)} - \alpha\left(1-\cos\xi_1^{(0)}\right) > 0 \,.$$

Combining all the above, we have thus proved our claim that $\tau_2 < \tau_1$ when $m \ll M$.

References

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[6] Admittedly this is not a good approximation to a real spring. However, we still chose to use this simplified model for two reasons: The effect of the mass of the spring can be most clearly seen in the model, and the kind of motion we consider here will induce a series of waves in a real spring, which tends to further obscure the physics involved.