# Excited $K^\pi = 0^+$ Bands in $^{168}$Er*

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A microscopic analysis using the projection before variation method is applied to study the $K^\pi = 0^+$ theoretical two-quasiparticle bands in $^{168}$Er. Among these calculated bands, the experimental observed bands are identified with remarkable agreement. It is suggested that the $K^\pi = 0^+$ excited bands in $^{168}$Er are pure two-quasiparticle bands and cannot be identified as collective O-vibration bands.

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## I. INTRODUCTION

IN a recent " Letter To The Editor [1], a microscopic analysis using the projection before variation method was applied to study the $K^\pi = 0^+$ two-quasiparticle bands in $^{168}$Er and remarkable agreement with experiment was obtain. It was suggested that the $K^\pi = 0^+$ excited bands in $^{168}$Er are pure two-quasiparticle bands and cannot be indentified as collective @vibration bands. In the Letter, concentration was focused mainly on the two experimentally observed low-lying $K^\pi = 0^+$ bands at 1.217 and 1.422 Mev. Besides, the description of the calculation was quite sketchy such that it may be not so apparent to those interested readers who are not really familiar with the field. Therefore, it is the purpose of this paper to report the detailed calculation of this work, and, besides the two $K^\pi = 0^+$ bands cited above, we shall report also other $K^\pi = 0^+$ bands which are theoretically possible.

Recent detailed experimental data on the low energy spectrum of $^{168}$Er[2], obtained from a high resolution Y-ray study following neutron capture, has stimulated many timely investigations[3,4,5] to look into the nature of the various rotational bands of this nucleus. These data are supposed to be complete below 2.2 Mev and have been used for a detailed test[6] on the validity of the interacting boson model[ ]. The results are interpreted to be in support of the model. However, the collectivity of the two excited $0^+$ bands at 1.217 Mev and 1.422 Mev have been put under questions[4,5] on the ground of their small B(E2) values to the ground state band and of the structure of the single-particle levels. Without addressing ourselves to the task of deciding which collective model is more generally acceptable, we report in this note some relevant results which may be of some help in elucidating the nature of these bands, These are obtained from a microscopic analysis using a standard proejction before variation treatment in a deformed intrinsic state[8,9,10]

The method of angular momentum proejction is well known in studies dealing with ground state rotational states in heavy deformed nuclei[8,9]. Lin, Faessler, and Dreizler[10] have extended the method to treat excited rotational states with even number of quasiparticles and applied it to negative parity states yeilding results in good agreement with experiment. However, its applicability to excited positive parity states remains untested. It is satisfying to see from the present work that this method indeed works well at least for the case of near lying $K^\pi = 0^+$ two-quasiparticle bands. This implies, in particular, that the method is capable of describing excited single quasiparticle states, and their couplings to the rotating core.

## II. THEORY

We start by taking for the deformed intrinsic states simple Nilsson BCS-type wave functions which can be written in the canonical representation as the following form

$$| \Phi > = \prod_{k>0} (U_k + V_k b_k^+ b_{\bar{k}}^+ ) | 0 > \tag{1}$$

The $b_k^+$'s are the creation operators of the Nilsson single particle states and $V_k^2 = 1 - U_k^2$ the occupation probabilities given as

$$V_k^2 = \tfrac{1}{2} \left[ 1 - \frac{\epsilon_k - \lambda}{\sqrt{(\epsilon_k - \lambda)^2 + \Delta^2}} \right] \quad \text{and} \quad \Sigma V_k^2 = N \tag{2}$$

where $\epsilon_k$ is the Nilsson single particle energy, $\lambda$ the Fermi energy of protons and neutrons, A the energy gap of protons or neutorns, and N the total number of particles of protons or neutrons. Wave functions $| \Phi >$ defined as equation (1) do not conserve either the particle number N or the angular momentum J. The usual way of restoring these broken symmetries is to introduce the projection methods for both the particle number and the angular momentum. The particle number projection operator is defined[7] as

$$Q_N = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{i(\hat{N}-N)\phi} \quad , \tag{3}$$

and the angular momentum projection operator is[8] (for axially symmetrical states)

$$\hat{P}_J = \frac{1}{2J + 1} \int_0^{2\pi} d\theta \sin\theta \, d_{00}^J (\theta) \cdot e^{-i\theta J_y} \tag{4}$$

The projected trial wave function is then

$$\Psi (\Delta_p, \Delta_n, \beta) = \hat{P}_J \hat{Q}_N(p) \hat{Q}_N(n) | \Phi > \tag{5}$$

with the pairing gap $\Delta_p$ (for protons), $\Delta_n$ (for neutrons) and deformation $\beta$ as variational parameters. The physical Hamiltonian in the laboratory system is taken to consist of a spherical shell-model part, which coincides with the Nilsson Hamiltonian at zero deformation, and an interaction part of the pairing-plus-quadrupole force

$$\hat{H} = \Sigma \epsilon_a C_a^+ C_a - 1/4 \Sigma G_\tau C_a^+ C_a^+ C_{\bar{b}} C_b - 1/4 \Sigma X_{\tau\tau'} (-)^\mu q_\mu^\tau (ac) q_\mu^{\tau'} (bd) C_a^+ C_b^+ C_d C_c \tag{6}$$

where $C_a^+$ is the creation operator of a shell-model state with energy $\epsilon_a$, and the strength constants for the pairing and the quadrupole forces are taken to be $G = 24/A$ Mev, $G_n = 17.41/A$ Mev and $X_{\tau\tau'} = 76\alpha_\tau \alpha_{\tau'} A^{-1.4}$ Mev with $\alpha_\tau = 1 \pm (N-Z)/(3A)$ for proton! (with the minus sign) and neutrons (with the

plus sign). The basis used includes N = 4 and 5 for protons and N = 5 and 6 for neutrons. To make up for the smaller basis used a quadrupole force reduction factor[12] is used and a small effective core moment of inertia $\theta_0$[9] is introduced by fitting the energy of the $2_1^+$ state (the smae value of $\theta_0$ is used for all states in the calculation). Trial wave functions $\Psi$ of equation (5) with good nucleon number and good angular momentum in the laboratory system are then used to calculate the expectation energies of the Hamiltonian (6). A variational search on the pairing gap A and the deformation $\beta$ is subsequently performed to determine the best wave functions for all the calculated $K^\pi = 0^+$ bands under considera- tion satisfying the condition

$$\delta < \Psi \left| \hat{H} \right| \Psi > \; = \; 0 \text{ for } \delta\Delta_n, \; \delta\Delta_p \text{ and } \delta\beta. \tag{7}$$

Since our projected wave functions have to reflect closely the detailed structure of near lying levels, we must make sure to have in hand a good intrinsic deformed model. Single-particle wave functions from Nillson[13], or Baranger and Kumar[12], for example, do not have good angular momenta. Therefore, without further modifications, empirically fitted parameter values based on these models may not be appropriate for our purpose. However, instead of performing a full scale, time consuming parameter search, it may be wise to take one of the well established models with the least possible change. In this work, we shall adopt the Nilson model with all the parameters having the same values as those given in ref. (13), except that the value of $\kappa$ is changed slightly by omiting the factor $\alpha_\tau$ in the $\ell \cdot$s and $\ell^2$ terms; the intrinsic Hamiltonian is given by

$$h = \hbar\omega(\beta)\,\alpha_\tau \left[ \; -\tfrac{1}{2}\Delta + \frac{r^2}{2} - \; \beta_2 Q_{20} \; - \; \beta_4 Q_{40} \; \right]$$

$$= -2k\hbar\,\omega_0 \left[ \; \ell \cdot s \; - \; \frac{\mu}{2} \; (\ell^2 - \; <\ell^2>_N) \; \right] \tag{8}$$

Here $\hbar\omega(\beta)$ is a function of $\beta$ and will be explained below. $\beta_2$ is the quadrupole deformation, $\beta_4$ the hexadecapole deformation, $Q_{20} = \gamma^2 Y_{20}, Q_{40} = \gamma^4 Y_{40}, \kappa = 0.0637, \mu = 0.6$ for protons and 0.42 for neutrons, $\hbar\omega_0 = 41 A^{-1/3}$ Mev. $<\ell^2>_N$ is the average of $\ell^2$ over one major shell which can be shown[14]) to be $N(N+3)/2$ with N the principle quantum number. The effect of mixing due to the $\Delta N = \pm 2$ terms is treated also as described in ref. (13). That is, this effect can be taken into account by a transformation which, as a result, replaces $\beta$ by $\beta' = (1 + \frac{0.95}{6}\beta)$, and $\hbar\omega_0$ by $\hbar\omega(\beta) = \hbar\omega_0(1 + \frac{1}{9}\epsilon^2)$ with $\epsilon = 0.95\beta(1 + \frac{0.95}{6}\beta)$.

Now, suppose that the Nilsson BCS wave function of equation (1) represents the intrinsic ground state of an axially symmetrical even-even nucleus, and its averaged total number of particles is given by equation (2). Then an excited two-quasiparticle state in the intrinsic system is given as (see for instance ref. (15)).

$$\left| kk' > \; = \; \alpha_k^+ \alpha_{k'}^+ \right| \phi > \; = \; \underset{\substack{j \neq k, k', \\ j > 0}}{\pi} \; (U_j \; + \; V_j b_j^+ b_{\bar{j}}^+) \; b_k^+ b_{k'}^+ \left| 0 > \right. \tag{9}$$

where $\alpha_k^+ = U_k b_k^+ - V_k b_{\bar{k}}$, and the particle number condition is given by

$$\underset{\substack{j \; \# k, k \\ j > 0}}{\Sigma} \; 2V_j^2 \; + \; 2 \; = \; N \tag{10}$$

This means that the distribution of particles in the excited state is in general not the same as that of the ground state system which satisfies equation (2). A special case of the two-quasiparticle excited state is when $k = \bar{k}$, which results an $K^\pi = 0^+$ excited state, we then have

$$|k\bar{k}\rangle = \alpha_k^+ \alpha_{\bar{k}}^+ |\Phi\rangle = \prod_{\substack{j \neq k \\ j > 0}} (U_j + V_j b_j^+ b_{\bar{j}}^+)(-V_k + U_k b_k^+ b_{\bar{k}}^+) |0\rangle \qquad (11)$$

with the particle number condition given by

$$\sum_{\substack{j \neq i \\ j > 0}} 2 V_j^2 + 2 U_i^2 = N. \qquad (12)$$

This mean that, to treat the state $|k\bar{k}\rangle$ is exactly the same as treating the ground state except for replacing $V_k$ by $U_k$ and $U_k$ by $-V_k$.

## III. NUMERICAL CALCULATION AND DISCUSSION

In performing practicle calculation for angular momentum and particle number projection, we introducing some numerical approximations. The integral operator (3) is reduced to a sum over two or four points [16]. A sum over two points ($\phi = 0^0, 180^0$) yields a wave function which does not contain components with $\Delta N = \pm 2, \pm 6, \pm 10$ etc. any more. An advantage of this approximation is that the wave functions remain real. If four points ($\phi = 45^0, 135^0, 225^0, 315^0$) are used, the components with $\Delta N = \pm 4, \pm 12, \pm 20$ etc. are removed in adition. This means that after a four-point projection the wave function has only admixtures with $\Delta N = \pm 8, \pm 16$, etc. which are supposed to be small.

For the angular momentum projection we use an interpolation technique. It can be shown that in the pairing + Quadrupole model the wave function overlap $n(\theta) = \langle \Phi | \hat{R}(\theta) | \Phi \rangle$ and the energy overlap $h(\theta) = \langle \Phi | \hat{H}\hat{R}(\theta) | \Phi \rangle$ can be expanded in the form [8]

$$\left.\begin{array}{c} n(\theta) \\ \\ h(\theta) \end{array}\right\} = A e^{-2\sin^2\theta} [ 1 + a_1 P_2 (\sin^2\theta) + a_2 P_2(\sin^2\theta) + a_3 P_3(\sin^2\theta) + . ] \qquad (13)$$

with the polynomials $p_n (\sin^2\theta)$ orthronormalized in the interval $0 \leqslant \sin^2\theta \leqslant 1 (0 \leqslant \theta \leqslant \pi/2)$. In the present work we calculate for five $\theta$-values and interpolate by equation (13). This procedure yields spectra which are numerically indentical with those resulting from an exact integration for angular momenta $J \leqslant 24$ [17].

In this manner, the expectation values of the Hamiltonian are varied with respect to $\Delta_p$ and $\Delta_n$ to determine the energies of the various J-states for the ground state band and the excited bands. To determine the stable deformation of the nucleus, variation of $\beta_2$ is also performed for the ground state band. The same value of $\beta_2$ is used for the excited bands, since for low-lying states the deformation is not expected to have sizable variations, especially for a good rotor like $^{168}$Er. For $\beta_4$, a value of 0.01, as given in ref. (13), is used throughout the whole calculation.

The calculation yields for the gound state band $\Delta_p = 0.82$ Mev, $\Delta_n = 0.651$ Mev, and $\beta_2 = 0.339$, all being in good accord with experiment. The value for the deformation $\beta_2$ is somewhat larger than the value of 0.3 used in other theoretical calculations [5,8] in which no projections of angular momentum are performed. In this connection, it is worth pointing out that our value ($\beta_2 = 0.339$) is in close agreement with the experimental value as extracted from B(E2; $2' \rightarrow 0^+$) value [19]. By examining our Nilsson single-particle energies in the ground state band of the intrinsic system at this value of deformation, it is found that there are five levels near the Fermi level, $\lambda_g$, of the ground state band. These are $h_{11/2, 7/2}, h_{9/2, 1/2}$ and $d_{3/2, 1/2}$ for Protons, and $i_{13/2, 7/2}$ and $p_{3/2, 1/2}$ for neutrons. However, when two-quasiparticles are excited, the particles arc redistributed over the whole spectrum of levels according to equation (12) and a new Fermi level, $\lambda_k$, emerges for each of the excited state bands. It is then found that there are only two excited states with $|\epsilon_k - \lambda_k|$ having a value of about 0.4 to 0.5 Mev. They are the $i_{13/2, 7/2}$ and the $p_{3/2, 1/2}$ of neutrons.
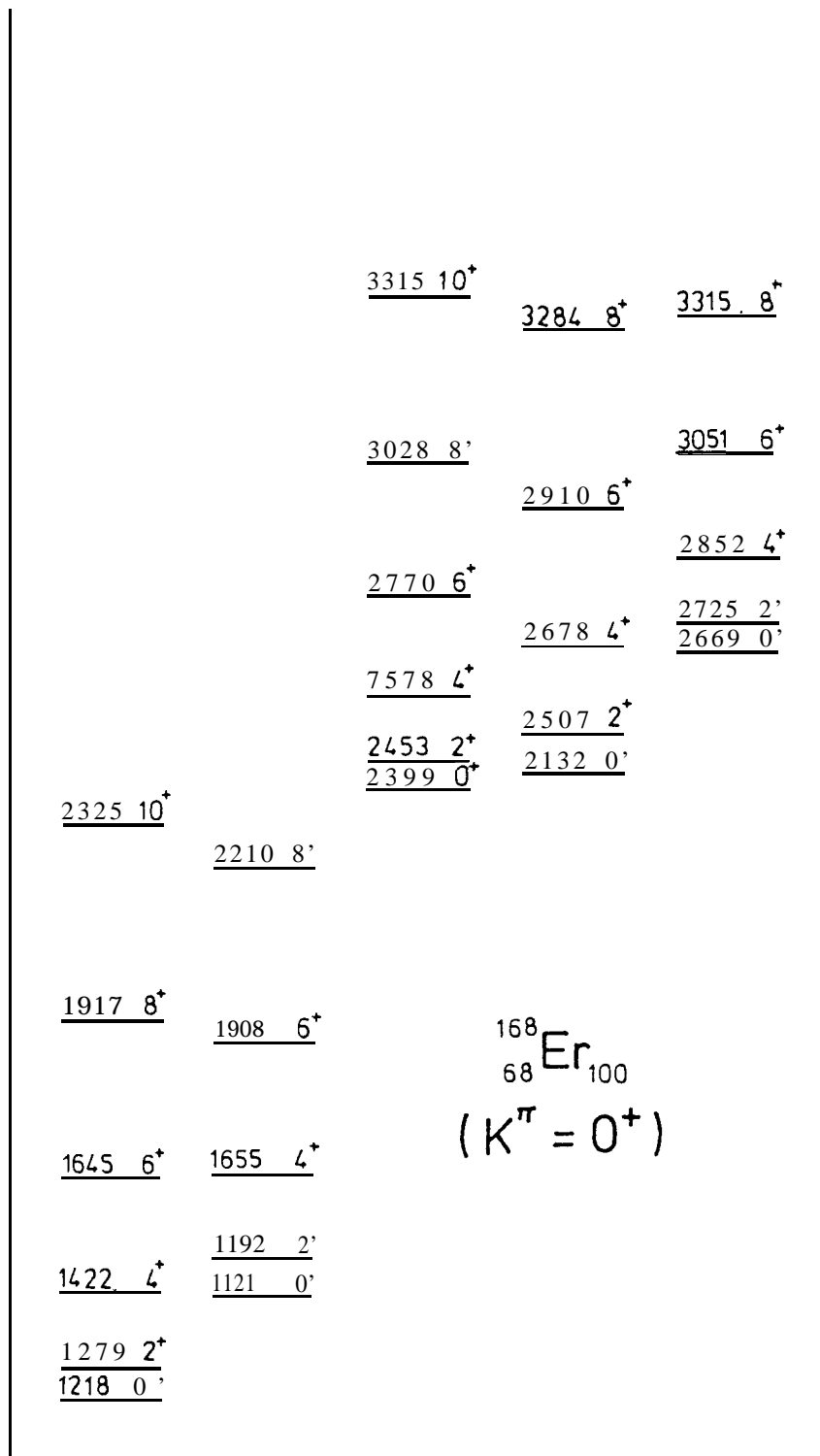
$\underline{3315\ 10^+}$

$\underline{3284\quad 8^+}$   $\underline{3315\ .\ 8^+}$

$\underline{3028\ \ 8'}$   $\underline{3051\quad 6^+}$

$\underline{2910\ 6^+}$

$\underline{2852\ \ 4^+}$

$\underline{2770\ 6^+}$

$\underline{2678\ \ 4^+}$   $\underline{2725\ \ 2'}$
$\underline{2669\ \ 0'}$

$\underline{7578\ \ 4^+}$

$\underline{2507\ \ 2^+}$

$\underline{2453\ \ 2^+}$   $\underline{2132\ \ 0'}$
$\underline{2399\ \ 0^+}$

$\underline{2325\ 10^+}$

$2210\ \ 8'$

$\mathbf{\underline{1917\ \ 8^+}}$

$\mathbf{\underline{1908\quad 6^+}}$

$^{168}_{68}\mathrm{Er}_{100}$

$(\ K^\pi = 0^+\ )$

$\underline{1645\ \ 6^+}$   $\underline{1655\quad 4^+}$

$\underline{1192\quad 2'}$
$\underline{1121\quad 0'}$

$\underline{1422.\ \ 4^+}$

$\underline{1279\ \ 2^+}$
$\underline{1218\ \ 0\ '}$

Fig. 1. Calculated energy spectra of the $K^\pi = 0^+$ theorectical excited bands in $^{168}$ Er.

Fig. 2. Comparison of the calculated energies with experiment for the ground state band and $K^\pi=0^+$ excited bands in $^{168}$Er.

All other states are about 1 Mev or more higher from their Fermi levels. Since the band head of an excited two-quasiparticle band can be written approximately as $E_k = 2[(\epsilon_k - \Delta_k)^2 + \Delta_k^2]^{\frac{1}{2}} + \eta_k$, where $\eta_k$ is a small correction term arising from the fact that the independent quasiparticle states in the-intrinsic system are not eigenstates in the laboratory system, we conclude- that there are only two low-lying $K^\pi = 0^+$ excited bands for this nucleus.

From the above mentioned five Nilsson single particle levels, we construct five $K^\pi = 0^+$ excited two-quasiparticle intrinsic states. Then we calculate the energy spectra of five $K^\pi = 0^+$ theoretical excited bands, from these intrinsic states and the results are shown in Fig. 1. We can see from this figure that there are only two low-lying theoretical bands in this nucleus which agrees completely with our theoretical analysis above as well as with experiments.

The calculated energy spectra of the ground state band and the two low-lying $K^\pi = 0^+$ excited bands are shown in Fig.2 along with the available experimental data. On comparison, it is suggestive to indentify the $K^\pi = 0_2^+$ band as the two-quasiparticle $K^\pi = 0^+$ excited band arising from the neutron $i_{13/2, 7/2}$ and the $K^\pi = 0_3^+$ band as that arising from the neutron $p_{3/2, 1/2}$. Under such a scheme, the calculation indicates that for the band heads we have $|\epsilon - \lambda_2| = 0.410$, $A$, $= 0.411$, $A$, $= 0.803$ Mev for the $K^\pi = 0_2^+$ band, and $|\epsilon - \lambda_3| = 0.51, \Delta_N = 0.369, \Delta_P = 0.803$ Mev for the $K^\pi = 0_3^+$ band. Accordingly, we have $\eta_2 = 65$ Kev and $\eta_3 = 170$ Kev. Since the parities of the quasiparticle states are different for the two theoretically assigned $K^\pi = 0^+$ excited bands, band mixing caused by a commonly used interaction, such as a Q-Q force, is rather insignificant. In fact, the interaction between the $i_{13/2, 7/2}$ and $p_{3/2, 1/2}$ two-quasiparticle Nilsson states can be calculated readily to be less than 10 Kev in the intrinsic system. Presumably, their interaction in the laboratory system will be of the same order of magnitude. This can only change the calculated energies at most by about one Kev, leaving our results essentially intact. The interactions with other states, which are all located at about 2.5 Mev or higher, are also exceedingly small. We conclude, therefore, that the two-quasiparticle $K^\pi = 0'$ bands must be rather pure two-quasiparticle bands.

From Fig.2, it is evident that the calculated spectra agree remarkable well with the experimental data. We make no attempts at calculating the B(E2) values to see if they show any collectivity, because the two calculated excited bands are theoretically constructed as pure twoquasiparticle bands, which is expected to show no collectivity.

## IV. CONCLUSION

In conclusion, we suggest that the two low-lying $K^\pi = 0^+$ excited bands in $^{168}$Er be identified as pure two-quasiparticle rotational bands, not as collective &vibration bands. Microscopic analysis based on the method of projection before variation does seem to be adequate for treating this problem. It is envisaged that there may be possibilities of extending the present treatment to more complicated structures, such as the y-bands, in heavy deformed nuclei.

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