Velocity-Anisotropy-Driven Bending Instability in the Galactic Stellar Disk

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Motivated by the results of a recent ultra-high-resolution simulation, which discovers the disk bending instability in a regime conventional thought to be stable, we analyze an idealized galactic stellar disk against such an instability, and obtain a reasonable agreement with the simulation results. The nature of this unstable bending perturbations differs from that predicted by Toomre, in that they tend to be of long wavelength. The success of our analysis relies fundamentally on the determination of two equations of states pertaining to two adiabatic invariants of the three-dimensional stellar motion in a disk. We find that the disk can be subject to the bending instability when β/Q is greater than unity, where β is the velocity anisotropy and the Toomre Q represents the ratio of disk kinetic energy to gravitational potential energy.

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I. Introduction

Anisotropy of stellar velocity dispersion may lead to bending of the galactic disk, in which the stars reside. It has been shown long ago by Toomre [1], with a heuristic analysis of an infinitely thin disk, that when the velocity anisotropy exceeds a certain threshold the instability of disk bending can occur. The threshold ratio $\sigma_z/\sigma\ h$ was shown to equal 0.3, where σ_h and σ_z represent the horizontal and vertical velocity dispersions, respectively, and the most unstable waves were also shown to be of short wavelength [1, 2].

Recent observations of several galaxies, including our own Milky Way [3, 4], indicate that the disk bending is a common phenomenon, not only in the outskirts of the galactic disk [5] but also in the main part of the disk [6, 7]. These observations have sparked revived interests in understanding how the stellar disk can become warped [8]. Recent works have even suggested that the bending of the stellar bar at the inner galaxy may produce the galactic bulge [9, 10]. In a series of works conducted in the past few years, various bendng mechanisms have been proposed. Aside from the ones resulting from the anisotropy of velocity dispersion, the misalignment of the disk rotational axis and halo axis is found to contribute to the disk bending [11, 12]. As far as the variations of the original Toomre's idea of bending instability are concerned, the inhomogeneous denisty distribution on the disk is also found to reinforce the disk bending [13, 14].

Most recently, an ultra-high-resolution simulation of the local galactic disk of uniform density demonstrates that the disk can undergo bending instability, even at a condition that is substantially below the threshold predicted by Toomre [15]. The unstable waves are of long horizontal wavelength. Though the instabilty in this regime appears to be relatively weak, a comparison of such a weakly unstable disk system with a stable disk system does clearly reveal striking difference in their long-term behaviors. The unstable disk becomes significantly thickened as compared with the stable disk. The disk thickening is likely due to the same quasi-linear relaxation processes as those leading to the formation of galactic bulge discovered several years ago.

This new simulation result prompts us to re-examine the theory of disk bending instability with an approach different from Toomre's non-rotating, infinitely-thin-disk model. We take into account the finite disk thickness and consider both vertical and horizontal dynamics consistently. In order to facilitate an eigen-value approach, we assume the disk slab to be in uniform rotation; the assumption retains the stabilizing epicyclic motion but ignores the swing-amplification. That is, our result gives a conservative estimate of the bending instability. In addition, we shall focus on the regime of long horizontal wavelength, where the waves are hydrodynamic in nature on the disk plane. In this regime, the bound orbits of particle vertical motion are rapidly oscillating across the finite thickness disk, thereby yielding a good adiabatically invariant action in the vertical direction. As will be shown later, the adiabatic invariants allow us to determine appropriate equations of state for this collisionless system, based on which our main results are obtained.

II. Mathematical fomulations

The self-gravitating uniform slab has a wellknown vertical density profile for isothermal particles:

$$\rho(z) = \rho_0 sech^2(z/z_0), \tag{1}$$

where ρ_0 is the central density and the vertical scale height $z_0 \equiv (T_z/2\pi G\rho_0)^{1/2}$. This equilibrium configuration remains approximately the same when the slab is in rotation, since the additional centrifugal force contributes only to the horizontal direction and this additional force is balanced by the horizotal gravity given by the matter interior to the rotating slab. Let the rotational frequency be . Without the rotation shear the particle epicyclic frequency is therefore $\kappa = 2$. On the other hand, the angular frequency of low energy particle vertical oscillation is $\sqrt{4\pi G\rho_0}$, but reduced to $\sqrt{2\pi G\rho_0}$ on the average. For all practical purposes, we have

$$\kappa^2 = 4 \ {}^2 < 2\pi G\rho_0, \tag{2}$$

or

$$\frac{2\pi G\rho_0}{\kappa^2} = \frac{\beta^2}{Q^2} \geq 1, \tag{3}$$

for any reasonable galaxy model, where Q is the Toomre Q and β is defined as the velocity anisotropy:

$$\beta^2 \equiv (\sigma_h/\sigma_z)^2,$$

which is greater than unity. The purpose of this work is to determine at what parameter set $(2\pi G\rho_0/\kappa^2, \beta)$ the bending instability can become marginally unstable.

The small-amplitude perturbations consist of the velocity perturbation $\delta\mathbf{v}$, density perturbation $\delta\rho$ and potential perturbation $\delta\phi$. The latter two quantities are related by the Poisson equation:

$$(\frac{d^2}{dz^2} - k^2)\delta\phi = 4\pi G\delta\rho \tag{4}$$

where we have let the perturbed quantities be proportional to $\exp[i\mathsf{k}\cdot\mathsf{r} - iwt]$ and both **k** and **r** are vectors in the horizontal direction. Note that the z dependence of the perturbed quantities must be solved consistently. In addition, the continuity equation is

$$-iw\delta\rho + \frac{d}{dz}(\rho\delta v_z) + i\rho\mathbf{k}\cdot\delta\mathbf{v}_{?} = 0. \tag{5}$$

Finally the three components of velocity perturbation satisfy

$$-iw\delta v_x - 2 \quad \delta v_y = \delta f_x, \tag{6}$$

$$-iw\delta v_y - 2 \quad \delta v_x = \delta f_y \tag{7}$$

and

$$-iw\delta v_z = \delta f_z \tag{8}$$

where δf_x, δf_y and δf_z are the perturbed force in the horizontal and z directions respectively. Since we are dealing with collisionless particles, this perturbed force should be different from that of the collisional gas in an essential way. We will attempt to determine this perturbed force in the remaining part of this section.

The essential difference between collisionless particles and collisional gases arises from the equations of state. While well-understood equations of state can be prescribed to the collisional gases, there is no universal equation of state for collisionless particles. For the latter, equations of state may be constructed only after the single-particle dynamics is relatively well-understood and an entropy-like quantity is also available. For the collisional gas, the different type of equations of state imply nothing more than different choices of entropy. As the entropy is a conserved quantity, the different choice of entropy simply correspond to the different physical conditions, under which a certain thermodynamics quantity remains a constant of time. Likewise, for collisionless particles, we would also like to identify the entropy-like conserved quantity, with which an equation of state can be constructed. This can be possible for problems with slowly-varying time-dependent forces (which is appropriate for our purposes because we are looking for the marginal instability). The particle energy is generally not a conserved quantity and instead the adiabatic invariants, or the actions, can be identified as the conserved entropy-like quantities.

For the problem at hand, the particle oscillations across the disk potential are rapid, and the action pertaining to this motion should be a good adiabatic invariant. In an equilibrium configuration, such oscillations proceed in the vertical direction. However, as the disk plane bends, the equi-potential contours should also be bent and therefore the action in question must be associated with the motion perpendicular to the distorted plane. The above consideration is only a half of the whole picture. The other half pertains to the way by which the disk plane bends. Since the perturbed vertical displacement of the bending instability is an even function of z, in contrast to an odd function of z for the density-wave perturbation, it is expected that to the leading order, the width of the potential trough remains unchanged. Combining these two factors together, we arrive at the conclusion that the particle energy perpendicular to the potential trough is an adiabatic invariant. This is because the conserved action is proportional to $\sigma_z z_0$ which equals $\sigma\Delta$ with Δ being the width of the potential trough and approximately equal to z_0; it thus follows that the velocity dispersion across the potential trough σ is approximately the same as the original σ_z. That is, on the slightly distorted disk plane, the particle anisotropic temperature tensor can be written as

$$\mathsf{T} = T_z \hat{z}'\hat{z}' + T_h(\mathsf{I} - \hat{z}'\hat{z}') \tag{9}$$

where

$$\hat{z}^0(\mathsf{r}) \equiv \hat{z} - \delta\theta(\mathsf{r}, z, t)\hat{k}, \tag{10}$$

$T_z \equiv \sigma_z^2$, $T_h \equiv \sigma_h^2$ and **I** is an identity tensor. The unit vector $\hat{z}'$ corresponds to the local direction perpendicular to the instantaneous potential trough, and the $\delta\theta$ measures the angle of local bending in the direction of the bending wave **k**. In fact, since the potential distortion arises from the vertical displacement, the angle of local bending can further be expressed as

$$\delta\theta\hat{k} = \nabla\xi_z = i\xi_z\mathsf{k} \tag{11}$$

where ξ_z is a vertical displacement of the potential contours. In other words, it satisfies

$$\delta\phi + \xi_z \frac{d\phi(z)}{dz} = 0, \tag{12}$$

which is different from the particle equation of motion. The quantity T_z in Eq. (9) is identified to be a conserved quantity associated with the motion across the disk gravitational potential.

How about the other quantity T_h, which corresponds to the average kinetic energy on the distorted plane? For a potential trough whose shape does not change but whose local orientation varies slowly in time and space, the average total kinetic energy should be conserved since the average energy can not change when the shape of the potential trough does not vary as there is no PdV work. Thus, the particle specific internal energy $tr[\mathsf{T}]/3 = (T_z + 2T_h)/3 = const.$, and it follows that $T_h \approx const.$, i.e., another conserved quantity.

The key results that have been obtained so far for the later use in the stability analysis are Eqs. (9), (10), (11) and (12), with T_z and T_h being the constants of motion. These results are new and markedly different from those obtained by previous works. They are obtained based on the assumptions that the spatial variation alone the disk plane is gentle, the temporal variation slow and also the vertical oscillation is faster than the epicyclic motion, i.e., Eqs. (2) and (3). These assumptions lead to two adiabatic invariants pertinent to two different types of motion as the eigenvalues of the anisotropic temperature tensor. Therefore, the self-consistent bending instability to be examined is limited to those long-wavelength perturbations on the galactic plane.

III. Stability analysis

The right-hand sides of Eqs. (6), (7) and (8) are

$$\delta \mathsf{f}_{?} = -i\mathsf{k}\left[\frac{T_h\delta\rho}{\rho} + \delta\phi\right] - i\mathsf{k}\left[(T_h - T_z)\frac{d\xi_z}{dz}\right] \tag{13}$$

and

$$\delta f_z = -\frac{d}{dz}\left[\frac{T_z\delta\rho}{\rho} + \delta\phi\right] - \left[k^2(T_z - T_h)\xi_z\right], \tag{14}$$

where the second squared brackets on the right of Eqs. (13) and (14) account for the force contributions from the tilt of the distorted plane, i.e., proportional to $-(T_z - T_h)\nabla(\hat{z}'\hat{z}')$. It is due to these forces that give rise to the tension of the disk plane. Finally, we may use Eq. (12) to express the vitual displacement ξ_z, which distorts the potential contours in terms of $\delta\phi$ as

$$\xi_z = -\frac{\delta\phi}{d\phi(z)/dz} \tag{15}$$

to complete the derivation of the anisotropic perturbed stress in terms of $\delta\rho$ and $\delta\phi$.

From the continuity equation, Eq. (5) and the z component of the momentum equation, Eq. (8), we can derive the following equation:

$$\begin{aligned} g'' + [2sech^2(z) + (\frac{w^2}{T_z} - 1)]g &= \frac{w}{T_z}(\mathsf{k}\cdot\delta\mathsf{v}_h)sech(z) + \frac{w^2}{T_z^2}sech(z)\delta\phi \\ &-\frac{k^2}{T_z}(T_h - T_z)[\frac{\delta\phi'}{2\sinh(z)} - sech(z)(1 + \frac{1}{2\sinh^2(z)})\delta\phi], \end{aligned} \tag{16}$$

where $g \equiv sech(z)\delta q$, with the non-adiabatic density perturbation $\delta q \equiv (\delta\rho/\rho) + (\delta\phi/T_z)$. Above, the length has been normalized to z_0 and the time normalized to $\sqrt{2\pi G\rho_0}$; in addition, the prime stands for the z derivative.

Equation (16) has three unknowns, $g, \delta\phi$ and $\mathbf{k}\cdot\delta\mathsf{v}_h$. The last quantity can be solved for from Eqs. (6) and (7) in terms of the other two, so that Eq. (16) can be re-expressed as

$$\begin{aligned} &g'' + [2sech^2(z) + (\frac{w^2}{T_z} - 1) - \frac{w^2k^2(T_h/T_z)}{w^2 - 4\ \ ^2}]g \\ &= -k^2(\frac{T_h}{T_z} - 1)[1 + \frac{w^2}{w^2 - 4\ \ ^2}](\frac{\psi}{2\tanh_z})' + [\frac{w^2}{T_z} - \frac{4k^2\ \ ^2}{w^2 - 4\ \ ^2}(\frac{T_h}{T_z} - 1)]\psi sech(z), \end{aligned} \tag{17}$$

where $\psi \equiv \delta\phi/T_z$. Moreover, the quantities g and ψ are related by the Poisson equation:

$$\psi'' + [2sech^2(z) + k^2]\psi = 2gsech(z). \tag{18}$$

Equations (17) and (18) are coupled second-order differential equations for the three-dimensional dynamics, and can be solved as an eigenvalue problem, with w being the eigenvalue. As the

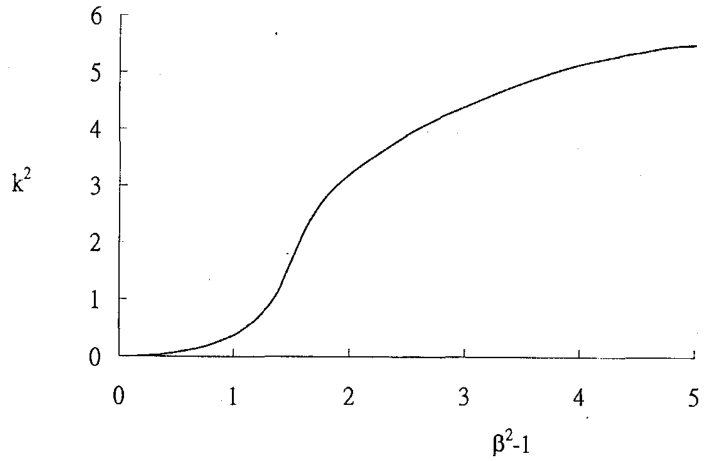

FIG. 1. The stability boundary in the $(\beta^2 - 1, k^2)$ phase space. Toward the small k^2 side is where the unstable regime lies. This stability diagram holds when Eq. (3) is satisfied.

marginal instability is of primary interest to us, we may set $w = 0$ and look for the relation between k^2 and the anisotropy $\beta \equiv T_h/T_z$. Equation (17) is simplified to be

$$g'' + [2sech^2(z) - 1]g = (\beta^2 - 1)k^2 sech(z)[\psi - (\frac{\psi}{2\tanh z})']. \tag{19}$$

The two equations, Eqs. (18) and (19), are solved by a shooting method, and the stability boundary is given in Fig. 1. The marginal unstable perturbations disappear when $\beta < 1$. Note that the instability favors long waves, and the stability boundary does not depend explicitly on the Toomre Q, but solely on the dispersion anisotropy. Clearly this result can not be the most general one, and a general result should have a Q dependence. In fact, this result is derived under an implicit assumption adopted to obtain the equations of state. We shall come back to discuss this issue further.

IV. Discussions and conclusions

In this work, we focus on the regime where the perturbations are of long wave, i.e., hydrodynamics, and the vertical oscillations of particle orbits are faster than the epicyclic motion. The effective equations of state are derived based on the identification of two entropy-like conserved quantities. The instability obtained in this analysis depends sensitively on these equations of state, which give rise to an anisotropic stess tensor, thereby yielding a negative tension on the slab to drive the instability.

Due to this critical dependence on the equations of state, it is worthwhile to examine the assumption of fast vertical oscillation in more details. Since the time-scale given by the instability growth rate can be chosen to be arbitrarily small, there remain basically two relevant time scales in this problem: the vertrical oscillation frequency $\sqrt{2\pi G\rho_0}$ and the epicyclic frequency 2 . The latter enters the problem only when a finite w perturbation is considered, and it drops out for the marginally unstable perturbations. In fact, Eqs. (17) and (19) are valid only up to the leading order of a small $\kappa^2/2\pi G\rho_0$ (c.f., Eq. (3)). The assumption of such a small quantity implicitly enters our analysis through the derivation of the equations of state, Eqs. (9), (10), (11) and (12). In particular, Eq. (12) assumes that the force potential entirely arises from the self-gravity, whereas that contributed by the epicyclic motion is negligible. When the latter is taken into account, the virtual displacement ξ_z that distorts the potential contours will not be related to the gravitational potential $\delta\phi$ by Eq. (12). The correction to Eq. (12) will be of order $\kappa^2/2\pi G\rho_0$, and this correction tends to suppress the instability. That is, the inequality, Eq. (3), is crucial for our result to be valid. We also notice that due to this inequality, Eqs. (18) and (19) are independent of rotation, and the stability boundary is the same as that with no rotation, though the growth rate will depend on rotation, as Eq. (17) indicates. The weak instability discovered by the ultra-high-resolution simulations mentioned in the Sec. (I) has a $Q = 1.3$ and $\beta^2 = 2$, yielding the ratio given in Eq. (3) equal to $1.2 > 1$. These parameters are marginal for Eq. (3) to be valid. We also note that in the same simulations, when $Q = 1.7$ and $\beta^2 = 2$, they yield the supposedly large parameter $(\beta/Q)^2 \approx 0.7 < 1$, and the instability is totally suppressed.

In sum, we have developed an analytical theory that addresses the linear instability of bending wave in the context of a self-gravitating, rotating uniform slab. The instability occurs in a new regime which is conventionally thought to be a stable regime [1, 2]. Our result for the stability boundary appears to agree with what has recently been found by numerical simulations [15]. There is, however, still a room for this theory to be improved, in that we shall be able to relax Eq. (3) and pin down the exact transition between the stable and unstable bending waves.

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