A Theory for Multifilamentary Pattern Formation in Semiconductors

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A theory for multifilamentary pattern formation in semiconductors with an S-shaped negative conductivity (SNDC) is presented. The theory is based on the basic equations in electrodynamics and the impact ionization mechanism with double impurity trap levels, which is widely accepted as the mechanism for SNDC. No extra physical quantities and/or parameters are introduced in the theory. The steady-state of the dynamical equations is a one-dimensional reaction-diffusion type equation, which provides stable multifilamentary pattern structures. The stability analysis for the steady-state N-filament (N=2, 3, and 4) solution is given.

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I. Introduction

It is well known that a semiconductor with an S-shaped current-density-field (J-E) characteristic (Fig. 1(a)) will exhibit a hysteresis loop in the current-voltage (I-V) curve (Fig. 1(b)), when the system is connected in series with a dc bias voltage V and a light load resistance R. The slope of the load line (LL in Fig. 1(a)) does not change if the load resistance R is a constant, but the position of the horizontal (vertical) intercept varies as the applied field (i.e., the bias voltage) varies. The holding voltage $V_h$ and the threshold voltage $V_{th}$ are, respectively, the bias voltage for which the load line passes through point Q or P in Fig. 1(a); in between P and Q the load line intercepts the J-E curve in three points. Outside P and Q, the load line intercepts the J-E curve in only one point. The J-E curve from P to Q has a negative differential conductivity (NDC) $dJ/dE < 0$, and the system is unstable when it is operating in the regime of NDC [1]. The system is therefore in a bistable state in the portion of AP or QB of the J-E curve, where the J-E curve intersects the load line three times, but only two of them are stable. The corresponding I-V curve (Fig. 1(b)) reflects the bistable nature which results in a hysteresis loop. When the bias voltage increases slowly from a low voltage and reaches the threshold voltage $V_{th}$, the system will jump abruptly from the low-current-density state P to the high-current-density state B (not along the path P to Q). In the reverse direction of decreasing the bias voltage slowly, the current changes gradually at $V_{th}$, but when the voltage reaches $V_h$ the system will drop abruptly from the high-current-density state Q to the low-current-density state A, thus forming a hysteresis loop.
FIG. 1. (a) $J-E$ characteristic for an SNDC semiconductor; and (b) the corresponding $I-V$ curve.

FIG. 2. Measured $I-V$ curve for low-temperature $n$-GaAs, reported by Mayer et al. in Ref. 2. The graph is reproduced from Ref. 2.

However the above picture is valid only when the system is always in a homogeneous steady state. In some systems the hysteresis loop may have more complicated structures if spatially inhomogeneous structures develop. For example, in low temperature $n$-GaAs [2] the measured hysteresis $I-V$ curve has a more complicated shape as shown in Fig. 2. The zigzag (consecutive small steps) in the bistable region of the $I-V$ curve signals the creation (or destruction) of a current
Theoretical explanation of the above multifilament observation is, so far, not satisfactory. One of the popular models is the one-dimensional reaction-diffusion (RD) equation known as the Schrödinger-type model [3], which has steady-state multifilamentary solutions. But it is known that these steady-state solutions are unstable [4], thus it is not a satisfactory model. To the authors’ knowledge, there have been two theories which tried to resolve the problem of stability of current filaments in bistable semiconductor systems [5, 6]. Both theories used the one-dimensional RD equation we have just mentioned, but with an additional constrain to the equation. Kardell et al. [5] introduced an additional variable to the RD equation, and thus becomes a two-variable model which can have stable multifilamentary solutions. But this model requires non-negligible hoppings (or tunnelings) of electrons between neighboring impurity sites, which seems not to be a practical assumption for an impurity concentration around $10^{15}/\text{cm}^3$. Recently Alekseev et al. [6] studied the RD equation with global coupling by external circuit. They found that in the strong coupling (large load resistance $R$) case the unstable mode may be stabilized. But in the weak coupling (small $R$) or no coupling ($R = 0$) case, all stationary non-uniform solutions are still unstable. Therefore their theory cannot explain the experimental result of Mayer et al. [2] which was performed in the weak coupling (small $R$) limit.

In this paper we propose a simple and realistic model which can support stable steady-state multifilamentary solutions in weak and no coupling ($R = 0$) limit. The theory is based on the basic equations in electrodynamics and the widely accepted two-trap-level model for SNDC materials with no extra assumptions. Electrons trapped in the impurity sites can gain or lose energy to change their energy level (ground, or first-excited, or the conduction level), but they cannot hop from one site to another. Our model has steady-state multifilamentary solutions which are proven to be stable. In the steady state, our model is the same as the RD equation. But in the time-dependent case (as unstable modes must be time-dependent) our model introduces an additional spatial dependent differential equation to couple the RD equation. Thus it imposes an additional constraint on the RD equation which is an important factor to stabilize the unstable steady-state multifilamentary solutions as previous theories suggested [5, 6]. It is worthwhile to point out that the constraint comes naturally from the basic equations of electrodynamics and we do not introduce it to our model artificially. For completeness, we do stability analysis for the steady-state solutions of our model at the end of the paper. The result confirms that our solutions are stable.

II. The model

A simple two-level model (the ground state and the first excited state of the impurity) is adequate to explain the origin of an S-shaped NDC [7, 8]. In this model there are three energy levels for an electron: the ground state and the first excited state of the donor impurities, in addition to the conduction level. We adapt this model in the following analysis. The electron densities in these three levels are governed by the generation-recombination (g-r) processes which give rise to the rate equations for the electron densities:

$$\frac{dn}{dt} = g_1(n, n_0, n_1, E),$$

(1)
\[ \frac{dn_0}{dt} = g_2(n, n_0, n_1, E), \quad (2) \]
\[ \frac{dn_1}{dt} = -g_1(n, n_0, n_1, E) - g_2(n, n_0, n_1, E), \quad (3) \]

where \( n, n_0, \) and \( n_1 \) are, respectively, the electron density of the conduction band, the ground state and the first excited state of the donors. The functions \( g_1 \) and \( g_2 \) are rate functions in the g-r processes and can be written as [7]

\[ g_1 = X_1^s n_1 - T_1^s(N_D - n_0 - n_1)n + X_1^s(E)n_1n + X_1(E)n_0n, \quad (4) \]
\[ g_2 = -X_0^s n_0 + T_0^s n_1 - X_1(E)n_0n, \quad (5) \]

where \( N_D \) is the effective total donor concentration. The physical meaning of the rate coefficients \( X_1^s, T_1^s \) etc. are shown in Fig. 3. Among these rate coefficients, \( X_1 \) and \( X_1^s \) are electric field dependent [7] which can be written as (Shockley lucky electron model) [9, 10]

\[ X_1(E) = C_0 e^{\varepsilon_0/(e\ell E)}, \quad (6) \]
\[ X_1^s(E) = C_1 e^{\varepsilon_1/(e\ell E)}, \quad (7) \]

where \( C_0 \) and \( C_1 \) are constants, \( \ell \) is the electron mean free path, and \( \varepsilon_0 \) and \( \varepsilon_1 \) are, respectively, the binding energy of the ground state and the first excited state of the donor impurity. In the homogeneous steady state \( g_1 = g_2 = 0 \), the condition of charge neutrality \( \rho(n, E) = eN_D - e(n + n_0 + n_1) = 0 \) will result in a multi-valued conduction electron density \( n = n(E) \) in a certain range of the electric field \( E \), which in turn produces an S-shaped \( J-E \) characteristic.

Now we construct the model which is able to describe the formation of multiple current filaments in semiconductors with an S-shaped NDC. All our starting equations are basic equations in electrodynamics:

\[ \frac{\partial n}{\partial t} - \frac{1}{e} \nabla \cdot J = g_1(n, n_0, n_1, E), \quad (8) \]

\[ \text{Fig. 3. Impact ionization parameters in the two-trap-level model.} \]
\[ J = e\mu n E + eD\nabla n, \]  
(9)

\[ \varepsilon \nabla \cdot E = \rho(r, E) \equiv eN_D - e[n(r, t) + n_0(r, t) + n_1(r, t)], \]  
(10)

\[ \nabla \times H = \frac{\varepsilon}{c} \frac{\partial E}{\partial t} + J \equiv J_0. \]  
(11)

Equation (8) is the equation of continuity (conservation of charge), where \( J \) is the conduction current density. \( J \) is the sum of the drift and the diffusion current densities (cf. Eq. (9)), where \( \mu \) and \( D \) are the electron mobility and the diffusion constant, respectively. Equation (10) is the Gauss’s law and Eq. (11) is the Ampere’s law that the total current density \( J_0 \) consists of the conduction and the displacement current densities, with \( \varepsilon \) the dielectric constant of the sample.

We consider a rectangular sample in the region \(-L_x \leq x \leq L_x, -L_y \leq y \leq L_y, \) and \(-L_z \leq z \leq L_z\). An external dc bias voltage \( V \) is applied in the \( z \)-direction, which is connected in series with a light load resistant \( R \) such that the system will be operating in the bistable state for a certain range of \( V \) (cf. Fig. 1(a)). To study filament formation we have to consider the conduction electron density \( n \) and the electric field \( E \) to be dependent on \( r \). It is a good approximation that there is no spatial \( z \)-dependence and we may write \( n = n(x, t) \) and \( E = E_x(x, t) \hat{\mathbf{x}} + E_z(t) \hat{\mathbf{z}} \), if we consider the case \( L_y \ll L_x \) thus \( y \)-dependence can be neglected. As in many applications [7], we consider the case that the relaxation of the g-r processes is much faster than the dielectric relaxation, and we can approximate \( g_1 \approx 0 \) and \( g_2 \approx 0 \). This is equivalent to say that the S-shaped \( J-E \) characteristic is an intrinsic property of the sample. The charge density \( \rho = \rho(n, E) = eN_D - e(n + n_0 + n_1) \) can then be obtained by eliminating \( n_0 \) and \( n_1 \) via Eqs. (2) and (3). The dynamical equatuins become

\[ \frac{\partial n}{\partial t} = \frac{\partial}{\partial x} \left( n\mu E_x + D \frac{\partial n}{\partial x} \right), \]  
(12)

\[ \frac{\varepsilon}{\partial x} E_x = \rho(n, E), \]  
(13)

\[ \frac{\partial E_z}{\partial t} = \frac{V}{AR} \left( \frac{L_z}{AR} + \frac{e\mu}{2L_x} \int_{L_x}^{L_x} n dx \right) E_z, \]  
(14)

where \( A = 4L_xL_y \) is the cross sectional area of the sample. Because \( E_x \) is an induced field and we expect that \( |E_x| \ll |E_z| \), therefore \( \rho(n, E) \approx \rho(n, E_z) \). The variable \( E_x \) can then be eliminated by Eq. (13), and we obtain the following dynamical equations:

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \mu f(u, E_z) + \frac{\partial u}{\partial x} \left( \mu \int_{L_x}^{L_x} f(u, E_z) dx \right) + D \frac{\partial u}{\partial x} \]  
(15)

\[ \frac{\partial E_z}{\partial t} = \frac{V}{eAR} \left( \frac{L_z}{eAR} + \frac{e\mu}{2eL_x} \int_{L_x}^{L_x} e^u dx \right) E_z, \]  
(16)
where we have defined $u \equiv \ln n$ and $f(u, E_z) \equiv \rho(e^u, E_z)/\epsilon$. The parameter values used in obtaining $\rho(n, E_z)$ are tabulated in Table I.

### III. Steady state solutions

We are looking for the steady-state solutions of Eqs. (15) and (16). We consider the extreme light load limiting case that $R \to 0$, as most of the experimental observations of the multifilamentary pattern were done in this limit. Equation (16) then tells us that $\partial E_z/\partial t = 0$ and $E_z = V/L_z$ is a constant for a dc bias $V$. Thus we may consider $E_z$ is the control parameter in Eq. (15). Note that the quantity in the parentheses of the right hand side of Eq. (15) is zero in the steady state. This can be understood as follows. The quantity is proportional to $J_x$ which is the $x$-component of the conduction current density (cf. Eq. (9)). Because the circuit is open in the $x$-direction, $J_x = 0$ at the boundaries $x = \pm L_x$ for all $t$. In the steady state $\partial n/\partial t = 0$, and therefore $\partial J_x/\partial x = 0$ by Eq. (8) ($g_1 = 0$). Thus $J_x = 0$ everywhere.

The equation we have to solve in the steady state is therefore the following:

$$D \frac{\partial^2 u}{\partial x^2} + \mu f(u, E_z) = 0. \tag{17}$$

In this equation $E_z$ is considered as a control parameter. Before solving this equation, it is helpful to study the functional behavior of $f(u, E_z)$ vs. $u$ for a given $E_z$. We find that the characteristic of the function $f(u, E_z)$ vs. $u$ depends on the value of $E_z$. In Fig. 4(a) we show that when $E_h < E_z < E_{th}$, $f(u) = 0$ has three roots, where $E_h$ ($E_{th}$) is the electric field corresponding to the bias voltage $V_h$ ($V_{th}$) shown in Fig. 1(b). In other range of $E_z$, $f(u) = 0$ has only one root.

### TABLE I. Typical material parameter values corresponding to n-GaAs at 4.2±K for the two-level generation-recombination mechanism [7].

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
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<tbody>
<tr>
<td>$T_1^n$</td>
<td>$10^6$ cm$^3$ s$^{-1}$</td>
</tr>
<tr>
<td>$T_2^n$</td>
<td>$10^6$ s$^{-1}$</td>
</tr>
<tr>
<td>$X^n_1$</td>
<td>$10^4$ s$^{-1}$</td>
</tr>
<tr>
<td>$X_1$</td>
<td>$2 \times 10^7$ s$^{-1}$</td>
</tr>
<tr>
<td>$X^n_2$</td>
<td>$5 \times 10^8 \exp(-6/E)$ cm$^3$ s$^{-1}$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$10^6 \exp(-1.5/E)$ cm$^3$ s$^{-1}$</td>
</tr>
<tr>
<td>$N_D$</td>
<td>$10^{15}$ cm$^3$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$10^4$ cm$^2$ V$^{-1}$ s$^{-1}$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$8.85 \times 10^{10}$ F cm$^{-1}$</td>
</tr>
<tr>
<td>$L_x$</td>
<td>$1.5 \times 10^{1.2}$ cm</td>
</tr>
</tbody>
</table>
Therefore if we define a function \( \phi(u, E_z) \) as \( (u_0 \) is an arbitrary value of \( u ) \)

\[
\phi(u, E_z) = \mu \int_{u_0}^{u} f(u^0, E_z) du^0,
\]

(18)

then the characteristic of \( \phi(u, E_z) \) vs. \( u \) has the form as shown in Fig. 4(b). It is important to note that the function \( \phi(u, E_z) \) vs. \( u \) has a local minimum when the electric field \( E_z \) is in the range \( E_h < E_z < E_{th} \), i.e., it looks like a potential well. When \( \phi(u) \) has the shape of a potential well, Eq. (17) can be solved in a rather easy way [11]. Because the formation of current filaments occurs only in the field range \( E_h < E_z < E_{th} \), it is sufficient for us to consider the case that \( \phi(u) \) has a local minimum.

Equation (17) can be written as [11]

\[
\frac{1}{2} D \left( \frac{du}{dx} \right)^2 + \phi(u, E_z) = \varepsilon,
\]

(19)

where \( \varepsilon \) is an integrating constant. To solve Eq. (19), we may interpret this equation in terms of the terminology in mechanics. If we imagine \( x \) as “time”; \( D, u, \) and \( \phi \), respectively, as the “mass”, “displacement”, and “potential energy” of the system, then \( \varepsilon \) plays the role of the “total energy”. From Fig. 4(b), we see that when \( \varepsilon \) lies in a certain range \( \varepsilon_{\min}(E_z) < \varepsilon < \varepsilon_{\max}(E_z) \), then there is a bounded solution in which the “motion” of \( u \) is confined in the region of the potential well \( (u_1(E_z) \leq u \leq u_2(E_z)) \). This is the solution we are looking for. In this solution \( u \) (and

FIG. 4. Plot of the function \( f(u, E_z) \) vs. \( u \) (a), and the function \( \phi(u, E_z) \) vs. \( u \) (b), for \( E_z \) in the range \( E_h < E_z < E_{th} \).
therefore the electron density $n$ varies with $x$ and thus current filaments are formed. Note that for $u$ outside the potential well, the only possible solution is $\varepsilon = \phi(u)$ which implies $du/dx = 0$, i.e., a homogeneous state. Otherwise the “motion” of $u$ will be unbounded, which is unphysical because in a system the electron density should be positive and finite with values within a certain region only. Therefore it is sufficient for us to consider the case that $\varepsilon_{\min}(E_z) < \varepsilon < \varepsilon_{\max}(E_z)$. In this range of $\varepsilon$, Eq. (19) can be numerically solved to give

$$x(u) = \pm \sqrt{D/2} \int_{u_1}^{u} \frac{du}{\sqrt{\varepsilon - \phi(u, E_z)}}.$$  \hspace{1cm} (20)

where the upper limit of the integration $u \leq u_2$.

However it is more instructive to solve Eq. (19) in the following way. We note that if $\varepsilon_{\min} < \varepsilon < \varepsilon_{\max}$, then the graph $du/dx$ vs. $u$ is a closed curve for $u_1 \leq u \leq u_2$, as shown in Fig. 5. It is reasonable to use the periodic boundary condition for $u(x)$ (with period $2L_z$), thus $u(-L_z) = u(L_z)$ and $du/dx(-L_z) = du/dx(L_z)$. Then we start from $x = -L_z$, which corresponds to some point in Fig. 5, and increase $x$ until $x = L_z$. The corresponding point (for each $x$) in Fig. 5 will move along the closed curve $du/dx$ vs. $u$, and returns to the starting point when $x = L_z$, because $u(L_z) = u(-L_z)$ and also $du/dx(L_z) = du/dx(-L_z)$. However it is possible that the corresponding point in Fig. 5 moves around the closed curve $N$ times when $x$ increases from $x = -L_z$ to $x = L_z$, where $N$ is some integer. Therefore we have a “quantization rule” as in the old theory of quantum mechanics:

$$\frac{2L_z}{N} = \frac{D}{\hbar} \int \frac{dq}{p}.$$  \hspace{1cm} (21)

where the “coordinate” $q$ is $u$ and the “momentum” $p$ is $D(du/dx)$. The integral is over one period in the $q$-space. There are $N$ periods as $x$ increases from $-L_z$ to $L_z$. The explicit expression for Eq. (21) is

$$\frac{L_z}{N} = \sqrt{\frac{D}{2} \int_{u_1(E_z)}^{u_2(E_z)} \frac{du}{\sqrt{\varepsilon - \phi(u, E_z)}}}.$$  \hspace{1cm} (22)

This equation implicitly suggests that the “energy” $\varepsilon$ depends on the number of period $N$, which may be denoted as $\varepsilon_N$. Thus when the number of period increases from $N$ to $N + 1$, the corresponding “energy” goes from $\varepsilon_N$ to $\varepsilon_{N+1}$, i.e., $\varepsilon$ is quantized as in the old theory of quantum mechanics. We note that $\varepsilon_N$ is a function of $E_z$ and denote it as $\varepsilon_N(E_z)$. With the aid of Eq. (22), the electron density $n$ as a function of $x$ is plotted in Fig. 6 for the cases of $N = 2$ and $3$.

The physical significance of the “quantization rule” imposed by Eq. (21) can be understood as follows. For a given bias voltage $V$ (then the field $E_z = V/L_z$), there are only several integers $N_i$ (say $i = 1, 2, \cdots, p$) which satisfy Eq. (21). This means that for this $V$ the system can only sustain $N_i$-filament state ($i = 1, 2, \cdots, p$). The conditions that $N_i$-filament may exit for a given $V$, will affect the structure of the hysteresis loop in the $I$-$V$ curve of the system. This is the topic we will discuss in the next section.
IV. Stepped hysteresis loop

In this Section we describe how the small steps in the hysteresis loop can be formed. We denote the integrated value of the integral in the right hand side of Eq. (22) as $S(\varepsilon, E_z)$:

$$S(\varepsilon, E_z) = \sqrt{\frac{D}{2}} \int_{\alpha_1(\varepsilon, E_z)}^{\alpha_2(\varepsilon, E_z)} \frac{du}{\sqrt{\varepsilon - \phi(u, E_z)}}.$$  \hspace{1cm} (23)

In this equation $\varepsilon$ is a continuous (not quantized) parameter which is restricted in the range $\varepsilon_{\min}(E_z) \leq \varepsilon \leq \varepsilon_{\max}(E_z)$. It can be shown that for a given $E_z$, $S(\varepsilon, E_z)$ has a minimum value $S_{\min}(E_z)$ when $\varepsilon = \varepsilon_{\min}(E_z)$, and a maximum value $S_{\max}(E_z)$ when $\varepsilon = \varepsilon_{\max}(E_z)$. From the quantization rule of Eq. (21) (where $\varepsilon = \varepsilon_{N}(E_z)$), we have

$$S_{\min}(E_z) \leq \frac{L_x}{N} \leq S_{\max}(E_z).$$  \hspace{1cm} (24)

This inequality tells us that for a given $E_z$, there may exist one or several $N$ which satisfy the condition (24). The number $N$, which satisfies the condition (24), is the number of filaments in the sample. The steady-state $N$-filament solution is denoted as $u^{(N)}(x)$, where $N$ must satisfy the inequality (24). In order to obtain the stepped hysteresis loop, we have to know the allowed $N$ as a function of $E_z$. We plot the $E_z$-dependence of $S_{\min}$ and $S_{\max}$ in Fig. 7. From Fig. 7 the allowed $N$ can be obtained for a given $E_z$ by using inequality (24). The $I$-$V$ hysteresis loop is plotted in Fig. 8, which contains several small steps. Figure 8 is qualitatively similar to the experimental results of Mayer et al. [2].
V. Stability analysis

Stability is an important question for theoretical models which have steady state multifilamentary solutions as we have pointed out in Introduction. The Schlögl-type reaction-diffusion (RD) equation [3] is

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u, \alpha),$$

where the nullcline of $f(u, \alpha) = 0$ is N-shaped ($\alpha$ is the control parameter). With the Neumann boundary conditions,

$$\left. \frac{\partial u(x, t)}{\partial x} \right|_{x=L_x} = \left. \frac{\partial u(x, t)}{\partial x} \right|_{x=0} = 0,$$

Eq. (25) is known to have steady-state multifilamentary solutions. However these solutions are unstable against small time-dependent fluctuations [4], therefore it is not a satisfactory model. Although the steady-state equation for our model Eq. (17) is quite similar to the Schlögl model, the time-dependent equation for our model (Eq. (15)) is different from the Schlögl model Eq. (25) as it introduces an additional constraint to the RD equation. Equation (15) is an integro-differential equation which comes from two differential equations, Eqs. (12) and (13). Therefore our model has an extra spatial-dependent differential equation for $E_x$, which is the additional constraint we find in our model. It is worthwhile to point out that the constraint comes naturally from basic equations of electrodynamics and we do not introduce it to our model artificially. The additional constraint will affects the stability of the multifilamentary solutions as suggested in previous studies [5, 6]. It turns out that the steady-state multifilamentary solutions of our model are stable, which we will show in the followings.
Without loss of generality, we consider the extreme light load limit $R \to 0$ as we have done in Sec. III. In this limit $E_z = V/L_z$ by Eq. (16), and $E_z$ can be considered as the control parameter, which is kept as a constant in the following analysis. We think it is more convenient to consider the dynamical equations for $n$ than those for $u$ ($u = \ln n$). The time-dependent equations of our model are Eqs. (12) and (13),

\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial x} \left( n \mu E_x + D \frac{\partial n}{\partial x} \right),
\]

\[
\epsilon \frac{\partial}{\partial x} E_x = \rho(n, E).
\]

We expand the variables $n$, $E_x$, and $\rho$ in terms of the Fourier components,

\[
n(x,t) = \sum_l n_l(t) e^{il\pi x/L_x},
\]

\[
E_x(x,t) = \sum_l E_{x,l}(t) e^{il\pi x/L_x},
\]

\[
\rho(n, E_z) = \sum_l \rho_l(t) e^{il\pi x/L_x}.
\]

Equations (12) and (13) become

\[
\frac{\partial n_l}{\partial t} = - \left( \frac{l \pi}{L_x} \right)^2 D n_l + i \mu l \frac{l \pi}{L_x} \sum_{l'} \rho_{l'} E_{x,l'} + \rho_l,
\]

and

\[
i \epsilon l \frac{\pi}{L_x} E_{x,l} = \rho_l.
\]

By eliminating $E_{x,l}$, we get

\[
\frac{\partial n_l}{\partial t} = - \left( \frac{l \pi}{L_x} \right)^2 D n_l + \mu l \frac{l \pi}{L_x} \sum_{l'} \rho_{l'} \frac{\rho_{l'} l \pi}{l' \pi - l \pi}.
\]

Because we are considering the stability of the steady-state solution with $N$-filament, we may decompose the components $n_l$ and $\rho_l$ into time-independent and time-dependent parts. Suppose $n(x,t)$ is very close to the steady-state $N$-filament solution $n^N(x) \equiv \exp[u^N(x)]$, we may write

\[
n(x,t) = n^N(x) + \delta n(N)(x,t),
\]

\[
\rho(x,t) = \rho^N(x) + \delta \rho^N(x,t),
\]
where $\delta n(N)(x,t)$ and $\delta \rho^{(N)}(x,t)$ denote small time-dependent fluctuations; and $\rho^{(N)}(x) \equiv \rho(n^{(N)}(x),E_z)$. The time-independent parts $n^{(N)}(x)$ and $\rho^{(N)}(x)$ are, respectively, expanded in terms of the Fourier components:

$$n^{(N)}(x) = \sum_l n_l^{(N)} e^{ilx},$$  \hspace{1cm} (34)

$$\rho^{(N)}(x) = \sum_l \rho_l^{(N)} e^{ilx}. \hspace{1cm} (35)$$

Similarly the time-dependent parts $\delta n^{(N)}(x,t)$ and $\delta \rho^{(N)}(x,t)$ can be expanded as

$$\delta n^{(N)}(x,t) = \sum_l \delta n_l^{(N)}(t) e^{ilx},$$  \hspace{1cm} (36)

$$\delta \rho^{(N)}(n) = \sum_l \delta \rho_l^{(N)}(t) e^{ilx} = \sum_l \rho_l^{(N)} \delta n_l^{(N)}(t) e^{ilx}, \hspace{1cm} (37)$$

where $\rho_l^{(N)}$ is the Fourier component of $\partial \rho(n^{(N)},E_z)/\partial n^{(N)}$:

$$\frac{\partial \rho(n^{(N)},E_z)}{\partial n^{(N)}} = \sum_l \rho_l^{(N)} e^{ilx}. \hspace{1cm} (38)$$

From these expansions, we see that $n_l(t) = n_l^{(N)} + \delta n_l^{(N)}(t)$ and $\rho_l(t) = \rho_l^{(N)} + \delta \rho_l^{(N)}(t)$.

We can then linearize Eq. (31) (by using Eq. (37)) and neglect higher order terms of $\delta n_l^{(N)}(t)$ to obtain

$$\frac{\partial \delta n_l^{(N)}}{\partial t} = \sum_{l'=1}^1 \frac{M_{l'0}^{(N)}}{\delta \rho_l^{(N)}} \delta n_{l'}^{(N)}, \hspace{1cm} (39)$$

where

$$M_{l0}^{(N)} = -\left( \frac{l\pi}{L_x} \right)^2 D \delta \rho_l + \frac{\mu_0 l}{\epsilon} \left[ (1 - \delta n_l^{(N)}) \frac{\rho_l^{(N)}}{(l - l_0)^2} + \sum_{l'=l}^{l_0} \frac{n_{l'}^{(N)} \delta \rho_l^{(N)}}{(l - l_0)} \right]. \hspace{1cm} (40)$$

Equation (39) is a set of coupled linear equations for $\delta n_l^{(N)}$ with the subscript $|l|$ runs from 1 to $\infty$. Note that it is easy to show, from Eq. (12), that $\delta n_0^{(N)} = 0$ identically for all time (by integration over $x$ and $y$, and the fact that the conduction current density $J_x = n\mu E_x + D(\partial n/\partial x)$ is zero at the boundaries $x = \pm L$). Therefore the $l = 0$ component does not appear in Eq. (39). In order that Eq. (39) to be soluble, we have to truncate the matrix elements $M_{l0}^{(N)}$. We let $M_{l0}^{(N)} = 0$ if either $|l|$ or $|l_0| > s > N$. The reason why the truncation is reasonable will be explained at the end of the Section. In the following calculations we calculate the cases for $N = 2$, 3, and 4 with $s = 10$ for all three cases. We write $\delta n_l^{(N)}(t) = \delta n_l^{(N)}(0)e^{\lambda t}$ and substitute into Eq. (39) to solve for the eigenvalues of the matrix $M_{l0}^{(N)}$. There are 2s different eigenvalues of $\lambda_l$ (|l| = 1, 2, \ldots, s). If
all the real parts of the \( \lambda \)'s are negative, then the steady-state solution \( n^{(N)}(x) \) is stable. In Fig. 8 we plot \( \lambda_{\text{max}} \) vs. \( E_z \) for \( N = 2, 3, \) and 4, where \( \lambda_{\text{max}} \) is defined as

\[
\lambda_{\text{max}} = \max \{ \text{Re}[\lambda_l], \ |l| = 1, 2 \ldots s \}.
\]

We see that all \( \lambda_{\text{max}} < 0 \) for \( N = 2, 3, \) and 4, and therefore all the corresponding steady-state multifilamentary pattern solutions are stable.

Finally it may be worthwhile to say some words about the approach we use in the stability analysis. We use the Fourier series expansion method to treat the spatial dependent problem. This is a reasonable approach because the steady-state solution \( n^{(N)}(x) \) satisfies the periodic boundary conditions \( n^{(N)}(-L_x) = n^{(N)}(L_x) \) and \( d n^{(N)}/d x|_{L_x} = d n^{(N)}/d x|_{-L_x} \). Therefore \( n^{(N)}(x) \) can be considered as an infinite extended periodic function with periodicity \( 2L_x \). Moreover \( n^{(N)}(x) \) is a smooth varying function (cf. Fig. 6), and therefore its Fourier series expansion is expected to converge rapidly, especially for small \( N \). It seems reasonable for us to truncate the series for \( l > 10 \) (for \( N = 2, 3, \) and 4) in the numerical calculation. One more point about the truncation is that if no filamentary state is stable, then the final stable state is a no-filament state, i.e., the uniform electron density state, which consists of only \( l = 0 \) component. Therefore higher-\( l \) components of \( \delta n_l^{(N)} \) should not play important roles in the stability analysis of \( n^{(N)}(x) \). This is because if a high-\( l \) component of \( \delta n_l^{(N)}(t) \) is unstable and becomes larger and larger as time \( t \) increases, then the original filamentary state \( n^{(N)}(x) \) will transform to another filamentary state which contains the \( l \)-component with a large \( l \). This is in contradiction to the stability analysis of the RD equation [4], which shows that no filamentary state is stable. Of course, the additional constraint in our theory may alter the above conclusion, and the truncation may not be a good approximation. But if this is the case, then it does no harm to our theory. Because filamentary states do exist, although it may become oscillatory states as studied in Ref. [6], in the large \( R \) limit. However we believe that the truncation is a good approximation, and stable stationary multifilamentary states exist which is consistent with the experimental result of Ref. [2] in the small \( R \) limit.

![FIG. 5. Plots of \( \lambda_{\text{max}} \) vs. \( E_z \) for \( N=2, 3, \) and 4. Note that all of them are negative.](image-url)
VI. Conclusions

We have established a theoretical model which can support stable steady-state multifilamentary pattern solutions in SNDC semiconductors. This model is based on the basic equations in electrodynamics and the widely accepted two-trap-level models for the SNDC materials. There are no extra assumptions in the model, and therefore we think that our model is simpler and more realistic than the existing models, which either needs an extra physical quantity to have stable steady-state solutions [5] or can not support stable steady-state solutions in the weak coupling (resistance $R \rightarrow 0$) limit [6]. For completeness, stability analysis of the steady-state solutions is given to show that these solutions are stable.

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References