IOPscience

HOME | SEARCH | PACS & MSC | JOURNALS | ABOUT | CONTACT US

Reconstruction of discontinuities in systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2007 J. Phys.: Conf. Ser. 73 012024

(http://iopscience.iop.org/1742-6596/73/1/012024)

The Table of Contents and more related content is available

Download details:
IP Address: 140.112.113.225
The article was downloaded on 23/06/2009 at 13:01

Please note that terms and conditions apply.

Journal of Physics: Conference Series **73** (2007) 012024 doi:10.1088/1742-6596/73/1/012024

Reconstruction of discontinuities in systems

Gunther Uhlmann[1] **and Jenn-Nan Wang**[2]

[1]Department of Mathematics, University of Washington, Box 354305, Seattle, WA 98195-4350, USA.

[2]Department of Mathematics, Taida Institute for Mathematical Sciences, and NCTS (Taipei), National Taiwan University, Taipei 106, Taiwan.

E-mail: `gunther@math.washington.edu, jnwang@math.ntu.edu.tw`

Abstract. We survey some recent results on the reconstruction of discontinuities by boundary measurements for elasticity and related systems in two dimensions. Our main tool is a new type of complex geometrical optics solutions.

1. Introduction

In this paper we consider the inverse boundary problem of reconstructing discontinuities inside a plane domain filled with elastic, thermoelastic media, or incompressible fluid. We are interested in designing a reconstruction algorithm which can determine as much information of the discontinuities as possible by boundary measurements. In this study, we assume that the medium is isotropic for those systems. A common feature of those systems is that they can be reduced to a larger system with Laplacian as the leading term.

In general, inverse boundary value problems are a class of inverse problems where one attempts to determine the internal parameters of body by making measurements only at the surface of the body. A prototypical example that has received a lot of attention is Electrical Impedance Tomography (EIT). In this inverse method one would like to determine the conductivity distribution inside a body by making voltage and current measurements at the boundary. This is also called Calderón problem [3]. The boundary information is encoded in the Dirichlet to Neumann map associated to the conductivity equation. Sylvester and Uhlmann [25] constructed complex geometrical optics (CGO) solutions for the conductivity equation. The phase functions of these solutions are linear. CGO optics have been used in EIT and have been instrumental in solving several inverse problems. We will not review these developments in detail here; see [27] and [26] for references; other reviews in EIT are [2] and [4]. There are many applications of EIT ranging from early breast cancer detection [29] to geophysical sensing for underground objects, see [16, 21, 22, 24]. The article [25] and the ones reviewed in [26] assumes that the measurements are made on the whole boundary. However, it is often possible to make the measurements only on part of the boundary; this is the partial data problem. This is the case for the applications in breast cancer detection and geophysical sensing mentioned above. Recently, new CGO solutions that are useful for the partial data problem were constructed in [18] for the conductivity equation and zeroth order perturbations of the Laplacian. The real part of the phase of these solutions are limiting Carleman weights. They have been generalized to first order perturbation of the Laplacian for scalar equations or systems in [5], [8], [23], and [28].

© 2007 IOP Publishing Ltd

Journal of Physics: Conference Series **73** (2007) 012024 doi:10.1088/1742-6596/73/1/012024

Constructions of CGO solutions for the conductivity equation and zeroth order perturbations of the Laplacian using hyperbolic geometry can be found in [15]; these have been applied to determine electrical inclusions in [9].

In two dimensions, when the underlying equation has Laplacian as the leading part, we have more freedom in choosing the complex phases for the CGO solutions. In particular any harmonic function is a limiting Carleman weight and can be the real part of a CGO solution. Motivated by this idea, a framework of constructing CGO solutions with general phases for systems with Laplacian as the leading term has been given in [30]. Applications of these CGO solutions to the reconstruction of discontinuities for the conductivity and the isotropic elasticity systems were given in [30] and [31], respectively. The main theme of this paper is to give an overview of the method used in [30] and [31]. To demonstrate the flexibility of our method, we also consider the thermoelasticity system here. This is a new application.

The method developed here shares the same spirit as Ikehata's enclosure method [10]. For the two-dimensional problem, we would like to mention a very interesting result by Ikehata in [12] where he introduced the Mittag-Leffler function in the object identification problem. This has the property that its modulus grows exponentially in some cone and decays to zero algebraically outside the same cone. Using the Mittag-Leffler function and shrinking the opening angle of the cone, one can reconstruct precisely the shapes of some embedded objects such as star-shaped objects. The numerical implementation of the Mittag-Leffler functions was carried out by Ikehata and Siltanen in [13]. The main restriction of the method using the Mittag-Leffler function is that it can be only applied to scalar equations with homogeneous background. That is, they probe the region with harmonic functions.

The novelty of our method is its flexibility in treating scalar equations, and even two-dimensional systems, with inhomogeneous background. Furthermore, for the object identification problem in such general systems, we are able to achieve for these general systems the analogous results as those in [12] and [13] for the conductivity equation with homogeneous background. We would also like to point out that the Mittag-Leffler function is in the form of infinite series. Therefore, to implement the Mittag-Leffler function numerically, one needs first to do a suitable truncation. This clearly introduces a priori errors in the input (Dirichlet) data. On the other hand, our special CGO solutions are in closed form. So we can prescribe the exact Dirichlet data in the inverse problem using our method.

We also would like to compare our method and that in [9]. The real parts of the phase functions of CGO solutions in [9] are radially symmetric. So their probing fronts are circles or spheres. Moreover, the construction of CGO solutions in [9] is based on the hyperbolic geometry. It has not been developed to studying more general equations or systems. The advantage of our method lies in the freedom of choosing the phase functions of CGO solutions. Consequently, we are able to determine more information in the object identification problem in the two dimensional case than [9] does. On the other hand, since the real parts of the phase functions in our CGO solutions are not necessarily radially symmetric, we can create different probing fronts by simply rotating the phase functions.

2. Elasticity and related systems

In this section we list the systems considered in this work. Let Ω be an open domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$.

2.1. Isotropic elasticity

The domain Ω is now modeled as an inhomogeneous, isotropic, elastic medium characterized by the Lamé parameters $\lambda(x)$ and $\mu(x)$. Assume that $\lambda(x) \in C^2(\overline{\Omega})$, $\mu(x) \in C^4(\overline{\Omega})$ and the following inequalities hold

$$\mu(x) > 0 \quad \text{and} \quad \lambda(x) + 2\mu(x) > 0 \quad \forall\, x \in \overline{\Omega} \quad \text{(strong ellipticity)}.$$

We consider the static isotropic elasticity system without sources

$$\mathrm{div}(\lambda(\mathrm{div}u)I + 2\mu S(\nabla u)) = 0 \quad \text{in} \quad \Omega. \tag{2.1}$$

Here and below, $S(A) = (A + A^T)/2$ denotes the symmetric part of the matrix $A \in \mathbb{C}^{2\times 2}$. Equivalently, if we denote $\sigma(u) = \lambda(\mathrm{div}u)I + 2\mu S(\nabla u)$ the stress tensor, then (2.1) becomes

$$\mathrm{div}\sigma = 0 \quad \text{in} \quad \Omega.$$

We will use the reduced system derived by Ikehata [11]. This reduction was also mentioned in [26]. Let $\begin{pmatrix} w \\ g \end{pmatrix}$ satisfy

$$\Delta \begin{pmatrix} w \\ g \end{pmatrix} + A(x) \begin{pmatrix} \nabla g \\ \mathrm{div}w \end{pmatrix} + Q(x) \begin{pmatrix} w \\ g \end{pmatrix} = 0,$$

where

$$A(x) = \begin{pmatrix} 2\mu^{-1/2}(-\nabla^2 + \Delta)\mu^{-1} & -\nabla \log \mu \\ (0^\dagger)^T & \frac{\lambda+\mu}{\lambda+2\mu}\mu^{1/2} \end{pmatrix}, \quad 0^\dagger = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$Q(x) = \begin{pmatrix} -\mu^{-1/2}(2\nabla^2 + \Delta)\mu^{1/2} & 2\mu^{-5/2}(\nabla^2 - \Delta)\mu\ \nabla\mu \\ -\frac{\lambda-\mu}{\lambda+2\mu}(\nabla\mu^{1/2})^T & -\mu\Delta\mu^{-1} \end{pmatrix}.$$

Here $\nabla^2 f$ is the Hessian of the scalar function f. Then

$$u := \mu^{-1/2}w + \mu^{-1}\nabla g - g\nabla\mu^{-1}$$

satisfies (2.1). A similar form was also used in [6] for studying the inverse boundary value problem for the isotropic elasticity system.

2.2. Stokes system

Let $\mu(x) \in C^4(\bar{\Omega})$ and $\mu(x) > 0$ for all $x \in \bar{\Omega}$. Here μ is called the viscosity function. In most literature μ is a constant. But our method works equally well even when μ is a variable function. Suppose that $u = (u_1, u_2)$ and p satisfy the Stokes system:

$$\begin{cases} \mathrm{div}(\mu S(\nabla u)) - \nabla p = 0 \quad \text{in} \quad \Omega, \\ \mathrm{div}u = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.2}$$

Here u and p represent the velocity field and the pressure, respectively. Motivated by the isotropic elasticity, we set $u = \mu^{-1/2}w + \mu^{-1}\nabla g - (\nabla\mu^{-1})g$ and

$$p = \nabla\mu^{1/2} \cdot w + \mu^{1/2}\mathrm{div}w + 2\Delta g = \mathrm{div}(\mu^{1/2}w) + 2\Delta g,$$

then (u, p) is a solution of (2.2) provided $\begin{pmatrix} w \\ g \end{pmatrix}$ satisfies

$$\Delta \begin{pmatrix} w \\ g \end{pmatrix} + A(x) \begin{pmatrix} \nabla g \\ \mathrm{div}w \end{pmatrix} + Q(x) \begin{pmatrix} w \\ g \end{pmatrix} = 0 \tag{2.3}$$

with

$$A(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2\mu^{-1} & -\mu^{-1}\nabla\mu \\ (0^\dagger)^T & \mu^{1/2} \end{pmatrix}$$

and

$$Q = \begin{pmatrix} -2\mu^{-1/2}\nabla^2\mu^{1/2} - \mu^{-1/2}\Delta\mu^{1/2} & -4\nabla^2\mu^{-1}\nabla\mu^{1/2} - 2\mu^{1/2}\mathrm{div}(\nabla\mu^{-1}) \\ \mu(\nabla\mu^{-1/2})^T & -\mu\Delta\mu^{-1} \end{pmatrix}.$$

The relation between the isotropic elasticity and the Stokes system was first observed in [7].

2.3. Thermoelasticity

We consider a linear elastic body with mechanically and thermically isotropic medium. Let q be the temperature and u the thermoelastic displacement. The static thermoelastic equations are described as follows:

$$\begin{cases} \mathrm{div}(\sigma(u) - \gamma q) = 0 \quad \text{in} \quad \Omega, \\ \Delta q = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{2.4}$$

where $\gamma = \frac{2\mu\alpha(1+\tau)}{3(1-2\tau)}$. Here α is the thermo expansion coefficient and τ is the Poisson ratio. Since the addition of temperature function q is a lower order in the first equation of (2.4), the same reduction for the isotropic elasticity still works for (2.4). Precisely, let $\begin{pmatrix} w \\ g \\ q \\ q \end{pmatrix}$ satisfy

$$\Delta \begin{pmatrix} w \\ g \\ q \\ q \end{pmatrix} + A(x) \begin{pmatrix} \nabla g \\ \mathrm{div} w \\ \nabla q \end{pmatrix} + Q(x) \begin{pmatrix} w \\ g \\ q \\ q \end{pmatrix} = 0,$$

where

$$A(x) = \begin{pmatrix} 2\mu^{-1/2}(-\nabla^2 + \Delta)\mu^{-1} & -\nabla\log\mu & -\gamma I_2 \\ (0^\dagger)^T & \frac{\lambda+\mu}{\lambda+2\mu}\mu^{1/2} & (0^\dagger)^T \\ 0^\ddagger & 0^\dagger & 0^\ddagger \end{pmatrix},$$

$$0^\ddagger = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and a suitable Q. Then $u = \mu^{-1/2}w + \mu^{-1}\nabla g - g\nabla\mu^{-1}$ and q satisfy (2.4).

In summary, all systems considered in this section can be transformed to the following general form:

$$PU := \Delta_x U + A_1(x)\partial_{x_1}U + A_2(x)\partial_{x_2}U + Q(x)U = 0 \quad \text{in} \quad \Omega, \tag{2.5}$$

where $U(x) = (u_1(x_1,x_2), \cdots, u_n(x_1,x_2))^T$ with $n \in \mathbb{N}$, $\Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2$, and A_1, A_2, Q are $n \times n$ matrices. To construct special solutions for systems (2.1), (2.2), and (2.4), it is enough to work on system (2.5).

3. Complex Geometrical Optics Solutions

A framework for constructing special CGO solutions for (2.5) has already given in [30]. We review the procedure here. Let $\tilde{\Omega}$ be an open bounded domain in $\mathbb{R}^2$ and $V(y) = V(y_1, y_2)$ satisfy

$$\Delta_y V + \tilde{A}_1\partial_{y_1}V + \tilde{A}_2\partial_{y_2}V + \tilde{Q}V = 0 \quad \text{in} \quad \tilde{\Omega}. \tag{3.1}$$

Assume that $\tilde{A}_1, \tilde{A}_2 \in C^2(\bar{\tilde{\Omega}})$ and $\tilde{Q} \in L^\infty(\tilde{\Omega})$. Given $\omega \in \mathbb{R}^2$ with $|\omega| = 1$, we can find $V(y)$ of (3.1) having the form

$$V(y) = e^{y\cdot(\omega+i\omega^\perp)/h}(\tilde{L} + \tilde{R}), \tag{3.2}$$

where $\tilde{L}$ is independent of h and $\tilde{R}$ satisfies

$$\|\partial^\alpha \tilde{R}\|_{L^2(\tilde{\Omega})} \le Ch^{1-\alpha}, \quad \forall\ |\alpha| \le 2. \tag{3.3}$$

To construct V having the form (3.2), (3.3), we can follow the approach in [8] and [28] which are based on [5] and [18]. The main tool are Carleman estimates.

With $V(y)$ at hand, we can construct CGO solutions with general phases. We now choose $\omega = (1, 0)$, $\omega^\perp = (0, 1)$, i.e., $y \cdot (\omega + i\omega^\perp) = y_1 + iy_2$, and denote $y = y_1 + iy_2$, $x = x_1 + ix_2$. Let Ω_0 be an open subdomain of Ω. Suppose that A_1, $A_2 \in C^2(\bar{\Omega}_0)$ and $Q \in L^\infty(\Omega_0)$. Let $y = \rho(x) = y_1(x_1, x_2) + iy_2(x_1, x_2)$ be a conformal map in Ω_0. Define $U(x) = V(y(x))$ and $\tilde{\Omega} = \rho(\Omega_0)$. Suppose that ρ^{-1} exists on $\tilde{\Omega}$. Let $\hat{A}_1(y) = (A_1\partial_{x_1} y_1 + A_2\partial_{x_2} y_1) \circ \rho^{-1}(y)$, $\hat{A}_2(y) = (A_1\partial_{x_1} y_2 + A_2\partial_{x_2} y_2) \circ \rho^{-1}(y)$, and $\hat{Q}(y) = (Q \circ \rho^{-1})(y)$ and $g(y) = |(\rho' \circ \rho^{-1})(y)|^2$. Now if we choose $V(y)$ satisfying

$$\Delta_y V + g(y)^{-1}\hat{A}_1(y)\partial_{y_1} V + g(y)^{-1}\hat{A}_2(y)\partial_{y_2} V + g(y)^{-1}\hat{Q}V = 0 \quad \text{in } \tilde{\Omega}, \tag{3.4}$$

then $U(x)$ satisfies (2.5) in Ω_0. According to the construction given previously, let $V(y)$ be a solution of (3.4) having the form

$$V(y) = e^{(y_1+iy_2)/h}(\tilde{L} + \tilde{R}),$$

where

$$\|\partial^\alpha \tilde{R}\|_{L^2(\tilde{\Omega})} \le Ch^{1-\alpha}, \quad \forall\ |\alpha| \le 2.$$

Denote $y_1(x_1, x_2) = \varphi(x_1, x_2)$ and $y_2(x_1, x_2) = \psi(x_1, x_2)$. We then obtain CGO solutions for (2.5) in Ω_0:

$$U(x) = e^{(\varphi+i\psi)/h}(L + R)$$

with $L = \tilde{L} \circ \rho$, $R = \tilde{R} \circ \rho$, and

$$\|\partial^\alpha R\|_{L^2(\Omega_0)} \le Ch^{1-\alpha}, \quad \forall\ |\alpha| \le 2. \tag{3.5}$$

Due to the conformality of ρ, φ and ψ are harmonic functions in Ω_0. Conversely, given any φ harmonic in Ω_0 with $\nabla\varphi \ne 0$ in Ω_0, we can find a harmonic conjugate ψ of φ in Ω_0 so that $\rho = \varphi + i\psi$ is conformal in Ω_0. The freedom of choosing φ plays a key role in our reconstruction method for the object identification problem. One useful example is $\rho = c_N(x - x_0)^N$ for $N \ge 2$, where $c_N \in \mathbb{C}$ with $|c_N| = 1$ and $x_0 \notin \bar{\Omega}$. Another issue is that we only construct CGO solutions in a domain Ω_0, which is a subdomain of Ω. We shall discuss how to extend these CGO solutions to Ω when we consider the concrete cases described in Section 2.

4. Inverse problem for the Stokes system

Using CGO solutions constructed above, we can investigate the reconstruction of inclusions or cavities embedded in a body with isotropic elastic medium. A complete discussion of this problem including numerical results will appear in a forthcoming paper [31]. In this work we consider the problem of reconstructing an obstacle immersed in an incompressible fluid by boundary measurements. The uniqueness of this inverse problem was proved in [1] and a reconstruction method based on complex spherical waves was given in [8]. Assume that D is a subset of Ω such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. Let $u = (u_1, u_2)$ be a vector-valued function satisfying

$$\begin{cases} \operatorname{div}(\mu S(\nabla u)) - \nabla p = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \operatorname{div} u = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ u = 0 \quad \text{on} \quad \partial D, \\ u = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

The boundary condition f satisfies the compatibility condition

$$\int_{\partial\Omega} f \cdot \mathbf{n} ds = 0, \tag{4.2}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. For any given $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution $(u, p) \in H^1(\Omega) \times L^2(\Omega)$ satisfying (4.1) provided

$$\int_{\Omega} p \; dx = 0. \tag{4.3}$$

From now on, we assume that p satisfies the normalizing condition (4.3). Therefore, we can define the Dirichlet-to-Neumann map (or velocity-to-force map) $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_D f = \sigma(u, p)\mathbf{n}|_{\partial\Omega},$$

where $\sigma(u, p) = \mu S(\nabla u) - pI_2$. The inverse problem we study here is to reconstruct D from the knowledge of Λ_D.

The starting point of our method is the following energy gap relation:

$$\int_D |S(\nabla u_0)|^2 dx \le \langle (\Lambda_D - \Lambda_0)(\bar{f}), f \rangle \le C\big(\int_D |S(\nabla u_0)|^2 dx + \int_D |u_0|^2 dx \big), \tag{4.4}$$

where Λ_0 is the Dirichlet-to-Neumann map associated with the unperturbed system:

$$\begin{cases} \operatorname{div}(\mu S(\nabla u_0)) - \nabla p_0 = 0 \quad \text{in} \quad \Omega, \\ \operatorname{div} u_0 = 0 \quad \text{in} \quad \Omega, \\ u_0 = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.5}$$

Next, we want to construct special solutions of (4.5) such that (4.4) will yield the information of ∂D.

Let $\rho_N = c_N(x - x_0)^N = \varphi_N + i\psi_N$ be defined as in the previous section. Without loss of generality, we take $x_0 = 0$. We observe that $\varphi_N > 0$ in some open cone Γ_N with an opening angle π/N. Pick a such cone Γ_N and assume that $\Gamma_N \cap \Omega = \Omega_0 \neq \emptyset$. As outlined in Section 2, let

$$U_{N,h}(x) = \begin{pmatrix} w_{N,h} \\ g_{N,h} \end{pmatrix} = e^{(\varphi_N + i\psi_N)/h}(L + R)$$

solve (2.3) in Ω_0. Then

$$u_{N,h} = \mu^{-1/2} w_{N,h} + \mu^{-1} \nabla g_{N,h} - g_{N,h} \nabla \mu^{-1}$$

with a suitable $p_{N,h}$ satisfy the unperturbed Stokes system (2.2) in Ω_0. Now to get solutions of (2.2) in the whole domain Ω, we use a cut-off technique. For $s > 0$, let $\ell_s = \{x \in \Gamma_N : \varphi_N = s^{-1}\}$. This is the level curve of ϕ_N in Γ_N. Let $0 < t < t_0$ such that

$$(\cup_{s \in (0,t)} \ell_s) \cap \Omega \neq \emptyset$$

and choose a small $\varepsilon > 0$. Define a cut-off function $\phi_{N,t}(x) \in C^\infty(\mathbb{R}^2)$ so that $\phi_{N,t}(x) = 1$ for $x \in \overline{(\cup_{s\in(0,t+\varepsilon/2)}\ell_s) \cap \Omega}$ and is zero for $x \in \bar{\Omega} \setminus (\cup_{s\in(0,t+\varepsilon)}\ell_s)$. We now define

$$u_{N,t,h}(x) = \phi_{N,t} e^{-t^{-1}/h} u_{N,h}$$

for $x \in (\cup_{s\in(0,t+\varepsilon)}\ell_s) \cap \Omega$. So $u_{N,t,h}$ can be regarded as a function in Ω which is zero outside of Ω_0. The pressure $p_{N,t,h}$ is defined similarly. Unfortunately, $(u_{N,t,h}, p_{N,t,h})$ is not a solution of (2.2) in Ω. Furthermore, the boundary value $u_{N,t,h}|_{\partial\Omega}$ can not be used in (4.5) since it does not satisfy the compatibility condition (4.2).

To set up appropriate Dirichlet conditions, we adopt the arguments used in [8]. We are going to use the Dirichlet data $f_{N,t,h} = u_{N,t,h}|_{\partial\Omega} - c_{N,t,h}\mathbf{n} = \phi_{N,t,h} e^{-t^{-1}/h} u_{N,h}|_{\partial\Omega} - c_{N,t,h}\mathbf{n}$, where

$$c_{N,t,h} = \int_{\partial\Omega} u_{N,t,h}|_{\partial\Omega} \cdot \mathbf{n} ds.$$

It is clear that $f_{N,t,h}$ satisfies (4.2). At the first look, the addition of the constant $c_{N,t,h}$ seems a bit awkward. So we may be able to omit $c_{N,t,h}$ in practice. But we can show that $c_{N,t,h}$ is actually decaying exponentially as $h \to 0$. To see this, we first observe that

$$\int_{\partial\Omega_{t+\varepsilon}} u_{N,h} \cdot \mathbf{n} ds = 0,$$

where $\Omega_{t+\varepsilon} = \cup_{s\in(0,t+\varepsilon)}\ell_s$. Thus we have that

$$\begin{aligned} c_{N,t,h} &= \int_{(\cup_{s\in(0,t+\varepsilon)}\ell_s)\cap\partial\Omega} \phi_{N,t,h} e^{-t^{-1}/h} u_{N,h} \cdot \mathbf{n} ds \\ &= \int_{\partial\Omega_{t+\varepsilon}} \phi_{N,t,h} e^{-t^{-1}/h} u_{N,h} \cdot \mathbf{n} ds \\ &= \int_{\partial\Omega_{t+\varepsilon}} (1 - \phi_{N,t,h}) e^{-t^{-1}/h} u_{N,h} \cdot \mathbf{n} ds. \end{aligned}$$

Using the decaying property of $e^{-t^{-1}/h} u_{N,h}$, we can derive the decaying behavior of $c_{N,t,h}$.

To handle the fact that $(u_{N,t,h}, p_{N,t,h})$ is not a solution for the Stokes system, we consider the boundary value problem:

$$\begin{cases} \mathrm{div}(\mu S(\nabla v_{N,t,h})) - \nabla q_{N,t,h} = 0 \quad \text{in} \quad \Omega, \\ \mathrm{div} v_{N,t,h} = 0 \quad \text{in} \quad \Omega, \\ v_{N,t,h} = f_{N,t,h} \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.6}$$

Using the regularity theorem for the Stokes system, arguing as in [8], we can show that

$$\|u_{N,t,h} - v_{N,t,h}\|_{H^1(\Omega)} \le C e^{-\varepsilon'/h} \tag{4.7}$$

as $h \to 0$.

Now going back to the integral identities (4.4), we define

$$E(N,t,h) = \langle (\Lambda_D - \Lambda_0) \overline{f}_{N,t,h}, f_{N,t,h} \rangle$$

and obtain

$$\int_D |S(\nabla v_{N,t,h})|^2 dx \le E(N,t,h) \le C\Big(\int_D |S(\nabla v_{N,t,h})|^2 dx + \int_D |v_{N,t,h}|^2 dx \Big). \tag{4.8}$$

Since the difference of $u_{N,t,h}$ and $v_{N,t,h}$ is exponentially small in h, we are allowed to replace $v_{N,t,h}$ by $u_{N,t,h}$ in (4.8). Using the method in [30] or [31], we can prove the following theorem.

Journal of Physics: Conference Series **73** (2007) 012024 doi:10.1088/1742-6596/73/1/012024

Theorem 4.1 *Let the curve ℓ_t be defined as above. Then we have:*
(i) *if $\ell_t \cap \bar{D} = \emptyset$ then there exist $C_1 > 0$, $\varepsilon_1 > 0$, and $h_1 > 0$ such that $E(N,t,h) \leq C_1 e^{-\varepsilon_1/h}$ for all $h \geq h_1$;*
(ii) *if $\ell_t \cap D \neq \emptyset$ then there exist $C_2 > 0$, $\varepsilon_2 > 0$, and $h_2 > 0$ such that $E(N,t,h) \geq C_2 e^{\varepsilon_2/h}$ for all $h \geq h_2$.*

Using Theorem 4.1, we can identify some part of ∂D by examining the asymptotic behavior of $E(N,t,h)$ in h for various t's. Moreover, by taking N arbitrarily large (the opening angle of Γ_N becomes arbitrarily small), we can reconstruct even more information of ∂D. For instance, theoretically, we are able to reconstruct the full information of a star-shaped obstacle D from Λ_D. To end this paper, we provide an algorithm of our method.

Step 1. Pick $x_0 \notin \bar{\Omega}$. Given $N \in \mathbb{N}$ and choose a cone Γ_N which intersects Ω.

Step 2. Start with $t > 0$ such that $\ell_t \cap \Omega \neq \emptyset$. Construct $u_{N,t,h}$ and determine the Dirichlet data $f_{N,t,h} = u_{N,t,h}|_{\partial\Omega} - c_{N,t,h}\mathbf{n}$.

Step 3. Evaluate $E(N,t,h)$.

Step 4. If $E(N,t,h)$ is arbitrarily small, then increase t and repeat Step 2 and 3; if $E(N,t,h)$ is arbitrarily large, then decrease t and repeat Step 2 and 3.

Step 5. Repeat Step 4 to get a good approximation of ∂D in Γ_N.

Step 6. Move the cone Γ_N around x_0 by taking a different c_N in $\varphi_N = \mathrm{Re}(c_N x^N)$. Repeat Step 2–5.

Step 7. Choose a larger N and a new cone Γ_N. Repeat Step 2–6.

Step 8. Pick a different x_0 and repeat Step 1–7.

Acknowledgments
The first was partially supported by NSF and a Walker Family Endowed Professorship. The second author was supported in part by the National Science Council of Taiwan (NSC 95-2115-M-002-003).

References

[1] Alvarez C, Conca C, Friz L, Kavian O, and Ortega J H 2005 *Inverse Problems* **21** 1531-52
[2] Borcea L 2002 *Inverse Problems* **18** R99-R136. Addendum 2003 *Inverse Problems* **19** 997-8
[3] Calderón A 1980 *Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Río de Janeiro* 65-73
[4] Cheney M, Isaacson D, and Newell J C 1999 *SIAM Review* **41** 85-101
[5] Dos Santos Ferreira D, Kenig C E, Sjöstrand J, and Uhlmann G *Comm. Math. Phys.* in press
[6] Eskin G and Ralston J 2002 *Inverse Problems* **18** 907-21
[7] Heck H, Li X, and Wang J N *Indiana Univ. Math. J.* to appear
[8] Heck H, Uhlmann G, and Wang J N 2007 *Inverse Problems and Imaging* **1** 63-76
[9] Ide T, Isozaki H, Nakata S, Siltanen S, and Uhlmann G *Comm. Pure Appl. Math.* in press
[10] Ikehata M 2001 *Int. Soc. Anal. Appl. Comput.* **9** 87-103
[11] Ikehata M A remark on an inverse boundary value problem arising in elasticity *preprint*
[12] Ikehata M 2004 *Contemp. Math. AMS* **348** 41-52
[13] Ikehata M and Siltanen S 2004 *Inverse Problems* **20** 1325-48
[14] Isozaki H 2004 *Amer. J. Math.* **126** 1261-1313
[15] Isozaki H and Uhlmann G 2004 *Advances in Math.* **188** 294-314
[16] Jordana J, Gasulla M, and Pallas-Areny R 2001 *Meas. Sci. Technol.* **12** 1061-68
[17] Kang H, Seo J K, and Sheen D 1997 *SIAM J. Math. Anal.* **28** 1389-1405
[18] Kenig C E, Sjöstrand J, and Uhlmann G *Ann. of Math.*, **156** 567–591
[19] Nakamura G and Uhlmann G 1994 *Invent. Math.* **118** 457-74
[20] Nakamura G and Uhlmann G 2003 *Invent. Math.* **152** 205-207
[21] Ramirez A, Daily W, LaBrecque D, Owen E, and Chesnut D 1993 *Water Resources Research* **29** 73-87

[22] Ramirez A, Daily W, Binley A, LaBrecque D, and Roelant D 1996 *J. Envir. Eng. Geophys.* **1** 189-203
[23] Salo M and Wang J N 2006 *Inverse Problems* **22** 2299-2309
[24] Slater L, Binley A M, Daily W, and Johnson R 2000 *Journal of Applied Geophysics* **44** 85- V102
[25] Sylvester J and Uhlmann G 1987 *Ann. of Math.(2)* **125** 153-169
[26] Uhlmann G 1999 *Harmonic Analysis and Partial Differential Equations (Essays in Honor of Alberto P. Calderón)* (Chicago:The University of Chicago Press) 295-345
[27] Uhlmann G 2006 *Selecta, papers of Alberto P. Calderón* edited by A. Bellow, C.E. Kenig and P. Malliavin
[28] Uhlmann G and Wang J N 2007 *SIAM J. Math. Anal.* in press
[29] Zou Y and Guo Z 2003 *Med. Eng. Phys.* **25** 79-90
[30] Uhlmann G and Wang J N 2007 Complex geometrical optics solutions and reconstruction of discontinuities *preprint*
[31] Uhlmann G, Wang J N, and Wu C T 2007 Reconstruction of inclusions in an elastic body *in preparation*