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Accurate pricing formulas for Asian options

Kuan-Wen Chen [a], Yuh-Dauh Lyuu [b,*]

[a] Department of Finance, National Taiwan University, No. 1, Sec. 4, Roosevelt Road, Taipei 106, Taiwan
[b] Department of Finance and Department of Computer Science and Information Engineering, National Taiwan University, No. 1, Sec. 4, Roosevelt Road, Taipei 106, Taiwan

Abstract

Asian options have payoffs that depend on the average price of the underlying asset such as stocks, commodities, or financial indices. As exact closed-form formulas do not exist for these popular options, how to price them numerically in an efficient and accurate manner has been extensively investigated. There are two types of Asian options, fixed-strike and floating-strike Asian options. Excellent lower-bound formulas for both types of options have been derived by Rogers and Shi. These formulas are extremely easy to calculate, but they restrict the option's maturity to exactly 1 year. This paper extends the Rogers–Shi formulas to general maturities. Numerical experiments are performed to compare the formulas with many other numerical methods in the literature and under a wide variety of situations. They confirm the extreme accuracy of the formulas.

Keywords: Option pricing; Asian option; Fixed strike; Floating strike; Geometric Browning motion; Approximation; Lognormal distribution

1. Introduction

Asian options have payoffs that depend on the arithmetic average price of the underlying asset, which can be stocks, commodities, or financial indices. They are therefore useful for hedging future transactions whose cost is related to the average price of the underlying asset. In fact, Asian options were originally issued in 1987 when Bankers Trust's Tokyo office used them for pricing average options on crude oil contracts, thus the name Asian option [1]. Today, they are commonly traded on currencies and commodity products. The price of the Asian option is less subject to price manipulation. Hence the averaging feature is popular in many thinly traded markets and embedded in complex derivatives such as the "refix" clauses in convertible bonds. This averaging feature furthermore makes Asian options enjoy lower volatilities than their underlying assets, thus cheaper relative to standard options on the same underlying assets.

Exact closed-form formulas have not been available for pricing Asian options since their introduction by Ingersoll [2]. The source of the difficulty lies in the technical fact that the average of lognormal random

* Corresponding author.
E-mail address: lyuu@csie.ntu.edu.tw (Y.-D. Lyuu).

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variables is not lognormally distributed. (A random variable is lognormal if its logarithm is normally distributed.) As a result, how to price Asian options numerically in an efficient and accurate manner has been extensively investigated in the literature. Approaches to the problem of valuing Asian options in the literature include:

1. Monte-Carlo simulation [3–5];
2. Binomial tree [6–8];
3. Convolution method [9];
4. Direct integration [10,11];
5. Partial differential equation (PDE) [12–15];
6. Fourier transform (FFT) [16];
7. Approximate analytic method [17–25].

All of the above methods involve some tradeoffs between numerical accuracy and computational efficiency. This paper follows the approximate analytic method, whose chief advantage is its high efficiency.

The asset price is assumed to follow the geometric Browning motion

$$S_t = S\mathrm{e}^{\sigma B_t + \alpha t},$$

where σ is the volatility, $\alpha = r - \sigma^2/2$, r is the risk-free interest rate, S is the current asset price at time 0, S_t is the asset price at time t, and B_t is a Browning motion with $B_0 = 0$. It is most intuitive to think of B_t as being normally distributed with mean 0 and variance t. The asset price is therefore lognormally distributed. This distributional assumption is standard in finance [26,27]. The particular form of the drift term α can be justified on economic grounds; any other forms result in arbitrage opportunities, which should disappear in efficient markets [28].

The payoff of a fixed-strike Asian call option at maturity date T is $\max(0, S_{\mathrm{ave}} - K)$, where $S_{\mathrm{ave}} = \frac{1}{T}\int_0^T S_t\,\mathrm{d}t$ denotes the average price of the underlying asset over the period $[0, T]$, and $K > 0$ is called the strike price. The payoff of the floating-strike Asian call option is similar. It is $\max(0, S_{\mathrm{ave}} - S_T)$, where S_T is the asset price at maturity. The arbitrage-free price of the Asian option equals its discounted expected payoff, that is, $\mathrm{e}^{-rT}E$[payoff]. This claim can again be justified by arbitrage considerations.

This paper generalizes the lower-bound formulas of Rogers and Shi [23] from $T = 1$ (year) to a general T by extending the techniques of Thompson [25]. The formulas will turn out to be very easy to evaluate. Extensive numerical experiments are then conducted to verify the extreme accuracy of the formulas as compared to many other well-known methods in the literature.

This paper is organized as follows. Section 2 introduces mathematical preliminaries for later use. Section 3 presents the pricing formulas. Section 4 describes the numerical results. Conclusions are given in Section 5.

2. Mathematical preliminaries

First the correlation matrix between $\frac{1}{T}\int_0^T B_s\,\mathrm{d}s$ and B_t is established.

Theorem 1. *The correlation matrix between* $\frac{1}{T}\int_0^T B_s\,\mathrm{d}s$ *and* B_t *equals*

$$\begin{bmatrix} \mathrm{Cov}(B_t, B_t) & \mathrm{Cov}\left(B_t, \frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) \\ \mathrm{Cov}\left(B_t, \frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) & \mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s, \frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) \end{bmatrix} = \begin{bmatrix} t & t\left(1 - \frac{t}{2T}\right) \\ t\left(1 - \frac{t}{2T}\right) & \frac{T}{3} \end{bmatrix},$$

where $0 \leqslant t \leqslant T$.

Proof. See Appendix. □

Next we establish the correlation matrix between $\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T$ and B_t.

Theorem 2. *The correlation matrix between* $\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T$ *and* B_t *equals*

$$\begin{bmatrix} \mathrm{Cov}(B_t,B_t) & \mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) \\ \mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) & \mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) \end{bmatrix} = \begin{bmatrix} t & \frac{-t^2}{2T} \\ \frac{-t^2}{2T} & \frac{T}{3} \end{bmatrix},$$

where $0 \leqslant t \leqslant T$.

Proof. See Appendix. □

Let $0 < \sigma_x, \sigma_y$ and $-1 < \rho < 1$. Suppose $(X, Y) \sim \phi(\mu_x, \mu_x, \sigma_x^2, \sigma_y^2, \rho)$ is a bivariate normal random variable with means μ_x and μ_y, variances σ_x^2 and σ_y^2, and correlation ρ. Then its density function is given by

$$f(x,y) = \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

for $-\infty < x, y < \infty$. The following is a known fact concerning bivariate normal random variables [29].

Fact 3. If $(X, Y) \sim \phi(\mu_x, \mu_x, \sigma_x^2, \sigma_y^2, \rho)$, then the conditional distribution of X given $Y = y$ is normal with mean $\mu_x + \frac{\rho\sigma_x}{\sigma_y}(y - \mu_x)$ and variance $\sigma_x^2(1 - \rho^2)$.

Let $X = B_t$ and $Y = \frac{1}{T}\int_0^T B_s\,\mathrm{d}s$. Theorem 1 and Fact 3 imply that the conditional distribution of X given $Y = y$ is normal with mean $\frac{3t(T-t/2)y}{T^2}$ and variance $t - \frac{3t^2}{T^3}\left(T - \frac{t}{2}\right)^2$, in other words,

$$B_t \text{ given } \frac{1}{T}\int_0^T B_s\,\mathrm{d}s = y \;\sim\; N\left(\frac{3t(T-t/2)y}{T^2}, t - \frac{3t^2}{T^3}\left(T - \frac{t}{2}\right)^2\right). \tag{1}$$

Similarly, Theorem 2 and Fact 3 imply that the distribution of X given $Y - B_T = z$ is normal with mean $-\frac{3t^2z}{2T^2}$ and variance $t - \frac{3t^4}{4T^3}$, in other words,

$$B_t \text{ given } \frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T = z \;\sim\; N\left(-\frac{3t^2z}{2T^2}, t - \frac{3t^4}{4T^3}\right). \tag{2}$$

Let $\Phi(\cdot)$ denote the standard normal distribution function and $I(\cdot)$ the indicator function. The moment generating function $M_X(A)$ of the random variable X is defined for all real values of A by

$$M_X(A) = E[\mathrm{e}^{AX}] = \int_{-\infty}^{\infty} \mathrm{e}^{Ax} f(x)\,\mathrm{d}x$$

if X is continuous with density function $f(x)$. The next result is standard in probability theory [29].

Fact 4. If $X \sim \phi(\mu, \sigma^2)$, then $M_X(A) = \exp\left(\mu A + \frac{\sigma^2 A^2}{2}\right)$ for all real values of A.

The following theorem will simplify our notations later.

Theorem 5. *Suppose* $X \sim \phi(\mu_x, \sigma_x^2)$, $Y \sim \phi(\mu_y, \sigma_y^2)$, *and* $c = \mathrm{Cov}(X, Y)$. *Then*

$$E[\mathrm{e}^X I(Y > 0)] = \mathrm{e}^{u_x + \frac{\sigma_x^2}{2}}\Phi\left(\frac{\mu_y + c}{\sigma_y}\right).$$

Proof. See Appendix. □

The next general theorem about expectations is critical to the development of our pricing formulas.

Theorem 6. *For any random variable X with density function* $f_X(x)$,

$$\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E(S_t - K, X > \gamma)\,\mathrm{d}t = -\frac{1}{T}\int_0^T E(S_t - K|X = \gamma) f_X(\gamma)\,\mathrm{d}t.$$

Proof. See Appendix. □

The final theorem can be proved in the same way as Theorem 6.

Theorem 7. *For any random variable X with density* $f_X(x)$,

$$\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E(S_t - S_T, X > \gamma)\,\mathrm{d}t = -\frac{1}{T}\int_0^T E(S_t - S_T|X = \gamma)f_X(\gamma)\,\mathrm{d}t.$$

3. The pricing formulas

In this section, we will derive lower-bound formulas for both fixed-strike and floating-strike Asian options. It is useful to recall that $S_t = S\cdot\exp(\sigma B_t + \alpha t)$ with $\alpha = r - \sigma^2/2$. We will use the simpler notation x^+ for the function $\max(x, 0)$.

3.1. An analytic formula for fixed-strike Asian options

Define the event set $A = \left\{\omega : \frac{1}{T}\int_0^T S_t\,\mathrm{d}t > K\right\}$. The value of the fixed-strike Asian option, V_{fixed}, equals

$$\begin{aligned} \mathrm{e}^{-rT}E\left[\left(\frac{1}{T}\int_0^T S_t\,\mathrm{d}t - K\right)^+\right] &= \mathrm{e}^{-rT}E\left[\left(\frac{1}{T}\int_0^T S_t\,\mathrm{d}t - K\right)I(A)\right] = \mathrm{e}^{-rT}E\left[\left(\frac{1}{T}\int_0^T (S_t - K)\,\mathrm{d}t\right)I(A)\right] \\ &= \mathrm{e}^{-rT}E\left[\left(\frac{1}{T}\int_0^T (S_t - K)I(A)\,\mathrm{d}t\right)\right] = \frac{\mathrm{e}^{-rT}}{T}\int_0^T E[(S_t - K)I(A)]\,\mathrm{d}t \\ &\geqslant \frac{\mathrm{e}^{-rT}}{T}\int_0^T E[(S_t - K)I(A')]\,\mathrm{d}t \end{aligned}$$

for any event set A'. The reason for the last inequality is $\frac{1}{T}\int_0^T S_t\,\mathrm{d}t \leqslant K$ for $\omega \in A' - A$. We will fix $A' = \left\{\omega : \frac{1}{T}\int_0^T B_t\,\mathrm{d}t > \gamma\right\}$ for a γ to be determined later. This particular event set is chosen for the tractability it provides in deriving the desired lower-bound formula, which will turn out to be extremely accurate.

We now seek the value of that maximizes the lower bound

$$\frac{\mathrm{e}^{-rT}}{T}\int_0^T E[(S_t - K)I(A')]\,\mathrm{d}t = \frac{\mathrm{e}^{-rT}}{T}\int_0^T E\left(S_t - K, \frac{1}{T}\int_0^T B_t\,\mathrm{d}t > \gamma\right)\mathrm{d}t.$$

Theorem 6 says any random variable X with density $f_X(x)$ satisfies

$$\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E(S_t - K, X > \gamma)\,\mathrm{d}t = -\frac{1}{T}\int_0^T E(S_t - K|X = \gamma)f_X(\gamma)\,\mathrm{d}t.$$

Thus the desired value of γ (call it γ^*) must satisfy

$$\frac{1}{T}\int_0^T E(S_t|X = \gamma^*)\,\mathrm{d}t = K.$$

Given $X = \frac{1}{T}\int_0^T B_t\,\mathrm{d}t$ as dictated by our choice for A', the above identity becomes

$$\frac{1}{T}\int_0^T S\cdot\exp\left\{\frac{3t(T - t/2)\gamma^*\sigma}{T^2} + \alpha t + \frac{\sigma^2}{2}\left[t - \frac{3t^2}{T^3}\left(T - \frac{t}{2}\right)^2\right]\right\}\mathrm{d}t = K$$

by Eq. (1) and Fact 4. This identity determines γ^* uniquely. We now have the lower bound:

$$V_{\text{fixed}} \geqslant \mathrm{e}^{-rT}\left\{\frac{1}{T}\int_0^T E\left[(S\mathrm{e}^{\sigma B_t + \alpha t} - K)I\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s > \gamma^*\right)\right]\mathrm{d}t\right\}. \tag{3}$$

It remains to calculate the expectation. Fix $t \in [0, T]$. Let $N_1 = \sigma B_t + \alpha t + \log S$, $N_2 = \frac{1}{T}\int_0^T B_t \, dt - \gamma^*$, $u_i = E(N_i)$, $\sigma_i^2 = \mathrm{Var}(N_i)$, and $c = \mathrm{Cov}(N_1, N_2)$. Then inequality (3) becomes

$$V_{\text{fixed}} \geqslant e^{-rT}\left\{\frac{1}{T}\int_0^T E[(e^{N_1} - K)I(N_2 > 0)]\,dt\right\} = e^{-rT}\left\{\frac{1}{T}\int_0^T e^{u_1 + \frac{\sigma_1^2}{2}}\Phi\left(\frac{u_2 + c}{\sigma_2}\right) - K\Phi\left(\frac{u_2}{\sigma_2}\right)dt\right\}$$

by Theorem 5. As $u_1 = \alpha t + \log S$, $u_2 = -\gamma^*$, $\sigma_1^2 = \sigma^2 t$, $\sigma_2^2 = T/3$, and $c = \sigma t(1 - t/2T)$ according to Theorem 1, we finally obtain the desired lower-bound formula:

$$V_{\text{fixed}} \geqslant e^{-rT}\left\{\frac{S}{T}\int_0^T e^{\alpha t + \frac{\sigma^2 t}{2}}\Phi\left(\frac{-\gamma^* + \sigma t\left(1 - \frac{t}{2T}\right)}{\sqrt{T/3}}\right)dt - K\Phi\left(\frac{-\gamma^*}{\sqrt{T/3}}\right)\right\}. \quad (4)$$

3.2. An analytic formula for floating-strike Asian options

The steps here parallel those in the previous section. Define the event set $A = \left\{\omega : \frac{1}{T}\int_0^T S_t \, dt > S_T\right\}$. The value of the fixed-strike Asian option, V_{floating}, equals

$$\begin{aligned} e^{-rT}E\left[\left(\frac{1}{T}\int_0^T S_t\,dt - S_T\right)^+\right] &= e^{-rT}E\left[\left(\frac{1}{T}\int_0^T S_t\,dt - S_T\right)I(A)\right] = e^{-rT}E\left[\left(\frac{1}{T}\int_0^T (S_t - S_T)\,dt\right)I(A)\right] \\ &= e^{-rT}E\left[\frac{1}{T}\int_0^T (S_t - S_T)I(A)\,dt\right] = \frac{e^{-rT}}{T}\int_0^T E[(S_t - S_T)I(A)]\,dt \\ &\geqslant \frac{e^{-rT}}{T}\int_0^T E[(S_t - S_T)I(A')]\,dt \end{aligned}$$

for any event set A'. The reason for the last inequality is $\frac{1}{T}\int_0^T S_t \, dt \leqslant S_T$ for $\omega \in A' - A$. We will fix $A' = \left\{\omega : \frac{1}{T}\int_0^T B_t \, dt - B_T > \gamma\right\}$ for a γ to be determined later. This particular event set is chosen for the tractability it provides in deriving the desired lower-bound formula, which will turn out to be extremely accurate.

We now seek the value of γ that maximizes the lower bound

$$\frac{e^{-rT}}{T}\int_0^T E[(S_t - S_T)I(A')]\,dt = \frac{e^{-rT}}{T}\int_0^T E\left(S_t - S_T, \frac{1}{T}\int_0^T B_t\,dt - B_T > \gamma\right)dt.$$

Table 1
Comparison with the tree algorithms of Hull and White [6] and Hsu and Lyuu [8] and the PDE method of Forsyth et al. [14]

n	Hull–White	PDE	Hsu–Lyuu	Eq. (4)
Case 1: S = 100, X = 100, r = 0.1, σ = 0.1, T = 0.25				
50	1.8486	1.8478	1.8714720	–
100	1.8501	1.8492	1.9095930	–
200	1.8508	1.8503	1.8891953	–
400	1.8512	1.8509	1.8703678	–
∞	1.8516	1.8514	1.8515402	1.851588
Case 2: S = 100, X = 100, r = 0.1, σ = 0.5, T = 5				
50	28.3899	28.3573	28.3893142	–
100	28.3972	28.3842	28.3973455	–
200	28.4011	28.3952	28.4013633	–
400	28.4031	28.4003	28.4032833	–
∞	28.4051	28.4054	28.4052033	28.364100

The parameters are from Tables 3 and 4 of Forsyth et al. [14]. The numbers quoted for the Hull–White method are based on calculations using the finest grids. The parameter n denotes the number of time periods. The ∞-row lists the extrapolated option values wherever available. The exact value in Case 1 is conjectured to be 1.8515 ± 0.0001 and that in Case 2 is conjectured to be 28.40525 ± 0.00015 [14].

Theorem 7 says any random variable X with density $f_X(x)$ satisfies

$$\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E(S_t - S_T, X > \gamma)\,\mathrm{d}t = -\frac{1}{T}\int_0^T E(S_t - S_T | X = \gamma) f_X(\gamma)\,\mathrm{d}t.$$

Thus the desired value of γ (call it γ^*) must satisfy

$$\frac{1}{T}\int_0^T E(S_t|X=\gamma^*)\,\mathrm{d}t = \frac{1}{T}\int_0^T E(S_T|X=\gamma^*)\,\mathrm{d}t.$$

Given $X = \frac{1}{T}\int_0^T B_t\,\mathrm{d}t - B_T$ as dictated by our choice for A', the above identity becomes

$$\frac{1}{T}\int_0^T S\cdot\exp\left\{\frac{3(-t^2)\gamma^*\sigma}{2T^2} + \alpha t + \frac{\sigma^2}{2}\left(t - \frac{3t^4}{4T^3}\right)\right\}\mathrm{d}t = S\cdot\exp\left(\alpha t - \frac{3\sigma\gamma^*}{2} + \frac{T\sigma^2}{8}\right)$$

Table 2
Comparison with Zhang [1,15] and Hsu and Lyuu [8]

X	σ	r	Exact	AA2	AA3	Hsu–Lyuu	Eq. (4)
95	0.05	0.05	7.1777275	7.1777244	7.1777279	7.178812	7.177726
100			2.7161745	2.7161755	2.7161744	2.715613	2.716168
105			0.3372614	0.3372601	0.3372614	0.338863	0.337231
95		0.09	8.8088392	8.8088441	8.8088397	8.808717	8.808839
100			4.3082350	4.3082253	4.3082331	4.309247	4.308231
105			0.9583841	0.9583838	0.9583841	0.960068	0.958331
95		0.15	11.0940944	11.0940964	11.0940943	11.093903	11.094094
100			6.7943550	6.7943510	6.7943553	6.795678	6.794354
105			2.7444531	2.7444538	2.7444531	2.743798	2.744406
90	0.10	0.05	11.9510927	11.9509331	11.9510871	11.951610	11.951076
100			3.6413864	3.6414032	3.6413875	3.642325	3.641344
110			0.3312030	0.3312563	0.3311968	0.331348	0.331074
90		0.09	13.3851974	13.3851165	13.3852048	13.385563	13.385190
100			4.9151167	4.9151388	4.9151177	4.914254	4.915075
110			0.6302713	0.6302538	0.6302717	0.629843	0.630064
90		0.15	15.3987687	15.3988062	15.3987860	15.398885	15.398767
100			7.0277081	7.0276544	7.0277022	7.027385	7.027678
110			1.4136149	1.4136013	1.4136161	1.414953	1.413286
90	0.20	0.05	12.5959916	12.5957894	12.5959304	12.596052	12.595602
100			5.7630881	5.7631987	5.7631187	5.763664	5.762708
110			1.9898945	1.9894855	1.9899382	1.989962	1.989242
90		0.09	13.8314996	13.8307782	13.8313482	13.831604	13.831220
100			6.7773481	6.7775756	6.7773833	6.777748	6.776999
110			2.5462209	2.5459150	2.5462598	2.546397	2.545459
90		0.15	15.6417575	15.6401370	15.6414533	15.641911	15.641598
100			8.4088330	8.4091957	8.4088744	8.408966	8.408519
110			3.5556100	3.5554997	3.5556415	3.556094	3.554687
90	0.30	0.05	13.9538233	13.9555691	13.9540973	13.953937	13.952421
100			7.9456288	7.9459286	7.9458549	7.945918	7.944357
110			4.0717942	4.0702869	4.0720881	4.071945	4.070115
90		0.09	14.9839595	14.9854235	14.9841522	14.984037	14.982782
100			8.8287588	8.8294164	8.8289978	8.829033	8.827548
110			4.6967089	4.6956764	4.6969698	4.696895	4.694902
90		0.15	16.5129113	16.5133090	16.5128376	16.512963	16.512024
100			10.2098305	10.2110681	10.2101058	10.210039	10.208724
110			5.7301225	5.7296982	5.7303567	5.730357	5.728161

The parameters are from Table 1 of [1]. The "Exact"-column is from [15], and the AA2 and AA3 columns are from [1]. The options are calls with $S = 100$ and $T = 1$.

by Eq. (2) and Fact 4. This identity determines γ^* uniquely. We now have the lower bound:

$$V_{\text{floating}} \geqslant e^{-rT}\left\{\frac{1}{T}\int_0^T E\left[(Se^{\sigma B_t+\alpha t} - S_T)I\left(\frac{1}{T}\int_0^T B_s\,ds - B_T > \gamma^*\right)\right]dt\right\}. \tag{5}$$

It remains to calculate the expectation. Fix $t \in [0, T]$. Let $N_1 = \sigma B_t + \alpha t + \log S$, $N_2 = \frac{1}{T}\int_0^T B_t\,dt - B_T - \gamma^*$, $u_i = E(N_i)$, $\sigma_i^2 = \text{Var}(N_i)$, and $c = \text{Cov}(N_1, N_2)$. Note that

$$E[e^{N_1}I(N_2 > 0)] = e^{u_1+\frac{\sigma_1^2}{2}}\Phi\left(\frac{u_2 + c}{\sigma_2}\right)$$

by Theorem 5. Inequality (5) becomes

$$V_{\text{fixed}} \geqslant e^{-rT}\left\{\frac{1}{T}\int_0^T E[(e^{N_1} - S_T)I(N_2 > 0)]\,dt\right\}.$$

As $u_1 = \alpha t + \log S$, $u_2 = -\gamma^*$, $\sigma_1^2 = \sigma^2 t$, $\sigma_2^2 = T/3$, and $c = -\sigma t^2/(2T)$ according to Theorem 2, the desired lower bound obtains:

Table 3
Comparison with Zhang [1,15] and Hsu and Lyuu [8] under a wide range of volatilities

X	σ	Exact	AA2	AA3	Hsu–Lyuu	Eq. (4)
95	0.05	8.8088392	8.80884	8.80884	8.808717	8.808839
100		4.3082350	4.30823	4.30823	4.309247	4.308231
105		0.9583841	0.95838	0.95838	0.960068	0.958331
95	0.1	8.9118509	8.91171	8.91184	8.912238	8.911836
100		4.9151167	4.91514	4.91512	4.914254	4.915075
105		2.0700634	2.07006	2.07006	2.072473	2.069930
95	0.2	9.9956567	9.99597	9.99569	9.995661	9.995362
100		6.7773481	6.77758	6.77738	6.777748	6.776999
105		4.2965626	4.29643	4.29649	4.297021	4.295941
95	0.3	11.6558858	11.65747	11.65618	11.656062	11.654758
100		8.8287588	8.82942	8.82900	8.829033	8.827548
105		6.5177905	6.51763	6.51802	6.518063	6.516355
95	0.4	13.5107083	13.51426	13.51182	13.510861	13.507892
100		10.9237708	10.92507	10.92474	10.923943	10.920891
105		8.7299362	8.72936	8.73089	8.730102	8.726804
95	0.5	15.4427163	15.44890	15.44587	15.442822	15.437069
100		13.0281555	13.03015	13.03107	13.028271	13.022532
105		10.9296247	10.92800	10.93253	10.929736	10.923750
95	0.6	–	–	–	17.406402	17.396428
100		–	–	–	15.128426	15.118595
105		–	–	–	13.113874	13.103855
95	0.8	–	–	–	21.349949	21.326144
100		–	–	–	19.288780	19.265518
105		–	–	–	17.423935	17.400803
95	1.0	–	–	–	25.252051	25.205238
100		–	–	–	23.367535	23.321951
105		–	–	–	21.638238	21.593393

The parameters are from Table 2 of [1]. The options are calls with $S = 100$, $r = 0.09$, and $T = 1$.

$$V_{\text{floating}} \geqslant \frac{Se^{-rT}}{T}\left\{\int_0^T \left[e^{\alpha t+\frac{\sigma^2 t}{2}}\Phi\left(\frac{-\gamma^* - \frac{\sigma t^2}{2T}}{\sqrt{T/3}}\right) - e^{\alpha T+\frac{\sigma^2 T}{2}}\Phi\left(\frac{-\gamma^* - \frac{\sigma T}{2}}{\sqrt{T/3}}\right)\right] dt\right\}$$
$$= Se^{-rT}\left[\frac{1}{T}\int_0^T e^{\alpha t+\frac{\sigma^2 t}{2}}\Phi\left(\frac{-\gamma^* - \frac{\sigma t^2}{2T}}{\sqrt{T/3}}\right) dt - e^{\alpha T+\frac{\sigma^2 T}{2}}\Phi\left(\frac{-\gamma^* - \frac{\sigma T}{2}}{\sqrt{T/3}}\right)\right]. \tag{6}$$

4. Numerical evaluation

We proceed to confirm the accuracy of our formulas by extensive numerical experiments. Because floating-strike Asian options are equivalent to fixed-strike Asian options [32], we will limit the evaluation to fixed-strike Asian options. Tables 1–6 are for fixed-strike options, which are our focus. Table 7 considers floating-strike options. To start with, Table 1 compares formula (4) with the trinomial tree algorithm of Hull and White [6], the PDE method of Forsyth et al. [14], and the binomial tree algorithm of Hsu and Lyuu [8]. All calculated option prices are close to each other despite their being based on different methodologies.

Table 2 compares formula (4) with the methods of Zhang [1,15] and Hsu and Lyuu [8]. Our formula produces results that are never more than 0.042% away from the PDE method of Zhang [15], whose data are generally accepted to be exact. Table 3 continues the experiments of Table 2 but under a very wide range of volatilities, up to $\sigma = 100\%$. The reason for these settings is that many pricing formulas deteriorate as the

Table 4
Comparison with Ju [22], Zhang [15], and Hsu and Lyuu [8]

X	σ	Exact	TE6	Hsu–Lyuu	Eq. (4)
95	0.05	15.1162646	15.11626	15.116230	15.116264
100		11.3036080	11.30360	11.304034	11.303605
105		7.5533233	7.55335	7.554073	7.553278
95	0.1	15.2138005	15.21396	15.213921	15.213761
100		11.6376573	11.63798	11.637813	11.637525
105		8.3912219	8.39140	8.391189	8.390833
95	0.2	16.6372081	16.63942	16.637276	16.636109
100		13.7669267	13.76770	13.767043	13.765476
105		11.2198706	11.21879	11.220047	11.217842
95	0.3	19.0231619	19.02652	19.023236	19.018567
100		16.5861236	16.58509	16.586222	16.581024
105		14.3929780	14.38751	14.393083	14.387081
95	0.4	21.7409242	21.74461	21.740973	21.729124
100		19.5882516	19.58355	19.588307	19.575938
105		17.6254416	17.61269	17.625501	17.612310
95	0.5	24.5718705	24.57740	24.571913	24.547903
100		22.6307858	22.62276	22.630828	22.606509
105		20.8431853	20.82213	20.843226	20.818216
95	0.6	–	–	27.425278	27.382898
100		–	–	25.655297	25.612978
105		–	–	24.013011	23.970344
95	0.8	–	–	33.031740	32.928877
100		–	–	31.535716	31.434324
105		–	–	30.133505	30.033063
95	1.0	–	–	38.361352	38.158663
100		–	–	37.085174	36.886054
105		–	–	35.881483	35.685358

The exact values are from Table 7 of [15]. TE6 stands for Ju's Taylor expansion method. The parameters are from Table 2 of [22] and Table 7 of [15]. The options are calls with $S = 100$, $r = 0.09$, and $T = 3$.

Table 5
Comparison with the one-dimensional PDE method of Večeř [30] and the binomial tree algorithm of Hsu and Lyuu [8]

X	σ	$T=1$					$T=3$				
		Exact	PDE1	PDE2	Hsu–Lyuu	Eq. (4)	Exact	PDE1	PDE2	Hsu–Lyuu	Eq. (4)
95	0.05	8.8088392	8.8088241	8.8088241	8.808717	8.8088389	15.1162646	15.1162526	15.1162526	15.116230	15.11626440
100		4.3082350	4.3080602	4.3080602	4.309247	4.3082311	11.3036080	11.3035792	11.3035792	11.304034	11.30360450
105		0.9583841	0.9583277	0.9583277	0.960068	0.9583309	7.5533233	7.5531978	7.5531978	7.554073	7.55327778
95	0.1	8.9118509	8.9118054	8.9118054	8.912238	8.9118360	15.2138005	15.2137661	15.2137661	15.213921	15.21376080
100		4.9151167	4.9150253	4.9150253	4.914254	4.9150753	11.6376573	11.6376011	11.6376011	11.637813	11.63752500
105		2.0700634	2.0700251	2.0700251	2.072473	2.0699297	8.3912219	8.3911498	8.3911498	8.391189	8.39083318
95	0.2	9.9956567	9.9956323	9.9956323	9.995661	9.9953622	16.6372081	16.6371770	16.6371770	16.637276	16.6361089
100		6.7773481	6.7773279	6.7773279	6.777748	6.7769994	13.7669267	13.7668950	13.7668950	13.767043	13.7654757
105		4.2965626	4.2964614	4.2964614	4.297021	4.2959409	11.2198706	11.2198412	11.2198412	11.220047	11.2178420
95	0.3	11.6558858	11.6558892	11.6558892	11.656062	11.6547575	19.0231619	19.0230953	19.0231388	19.023236	19.0185667
100		8.8287588	8.8287699	8.8287699	8.829033	8.8275482	16.5861236	16.5860134	16.5861083	16.586222	16.5810236
105		6.5177905	6.5178134	6.5178134	6.518063	6.5163551	14.3929780	14.3927638	14.3929591	14.393083	14.3870805
95	0.4	13.5107083	13.5107373	13.5107373	13.510861	13.5078924	21.7409242	21.7359140	21.7409067	21.740973	21.7291244
100		10.9237708	10.9238047	10.9238049	10.923943	10.9208908	19.5882516	19.5801909	19.5882367	19.588307	19.5759378
105		8.7299362	8.7299785	8.7299789	8.730102	8.7268042	17.6254416	17.6129231	17.6254290	17.625501	17.6123103
95	0.5	15.4427163	15.4427436	15.4427631	15.442822	15.4370694	24.5718705	24.5164835	24.5718583	24.571913	24.5479028
100		13.0281555	13.0281668	13.0282104	13.028271	13.0225321	22.6307858	22.5534589	22.6307744	22.630828	22.6065085
105		10.9296247	10.9295940	10.9296853	10.929736	10.9237503	20.8431853	20.7378307	20.8431724	20.843226	20.8182163
95	0.6	–	17.4057119	17.4063840	17.406402	17.3964280	–	27.1922830	27.4252385	27.425278	27.3828984
100		–	15.1272033	15.1284092	15.128426	15.1185950	–	25.3547907	25.6552489	25.655297	25.6129780
105		–	13.1117954	13.1138637	13.113874	13.1038552	–	23.6323908	24.0129680	24.013011	23.9703437
95	0.8	–	21.3206229	21.3500057	21.349949	21.3261438	–	31.8446547	33.0316957	33.031740	32.9288767
100		–	19.2465024	19.2888389	19.288780	19.2655176	–	30.1240393	31.5356736	31.535716	31.4343240
105		–	17.3646285	17.4239955	17.423935	17.4008033	–	28.4742281	30.1334450	30.133505	30.0330626
95	1.0	–	25.0465250	25.2521580	25.252051	25.2052379	–	35.4451734	38.3595938	38.361352	38.1586628
100		–	23.1006194	23.3676388	23.367535	23.3219514	–	33.7509030	37.0830464	37.085174	36.8860543
105		–	21.2980435	21.6383464	21.638238	21.5933927	–	32.1020703	35.8789184	35.881483	35.6853580

PDE1 is based on the 100×2000 grid over $[0, T] \times [-1, 1]$. PDE2 is based on the $100 \times 10{,}000$ grid over $[0, T] \times [-1, 9]$. The parameters and numerical data for "Exact" and Hsu–Lyuu are from Tables 3 and 4. The numerical data for PDE1 and PDE2 are from [31]. The options are calls with $S = 100$ and $r = 0.09$.

Table 6
Comparison with Fusai [32], Zhang [15], and Hsu and Lyuu [8]

X	σ	Hsu–Lyuu	Fusai	Exact	Monte Carlo	Eq. (4)
95	0.05	8.808717	8.80885	8.8088392	8.81	8.808839
100		4.309246	4.30824	4.3082350	4.31	4.308231
105		0.960069	0.95839	0.9583841	0.95	0.958331
95	0.10	8.912238	8.91185	8.9118509	8.91	8.911836
100		4.914254	4.91512	4.9151167	4.91	4.915075
105		2.072473	2.07007	2.0700634	2.06	2.069930
90	0.30	14.984037	14.98396	–	14.96	14.982782
100		8.829033	8.82876	8.8287588	8.81	8.827548
110		4.696895	4.69671	–	4.68	4.694902
90	0.50	18.188933	18.18885	–	18.14	18.182957
100		13.028271	13.02816	13.0281555	12.98	13.022532
110		9.124414	9.12432	–	9.10	9.117950
90	0.60	19.964542	–	–	19.94	19.954163
100		15.128426	–	–	15.13	15.118595
110		11.342769	–	–	11.36	11.332282
90	0.80	23.622784	–	–	23.61	23.598025
100		19.288780	–	–	19.33	19.265518
110		15.739790	–	–	15.74	15.716408
90	1.00	27.305012	–	–	27.25	27.256476
100		23.367535	–	–	23.36	23.321951
110		20.051542	–	–	20.03	20.006949

The parameters and Monte Carlo results for $\sigma \leqslant 0.5$ are from Table 1 of [25]. The Monte Carlo results for $\sigma > 0.5$ are based on 2×10^6 simulation paths. The options are calls with $S = 100$, $r = 0.09$, and $T = 1$. The Hsu–Lyuu algorithm's computed option values and the exact option values are from Table 3. The "Fusai" column is from Table 3 of [32] with the most computing times.

volatility rises. The table shows our formula agrees very well with other methodologies even under high volatilities. In fact, for $\sigma \in [60\%, 100\%]$, the formula produces results that are never more than 0.21% away from what the Hsu–Lyuu algorithm generates. (The Hsu–Lyuu algorithm is known to be extremely accurate under high volatilities.) Hence the formula's performance is degraded only slightly by high volatility.

Table 4 compares formula (4) with the approximate formula of Ju [22], the PDE method of Zhang [15], and the binomial tree algorithm of Hsu and Lyuu [8]. This time, we not only let the volatility go up to 100% but also raise the maturity to 3 years. The reason for these settings is that many formulas deteriorate as the maturity increases. Again our formula agrees very well with other methodologies. In fact, even for $\sigma \in [60\%, 100\%]$, the formula produces results that are never more than 0.55% away from what the Hsu–Lyuu algorithm generates. (The Hsu–Lyuu algorithm is known to be extremely accurate for long maturities.)

Table 5 compares formula (4) with the one-dimensional PDE method of Večeř [30] as implemented by Hsu [31] and the binomial tree algorithm of Hsu and Lyuu [8]. The two algorithms are currently the fastest general pricing algorithms for Asian options. Here we let the volatility σ go up to 100% and the maturity T stand at 1 and 3 years. Table 6 compares formula (4) with Zhang [15], Monte Carlo simulation, Hsu and Lyuu [8], and the transform method of Fusai [33] with the volatility as high as 100%. Again the formula agrees very well with all the methodologies in both tables.

In summary, our formula for the fixed-strike Asian option approximates the true value very well. It deteriorates with increasing volatility and/or maturity, but only very slightly. Our formula is furthermore extremely efficient.

5. Conclusions

This paper generalizes the lower-bound pricing formulas of Rogers and Shi [23] and Thompson [25] for fixed-strike and floating-strike Asian options. Extensive numerical comparisons with other known methods in the literature confirm the extreme accuracy of our efficient formulas. This holds even under difficult situations where the maturity is long and/or the volatility is high. We conclude that the simple formulas (4) and (6) are extremely efficient and accurate in pricing Asian options. The results have practical applications in hedging such options.

Appendix

In this appendix, we will prove the theorems stated but left unproven in the main text.

Theorem 1. *The correlation matrix between* $\frac{1}{T}\int_0^T B_s\,\mathrm{d}s$ *and* B_t *equals*

$$\begin{bmatrix} \mathrm{Cov}(B_t,B_t) & \mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) \\ \mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) & \mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) \end{bmatrix} = \begin{bmatrix} t & t\left(1-\frac{t}{2T}\right) \\ t\left(1-\frac{t}{2T}\right) & \frac{T}{3} \end{bmatrix},$$

where $0 \leqslant t \leqslant T$.

Proof. First, $\mathrm{Cov}(B_t,B_t) = \mathrm{Var}(B_t) = t$ by the definition of Brownian motion. Next,

$$\begin{aligned}\mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) &= E\left(B_t\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) - E(B_t)E\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) = E\left(\frac{1}{T}\int_0^T B_tB_s\,\mathrm{d}s\right)\\ &= \frac{1}{T}\int_0^T E(B_tB_s)\,\mathrm{d}s = \frac{1}{T}\int_0^t E(B_tB_s)\,\mathrm{d}s + \frac{1}{T}\int_t^T E(B_tB_s)\,\mathrm{d}s = \frac{1}{T}\int_0^t s\,\mathrm{d}s + \frac{1}{T}\int_t^T t\,\mathrm{d}s\\ &= \frac{1}{T}\left[\frac{t^2}{2} + t(T-t)\right] = t\left(1-\frac{t}{2T}\right).\end{aligned}$$

Finally, note that

$$\mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) = \mathrm{Var}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s\right) = \frac{1}{T^2}\mathrm{Var}\left(\int_0^T B_s\,\mathrm{d}s\right).$$

It is a fact that

$$\mathrm{Var}\left[\int_a^b f'(t)(B_t - B_a)\,\mathrm{d}t\right] = \int_a^b [f(t)-f(b)]^2\,\mathrm{d}t$$

(see [34]). Hence,

$$\frac{1}{T^2}\mathrm{Var}\left(\int_0^T B_s\,\mathrm{d}s\right) = \frac{1}{T^2}\int_0^T (s-T)^2\,\mathrm{d}s = \frac{T}{3}. \qquad \square$$

Theorem 2. *The correlation matrix between* $\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T$ *and* B_t *equals*

$$\begin{bmatrix} \mathrm{Cov}(B_t,B_t) & \mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) \\ \mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) & \mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) \end{bmatrix} = \begin{bmatrix} t & \frac{-t^2}{2T} \\ \frac{-t^2}{2T} & \frac{T}{3} \end{bmatrix},$$

where $0 \leqslant t \leqslant T$.

Proof. First,

$$\begin{aligned}\mathrm{Cov}\left(B_t,\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) &= E\left[B_t\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right)\right] - E[B_t]E\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right)\\ &= E\left[B_t\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right)\right] = \frac{1}{T}\int_0^T E(B_tB_s)\,\mathrm{d}s - t = \frac{t}{T}\left(T-\frac{t}{2}\right) - t = \frac{-t^2}{2T}.\end{aligned}$$

Next,

$$\begin{aligned}\mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T, \frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) &= \mathrm{Var}\left(\frac{1}{T}\int_0^T B_s\,\mathrm{d}s - B_T\right) = \frac{1}{T^2}\mathrm{Var}\left(\int_0^T B_s\,\mathrm{d}s - B_T\right)\\ &= \frac{1}{T^2}\left[\mathrm{Var}\left(\int_0^T B_s\,\mathrm{d}s\right) + \mathrm{Var}(B_T) - 2\mathrm{Cov}\left(\int_0^T B_s\,\mathrm{d}s, B_T\right)\right]\\ &= \frac{1}{T^2}\left(\frac{T^3}{3} + T - 2\frac{T}{2}\right) = \frac{T}{3},\end{aligned}$$

where the next-to-last identity is due to Theorem 1. □

Theorem 5. *Suppose* $X \sim \phi(\mu_x, \sigma_x^2)$, $Y \sim \phi(\mu_y, \sigma_y^2)$, *and* $c = \mathrm{Cov}(X, Y)$. *Then*

$$E[\mathrm{e}^X I(Y > 0)] = \mathrm{e}^{u_x + \frac{\sigma_x^2}{2}} \Phi\left(\frac{\mu_y + c}{\sigma_y}\right).$$

Proof. From the definition of expectation,

$$E[\mathrm{e}^X I(Y > 0)] = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\mathrm{e}^x}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}}\,\mathrm{d}y\,\mathrm{d}x.$$

Let $u = (x - \mu_x)/\sigma_x$ and $v = (y - \mu_y)/\sigma_y$. Then $\mathrm{d}x\,\mathrm{d}y = \sigma_x\sigma_y\,\mathrm{d}u\,\mathrm{d}v$, and the above formula becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\mu_y/\sigma_y}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \mathrm{e}^{u\sigma_x+\mu_x}\mathrm{e}^{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}}\,\mathrm{d}v\,\mathrm{d}u &= \frac{\mathrm{e}^{\mu_x}}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\mu_y/\sigma_y}^{\infty} \mathrm{e}^{u\sigma_x}\mathrm{e}^{-\frac{(u-\rho v)^2+v^2(1-\rho^2)}{2(1-\rho^2)}}\,\mathrm{d}v\,\mathrm{d}u\\ &= \frac{\mathrm{e}^{\mu_x}}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\mu_y/\sigma_y}^{\infty} \mathrm{e}^{u\sigma_x}\mathrm{e}^{-\frac{(u-\rho v)^2}{2(1-\rho^2)}-\frac{v^2}{2}}\,\mathrm{d}v\,\mathrm{d}u.\end{aligned}$$

Change the variable again: $w = (u - \rho v)/\sqrt{1-\rho^2}$. Then $\mathrm{d}w = \mathrm{d}u/\sqrt{1-\rho^2}$, and the above formula becomes

$$\frac{\mathrm{e}^{\mu_x}}{2\pi}\int_{-\mu_y/\sigma_y}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{w^2}{2}-\frac{v^2}{2}}\cdot\mathrm{e}^{(\sqrt{1-\rho^2}w+\rho v)\sigma_x}\,\mathrm{d}w\,\mathrm{d}v = \frac{\mathrm{e}^{\mu_x+\frac{\sigma_x^2}{2}}}{\sqrt{2\pi}}\int_{-\mu_y/\sigma_y}^{\infty}\mathrm{e}^{-\frac{(v-\sigma_x\rho)^2}{2}}\,\mathrm{d}v\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{(w-\sigma_x\sqrt{1-\rho^2})^2}{2}}\,\mathrm{d}w.$$

As $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(w-\sigma_x\sqrt{1-\rho^2})^2}{2}}$ is a density function, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{(w-\sigma_x\sqrt{1-\rho^2})^2}{2}}\,\mathrm{d}w = 1$. The above equation now becomes

$$\frac{\mathrm{e}^{\mu_x+\frac{\sigma_x^2}{2}}}{\sqrt{2\pi}}\int_{-\mu_y/\sigma_y}^{\infty}\mathrm{e}^{-\frac{(v-\sigma_x\rho)^2}{2}}\,\mathrm{d}v.$$

With one more change of variable $k = v - \sigma_x\rho$, we have $\mathrm{d}k = \mathrm{d}v$, and the above equation finally becomes

$$\frac{\mathrm{e}^{\mu_x+\frac{\sigma_x^2}{2}}}{\sqrt{2\pi}}\int_{-\frac{\mu_y}{\sigma_y}-\sigma_x\rho}^{\infty}\mathrm{e}^{-\frac{k^2}{2}}\,\mathrm{d}k = \mathrm{e}^{\mu_x+\frac{\sigma_x^2}{2}}\Phi\left(\frac{\mu_y}{\sigma_y}+\sigma_x\rho\right) = \mathrm{e}^{\mu_x+\frac{\sigma_x^2}{2}}\Phi\left(\frac{\mu_y+c}{\sigma_y}\right). \qquad \square$$

Theorem 6. *For any random variable X with density function* $f_X(x)$,

$$\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E(S_t - K, X > \gamma)\,\mathrm{d}t = -\frac{1}{T}\int_0^T E(S_t - K|X = \gamma)f_X(\gamma)\,\mathrm{d}t.$$

Proof. By the definition of expectation,

$$\frac{\partial}{\partial \lambda}\frac{1}{T}\int_0^T E(S_t - K, X > \gamma)\,\mathrm{d}t = \frac{\partial}{\partial \gamma}\frac{1}{T}\int_0^T \int_{-\infty}^{\infty}\int_{\gamma}^{\infty}(S_t - K)f_{B_t,X}(B_t, X)\,\mathrm{d}X\,\mathrm{d}B_t\,\mathrm{d}t.$$

Exchange the integral and the partial derivative to get

$$\frac{1}{T}\int_0^T \int_{-\infty}^{\infty}\frac{\partial}{\partial \gamma}\int_{\gamma}^{\infty}(S_t - K)f_{B_t,X}(B_t, X)\,\mathrm{d}X\,\mathrm{d}B_t\,\mathrm{d}t.$$

Above, $f_{B_t,X}(B_t, X)$ is the joint density function of B_t and X. (Technically, the integral must be uniformly convergent for the interchange to be valid. It can be shown to be the case with our options.) By Leibniz's rule, the above equals

$$\begin{aligned}\frac{1}{T}\int_0^T \int_{-\infty}^{\infty} -(S_t - K)f_{B_t,X}(B_t, \gamma)\mathrm{d}B_t\,\mathrm{d}t &= \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} -(S_t - K)f_{B_t,X}(B_t, \gamma)\frac{f_X(\gamma)}{f_X(\gamma)}\mathrm{d}B_t\,\mathrm{d}t \\ &= \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} -(S_t - K)f_{B_t,X}(B_t|\gamma)f_X(\gamma)\mathrm{d}B_t\,\mathrm{d}t \\ &= -\frac{1}{T}\int_0^T E(S_t - K|X = \gamma)f_X(\gamma)\,\mathrm{d}t. \qquad \square\end{aligned}$$

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