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Properties of the Geometrically Averaged Ratio, an Alternative Standardized Measure

Wen-Chung Lee

In epidemiology, the comparative mortality figure and the standardized mortality ratio are standardized measures in common use. Both are weighted averages of rate ratios (or observed/expected death count ratios) on the arithmetic scale. I propose a new standardized measure, the geometrically averaged ratio (GAR), which is defined through simple averaging on the logarithmic scale. I show that, in addition to providing a valid comparison between populations, the geometrically averaged ratio possesses the following desirable properties: (1) invertibility and invariance of standardized sex ratios and (2) interpopulational comparability with different standards. (Epidemiology 1999;10:456–459)

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Standardization of rates has been a basic tool for epidemiologists.[1–3] The two commonly used standardized measures are the comparative mortality figure (CMF) and the standardized mortality ratio (SMR), although several other methods have also been proposed.[4,5]

The formulas for CMF and SMR are:

$$\mathrm{CMF} = \frac{\sum_{i=1}^{K} P_i^\star M_i}{\sum_{i=1}^{K} P_i^\star M_i^\star} = \frac{\sum_{i=1}^{K} O_i^\star \frac{M_i}{M_i^\star}}{\sum_{i=1}^{K} O_i^\star} = \frac{\sum_{i=1}^{K} O_i^\star \frac{O_i}{E_i}}{\sum_{i=1}^{K} O_i^\star} \quad (1)$$

and

$$\mathrm{SMR} = \frac{\sum_{i=1}^{K} O_i}{\sum_{i=1}^{K} E_i} = \frac{\sum_{i=1}^{K} E_i \frac{O_i}{E_i}}{\sum_{i=1}^{K} E_i} = \frac{\sum_{i=1}^{K} E_i \frac{M_i}{M_i^\star}}{\sum_{i=1}^{K} E_i}, \quad (2)$$

respectively, where the age groups are indexed by $i (i = 1,2,\ldots,K)$. The $O_i^\star$, $P_i^\star$, and $M_i^\star$ ($M_i^\star = O_i^\star/P_i^\star$) denote the death number, population number, and mortality of the ith age group in the standard population, and the O_i, P_i, and M_i ($M_i = O_i/P_i$) denote the corresponding figures in the study population (population to be age-standardized). E_i ($E_i = P_i \cdot M_i^\star$) represents the expected death count. From Eq 1 and Eq 2, we see that both the CMF and SMR are weighted averages of rate ratios [or observed/expected death count (O/E) ratios] in the arithmetic scale.

However, the weighting systems these two indices take are fundamentally different: the weightings adopted by the CMF index (the $O_i^\star$) remain unchanged as long as the same standard population is used, whereas the SMR changes its weights (the E_i) according to which population is to be age-standardized. Using an identical (unchanged) weighting system throughout has the advantage that if a population has higher rates than the corresponding rates of another population at all ages, its weighted average will also be higher. By contrast, a changing weighting system will not guarantee this dominance. This principle provides a theoretical basis for the well-known doctrine that the CMF is suitable for interpopulational comparison, whereas the SMR is not.[1–3]

The Geometrically Averaged Ratio

In this paper, I propose a new standardized measure, the geometrically averaged ratio (GAR). Using the same notations as above, the new measure can be expressed as:

$$\mathrm{GAR} = \exp\left(\frac{1}{K}\sum_{i=1}^{K} \log \frac{M_i}{M_i^\star}\right) = \left(\prod_{i=1}^{K} \frac{M_i}{M_i^\star}\right)^{\frac{1}{K}} \quad (3)$$

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TABLE 1. Population Structures and Mortality by Sex of Rural Districts in Taiwan, 1990, Age-Standardized Sex Ratios According to Standardized Measure

	Males (M)		Females (F)		Total (T)	
Age (Years)	Population	Mortality*	Population	Mortality*	Population	Mortality*
0–9	561,615	1.394	527,994	1.042	1089,609	1.223
10–19	594,374	1.235	564,246	0.519	1158,620	0.886
20–29	675,570	2.032	596,149	0.882	1271,719	1.493
30–39	528,865	3.116	442,094	1.348	970,959	2.311
40–49	298,447	5.894	269,129	2.423	567,576	4.248
50–59	288,330	10.970	257,938	5.470	546,268	8.373
60–69	247,340	23.122	184,812	15.102	432,152	19.692
70+	125,424	74.515	128,698	66.769	254,122	70.592

	Age-Standardized Sex Ratios			
	M/F	Inverse of M/F	F/M	(M/T)/(F/T)
SMR†	1.476	0.678	0.691	1.457
CMF†	1.447	0.691	0.677	1.462
GAR†	1.857	0.539	0.539	1.857

* Per 1,000 population.
† SMR = standardized mortality ratio; CMF = comparative mortality figure; GAR = geometrically averaged ratio.

$$= \frac{\left(\prod_{i=1}^{K} M_i\right)^{\frac{1}{K}}}{\left(\prod_{i=1}^{K} M_i^*\right)^{\frac{1}{K}}} \quad (4)$$

$$= \left(\prod_{i=1}^{K} \frac{O_i}{E_i}\right)^{\frac{1}{K}}. \quad (5)$$

Eq 3 shows that the GAR is an unweighted geometric average of the age-specific rate ratios (the exponential of the simple arithmetic average on the logarithmic scale). It can also be expressed as a ratio of two geometrically averaged rates (both unweighted) (Eq 4) or alternatively as an unweighted geometric average of the O/E ratios (Eq 5).

Because the GAR adopts an unchanged weighting system, it can provide a valid comparison between populations. Moreover, the following examples show that the GAR possesses some additional desirable properties.

Example 1: Invertibility and Invariance of Standardized Sex Ratios

Table 1 shows the population size and mortality by sex for rural districts in Taiwan in 1990. We compare the mortality between males and females by calculating the age-standardized mortality sex ratio. Treating the females (F) as the referent, Table 1 gives the CMF, SMR, and GAR (labeled M/F).

Suppose we wish to present the results as a ratio of females to males instead. Can we simply invert the male-to-female sex ratios? Or should we go over the calculations again with males this time taken as the referent? As Table 1 shows, the GAR does permit a simple inversion. By contrast, neither the SMR nor the CMF possesses this invertibility property.

Next, we examine the consequences of using the total rural population (T) as the reference. Using this standard, we can calculate the ratio for males (M/T) and the ratio for females (F/T). The ratio of these two indices, (M/T)/(F/T), defines yet another male-to-female sex ratio (Table 1). Comparing the sex ratio M/F and the sex ratio (M/T)/(F/T), we find that the GAR possesses an invariance property such that its value will not change [M/F = (M/T)/(F/T)] because the referent group has been changed.

Example 2: Interpopulational Comparability with Different Standards

Suppose that a certain rural township in Taiwan is exposed to an environmental hazard, such as polluted underground waters, radiation, etc. Men and women in the township received equal amount of exposure. The exposure, however, has differential impacts on both sexes. Table 2 presents the observed (O) and expected (E) deaths of this township by sex and age. The expected deaths for males (m) and females (f) in this township are derived using mortality rates of the rural Taiwanese males (M) and females (F) (Table 1), respectively (here we assume that the particular exposure is unique to the township such that using the general population rates to derive the expected deaths will not produce bias[6]).

From Table 2, one finds that the O/E ratios in men are greater than the corresponding ratios in women for every age category, indicating that men are more susceptible to the exposure. Nevertheless, standardization using SMR and CMF reveals otherwise. Both erroneously report that women are more susceptible (compare m/M and f/F in Table 2). On the other hand, the GAR summarizes the data without distortion (2.987 > 2.817).

TABLE 2. Population Structures and Mortality of a Hypothetical Exposed Population in Taiwan*

Age (Years)	Males (m)				Females (f)			
	Population	O	E	O/E	Population	O	E	O/E
0–9	3,000	14	4.2	3.3	5,000	16	5.2	3.1
10–19	5,000	15	6.2	2.4	5,000	6	2.6	2.3
20–29	5,000	24	10.2	2.4	4,000	8	3.5	2.3
30–39	5,000	34	15.6	2.2	4,000	11	5.4	2.0
40–49	5,000	73	29.5	2.5	3,000	17	7.3	2.3
50–59	1,000	33	11.0	3.0	2,000	31	10.9	2.8
60–69	500	47	11.6	4.1	2,000	115	30.2	3.8
70+	500	186	37.3	5.0	1,000	320	66.8	4.8

	Exposure Impact	
	m/M	f/F
SMR	3.398	3.972
CMF	3.875	4.030
GAR	2.987	2.817

* O = observed death count; E = expected death count, using the rural Taiwanese male (M) and female (F) populations (see Table 1) as the standard, respectively; SMR = standardized mortality ratio; CMF = comparative mortality figure; GAR = geometrically averaged ratio.

Confidence Intervals for Geometrically Averaged Ratio

The approximate confidence intervals of the GAR can be derived using a Taylor series expansion.[7] Assuming that the population and mortality of the standard population are error-free, the 95% confidence intervals of the GAR can be expressed as:

$$95\% \text{ CI of GAR} = \exp\left(\frac{1}{K}\sum_{i=1}^{K}\log\frac{O_i}{E_i} \pm \frac{1.96}{K}\sqrt{\sum_{i=1}^{K}\frac{1}{O_i}}\right). \quad (6)$$

If the random error in the standard is not negligible, the formula becomes:

$$95\% \text{ CI of GAR} = \exp\left(\frac{1}{K}\sum_{i=1}^{K}\log\frac{O_i}{E_i} \pm \frac{1.96}{K}\sqrt{\sum_{i=1}^{K}\left(\frac{1}{O_i}+\frac{1}{O_i^*}\right)}\right). \quad (7)$$

These formulas are accurate only when the O_is (and O_i^*s) are sufficiently large. If some of them are small (say, <5), one should use the bootstrap approach[8] instead.

Table 3 presents the data of respiratory cancer deaths and person-years of follow-up among the cohort of Montana smelter workers, cross-classified by age and employment year (taken from Appendix V in Reference 2). Using the age- and period-specific respiratory cancer mortality rates in the United States[2,Table 3.2] as the standard rate, we calculate the expected death and the O/E ratio for each cell. The results are then summarized by the GAR index for workers employed before 1925 as well as for those employed after 1925 (Table 3). The approximate and bootstrap confidence intervals are also presented (neglecting the error in the standard). It can be seen that for post-1925 workers, the approximation based on Taylor series expansion is good, whereas for pre-1925 workers (one cell containing only four deaths), the approximation is not satisfactory. For a comparison, the SMR is 3.64 (3.03–4.37) for pre-1925 workers and 1.64 (1.40–1.91) for post-1925 workers.

Discussion

The properties of invertibility and invariance of standardized measure have already been discussed (the

TABLE 3. Respiratory Cancer Deaths among Montana Smelter Workers, 1938–1977*

Age (Years)	Workers Employed before 1925				Workers Employed in 1925 or Later			
	Person-Years	O	E	O/E	Person-Years	O	E	O/E
40–49	5,793.8	4	0.96	4.17	43,568.9	17	11.70	1.45
50–59	8,538.8	21	5.77	3.64	33,272.5	59	34.37	1.72
60–69	8,711.1	56	13.57	4.13	15,682.6	61	37.65	1.62
70–79	5,127.2	34	11.32	3.00	4,296.2	24	14.73	1.63

	Estimate	Approximate CI	Bootstrap CI	Estimate	Approximate CI	Bootstrap CI
GAR	3.70	2.78–4.94	2.33–4.66	1.60	1.34–1.92	1.29–1.87

* O = observed death count; E = expected death count, based on age- and period-specific respiratory cancer mortality rates in the U.S.; GAR = geometrically averaged ratio; approximate CI = confidence intervals based on Taylor series expansion; bootstrap CI = confidence intervals based on 10,000 bootstrap simulations.

invariance property under the alternative name, Yule's criterion).[4,5] These properties are helpful when no "natural" standard exists (for example, for the sex-ratio problem, should the standard be males, females, or totals?). These properties may offer new insights about what constitutes a proper "grand" standard (the "old" world standard population, or perhaps a "new" world standard[9]). Interpopulational comparability with different standards is helpful when one wishes to compare the "relative" impact of a particular exposure on different populations, each with a different background risk. The differential susceptibility between two sexes as demonstrated in this paper is one such example. In situations such as these, one should consider using the GAR as a standardized measure.

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