台灣管理學刊
第4卷 第2期，2004年8月 pp.123-140

Enhancing the Computational Efficiency for the Monte-Carlo Simulation Approach[1]

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ABSTRACT

Before 1993, there were only few papers using the Monte Carlo simulation approach to value American options. Since then, a number of articles developed alternative computational skills for the Monte Carlo simulation to value these options. Recently, Grant, Vora and Weeks (1996) successfully developed a technique which can simply and directly determine "whether early exercise is optimal or not for American options when a particular asset value is reached at a given time using the Monte Carlo approach ". In this paper we first use the Geske and Johnson (1984) method to improve the computational efficiency for the Grant, Vora and Weeks method for valuing plain vanilla American options. We then extend our computational algorithm to the case of American options on maximum or minimum of two risky assets, whose prices are jointly lognormal distributions. We also show how to calculate the hedge ratios using the Monte Carlo simulations. Furthermore, we investigate how the key parameters affect the values of options on maximum or minimum of two risky assets.

Keywords: Monte Carlo simulation approach, American options, Values of options on maximum or minimum of two risky assets

[1] The earlier version of this paper was presented at the Seventh Conference On Theories and Practices of Security and Financial Markets, the 1999 Financial Management Annual Conference, and the 2000 NTU international conference on Finance. We are especially grateful to Gang Shyy and Her-Jiun Sheu for helpful comments and discussions.

I. Introduction

There are now an increasing number of important security models where analytical solutions are not available. These complex options require a flexible valuation method accommodating multiple assets, multiple types of uncertainty, and path dependence, including American-style options. Monte Carlo simulation is an inherently flexible valuation method and has been used widely to value complex European-style options, but not American-style options, even since Boyle (1977) first introduced Monte Carlo simulation as a numerical method to value contingent claims.

Despite the popularity of the Monte Carlo simulation approach, the valuation of early-exercise features remains a tough problem in many important settings, particularly for multifactor models[2] (such as the development of natural resources (Brennan and Schwartz (1995), the adoption of technological innovation (Grenadier and Weiss (1994), etc.). Hence, it is important to develop an efficient computational algorithm for the Monte Carlo simulations to value American options on several state variables.

Tilly (1993) recently succeeded in developing a procedure for incorporating an early exercise in a Monte Carlo simulation. His method needs storage of the paths followed by the underlying asset prices, their ranking, and their re-ranking at each possible early-exercise date. He also illustrated his method for valuing a plain vanilla American put option. In his paper, Tilly reported, but did not demonstrate, that it was possible to handle two or more state variables influencing the early-exercise decisions.

Barraquand and Martineau (1995) subsequently proposed an approach that could track the conditional probabilities of path-specific outcomes in a Monte Carlo simulation. They used these values to make early-exercise decisions and to value American put options on one, three, and ten underlying homogenous asset prices. Writing the put options on the maximum of multiple assets rapidly reduces the relevant state space when the initial asset prices are close to the exercise prices. This can simplify the problem, because it narrows the state space that the users must examine.

Broadie and Glasserman (1996) also developed a simulation algorithm which produces two estimators for the true option value, one biased high and the other biased low, and both asymptotically unbiased as the number of simulations tends to infinity. By convergence principle, these two estimates provide a conservative confidence interval for the option values. In their paper, they achieve a high degree of accuracy using 100 iterations of a three-period tree (two early exercise dates) and 50 branch nodes.

Grant, Vora and Weeks (1996)[3] (hereafter GVW) more recently demonstrated how to simply and directly determine “whether early exercise is optimal when a particular price is reached at a particular time." Hence, their model is able to

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[2] For excellent survey papers on the recent developments of the Monte Carlo simulation approach, please refer to Boyle, Broadie and Glasserman (1997), and Broadie and Glasserman (1997).

[3] In another paper, they extended their approach to develop a computational algorithm for calculating American-style Asian options.

incorporate the early-exercise feature into Monte Carlo simulations in such a way that it is readily extendible to value very complex options. They showed that their approach could accurately value plain vanilla American options and also argued that it is easy to extend their approach to value American options depending on multiple assets, but they did not show how to do it.

The contributions in this paper are threefold. We first use the Geske and Johnson (1984) approach to improve the computational efficiency of the GVW method for valuing plain vanilla American options. Secondly, we extend our computational algorithm to the case of American options on the maximum or minimum of two risky assets[4] , whose prices are jointly lognormal distributions. Our approach can thus be viewed as an alternative of Broadie and Glasserman (1996) method. Thirdly, practitioners are more concerned about hedging, hence we also illustrate how to calculate the hedge ratios using Monte Carlo simulations in this paper.

Our paper continues in five sections. Section 2 describes the basic ideas of Monte Carlo simulations and briefly reviews the GVW algorithm. Section 3 illustrates how to incorporate the Geske and Johnson (1984) approach into the GVW method to improve computational efficiency for the case of plain vanilla American options. We then extend the technique to the case of options on the maximum or minimum of two risky assets whose payoff follows jointly lognormal distribution. We also describe how to use the Monte Carlo simulation approach to calculate the option's hedge ratio. Section 4 presents and analyzes the numerical results. We then draw conclusions in Section 5.

II. Literature Review

2.1 Monte-Carlo Simulations

Boyle (1977) introduced Monte Carlo simulation as a numerical method to value contingent claims and we summarize the procedures for implementing that simulation approach as follows. We first simulate sample paths of the underlying state variables over the relevant time horizon in a risk-neutral world. Secondly, we evaluate the discounted cash flow of a security on each sample path, as determined by the structure of the security in question. Thirdly, we calculate the average value of the discounted cash flow over sample paths. Hence, we obtain the desired values of contingent claims. Bascially, Boyle's Monte-Carlo Technique can only value European options.

Grant, Vora and Weeks (1996) extended the Boyle's Monte-Carlo simulation methods to value American options. Suppose that the value of a derivative depends on a non-dividend paying underlying asset. Its price S follows a standard geometric Brownian Motion in a risk-neutral world given as

$$\frac{dS}{S} = rdt + \sigma dz, \tag{1}$$

where r, σ and dZ are the risk-free interest rate, the instantaneous standard deviation of the rate of return, and the standard Brownian Motion process, respectively. Applying Ito's lemma, we rewrite equation (1) as follows:

[4] The extension of our approach to the case of options on several assets is straightforward.

$$d \ln S = (r - \sigma^2 / 2)dt + \sigma dZ. \tag{2}$$

Equation (2) illustrates that the change of ln *S* during an interval of time (e.g. *t* to *T*) follows a normal distribution with the following property:

$$\ln S_T - \ln S_t \sim \Phi\left[(r - \sigma^2 / 2)(T - t), \sigma\sqrt{T - t}\right] \tag{3}$$

where $\Phi[m, s]$ is the normal distribution with mean *m* and standard deviation *s*. It follows that

$$S_T = S_t \cdot \exp\left[(r - \sigma^2 / 2)(T - t) + \sigma\sqrt{T - t} \cdot \varepsilon\right], \tag{4}$$

where ε is a random variable drawn from a standardized normal distribution. Employing equation (4) and drawing a sample from a standard normal distribution, we can compute the stock prices at time *T*.

Grant, Vora and Weeks (1996) attempted to identify the critical price $S^*_{t_i}$ at selected instants t_i, i =1, 2, ... , N - 1, between the current time *t* and expiration time *T*. The determination of the critical price $S^*_{t_i}$ is done by simulation at successive time steps proceeding backward in time. Once the critical price $S^*_{t_i}$ is identified, the value of a derivative can be computed by the usual simulation procedures, respecting the early exercise strategy as dictated by the known exercise boundary.

We now illustrate the procedures using an American option depending on one underlying asset as an example. At any time *t*, the value of the American put is P_t, and to calculate the value of an American put option it is necessary to identify the critical price $S^*_{t_i}$ at all dates between *t* and *T*. We state the valuation problem formally as follows:

$$P_t(S^*_t) = \max_{\{S_t\}} Q_t[S_t], \tag{5}$$

where $Q_t = X - S_t$ when early exercise is optimal, i.e., $S_t < S^*_t$ or $Q_t = E\left[P_{t+\tau}(S^*_{t+\tau})\right]e^{-r\tau}$ when holding the option is optimal, i.e., $S_t \geq S^*_t$. $E[*]$ is the expectation operator and τ is an arbitrarily small unit of time.

As mentioned above, the early-exercise decision at each date depends on the knowledge of the optimal early-exercise decisions at all future dates. We must thus employ the backward recursion of dynamic programming, beginning with the terminal condition. At time *T*, it is optimal to exercise if the put is in-the-money, when its value is $P_T = \max(0, X - S_T)$, i.e., $S^*_T = X$. The optimization process begins at the last date before the put expires, $T - \tau$. The holder of the put option can exercise early or hold until the expiration of the option. At time $T - \tau$, the value of the put is $X - S_{T-\tau}$ for $S_{T-\tau} < S^*_{T-\tau}$ or $E[P_T]e^{-r\tau}$ for $S_{T-\tau} \geq S^*_{T-\tau}$.

We now identify the critical price by finding the price for which $X - S^*_{T-\tau} = E[P_T]e^{-r\tau}$. Once the critical price at each date has been identified, we estimate the value of the option through a simulation initiated at time 0, for the appropriate initial conditions. Early exercise occurs on the first date when the stock price falls below the critical price. Apart from a pure diffusion process, GVW also showed how to calculate the American put option value under the assumption that the underlying asset follows a jump-diffusion process. Since the computational procedures of a jump-diffusion process are similar to those of a pure diffusion process, we do not review them.

2.2 Richardson Extrapolation Techniques

In an important contribution, Geske and Johnson (1984) showed that it was possible to value an American-style option by using a series of options exercisable at one of a finite number of exercise points. They employed Richardson extrapolation techniques to derive an efficient computational formula using the values of Bermuda options. The Richardson extrapolation techniques were afterwards used to enhance the computational efficiency and/or accuracy of American option pricing. For example, Bunch and Johnson (1992) argue that there is nothing in the Geske and Johnson's approach that require the exercise points to be equally spaced. Therefore, they propose a model that has the optimal exercise points, i.e. to choose the exercise points as a wealth maximizer would. Hence they can only use the maximum value of once and twice exercisable option to value American options.

However, as pointed out by Omberg (1987), there may in the case of some options be the problem of non-uniform convergence. A plausible example of a non-uniform convergence is a deep-in-the-money put option written on a low volatility, high dividend stock going ex-dividend once during the term of the option at time $T/2$ (T is the time to maturity of the option). Chang, Chung and Stapleton (2001) show how to solve the problem of non-uniform convergence encountered in the original Geske and Johnson (1984) model. They make the exercise points of the n-point exercisable options cover those exercise points of the m-point exercisable options where n is great than m. However, in general, the non-uniform convergence problem is not very serious for most of the cases.

III. The Extended Grant, Vora and Weeks Method

3.1 The Case of Plain Vanilla American Put Options

As mentioned earlier, GVW tactfully incorporated a backward recursion approach into the traditional Monte Carlo simulation method to value American put options. However, they had to consider the number of early-exercise dates over the range from 12 to 20 to get accurate approximated values of American put options. To calculate a plain vanilla America put option, the CPU time of their approach with 20 early-exercise dates, and 1,000 iterations to estimate the critical prices, and 200,000 iterations to estimate the put value, is about 2 minutes and 10 seconds. The computational time of their approach is relatively slow to other numerical methods, such as the binomial method. Apparently, their approach has limitations on the applications to the case of options depending on more than one underlying asset. In this section, we will show how to employ the Geske and Johnson (1984) approach to improve the computational efficiency of the GVW method.

In their original paper, Geske and Johnson (1984) showed that an American put option can be calculated with a high degree of accuracy using a Richardson extrapolation approach. If $P(n)$ is the price of a Mid-Atlantic option exercisable at one of n equally spaced dates, then, for example, using $P(1)$, $P(2)$ and, $P(3)$, the price of American put is approximately

$$\hat{P} = P(3) + \frac{7}{2}(P(3) - p(2)) - \frac{1}{2}(P(2) - P(1)). \tag{6}$$

Term $\hat{P}$ denotes the approximated value of the American put option, $P(1)$ denotes the value of a European option, $P(2)$ denotes the value of a Mid-Atlantic option permitting exercise at time $T/2$ or T, and $P(3)$ denotes the value of a Mid-Atlantic option permitting exercise at time $T/3$, $2T/3$, or T.

Incorporating the Geske and Johnson (1984) approach into the GVW method, we do not have to consider 12 to 20 early-exercise dates to estimate the put values. In fact, we only have to use the closed-form solutions for European options to calculate $P(1)$ and the GVW computational technique to compute $P(2)$ and, $P(3)$. Hence, using equation (6), we can obtain the approximated American put values with less computational efforts.

To improve the computational efficiency of the GVW method in another dimension, we use the closed-form solutions for European options to determine the critical prices for $P(2)$ and $P(3)$ at time $T/2$ and $2T/3$. The reason behind this application is that the twice (thrice) exercisable option becomes a pure European option at time $T/2(2T/3)$. The advantages in using closed-form solutions to calculate the critical prices, instead of using 1000 paths in the GVW paper, are twofold. First, we can obtain more accurate critical prices. Second, we can save a lot of time to compute the critical prices.

3.2 The Case of Options on the Maximum or Minimum of Two Risky Assets:

In this subsection we extend the computational algorithm developed in the previous subsection to the case of options on the maximum or minimum of two risky assets. There are four types of maximum and minimum options on n-risky assets. We summarize the payoff of these options as follows:

1. The payoff function of a maximum call option :
   $Max[0, Max(S_{1T}, S_{2T},..., S_{nT}) - K]$
2. The payoff function of a maximum put option:
   $Max[0, K - Max(S_{1T}, S_{2T},..., S_{nT})]$
3. The payoff function of a minimum call option:
   $Max[0, Min(S_{1T}, S_{2T},..., S_{nT}) - K]$
4. The payoff function of a minimum put option:
   $Max[0, K - Min(S_{1T}, S_{2T},..., S_{nT})]$

We use the maximum call option and minimum put option as examples to show how to extend the GVW and the extended GVW models to the case of options depending on two correlated state variables. It is noted that the critical price in a one-state variable case is a "point" while the critical prices of two-state variable case are trajectories expanded by the two state variables.

Consider an American minimum put option depending on two underlying stocks, S_1 and S_2, and strike price X. Both underlying stock prices follow a geometric Brownian Motion in a risk-neutral world as shown in equations (7) and (8),

$$dS_2 / S_2 = rdt + \sigma_1 dZ_1 \quad (7)$$

$$dS_2 / S_2 = rdt + \sigma_2 dZ_2 \quad (8)$$

where $\sigma_1(\sigma_2)$ denotes the instantaneous volatility of the rate of return of $S_1(S_2)$, and dZ_1 and dZ_2 are the standard Brownian Motion, respectively.

Furthermore, the correlation between dZ_1 and dZ_2 is as follows:

$$(dZ_1)(dZ_2) = \rho_{12}dt . \tag{9}$$

The magnitudes of correlation coefficients affect the trajectories of critical prices.

Based on the study of Tan and Vetzal(1995), the critical price space of the two risky assets has the property as shown in Figure 1. The region "A" in Figure A indicates that early exercise is optimal, while the region "B" in Figure 1 suggests that the holders of the options should keep the option alive.

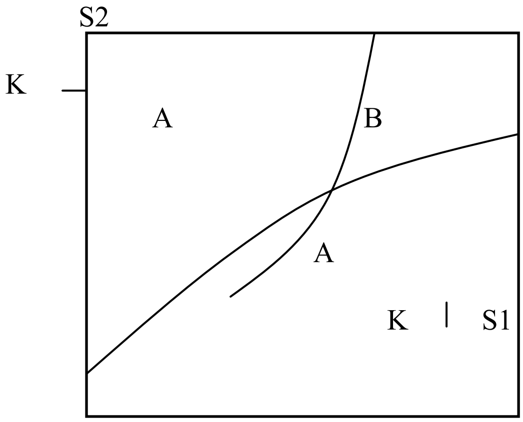

Figure 1 The regions of Early Exercise for the Minimum Put Options

In our computational algorithm, we divided the S_1-S_2 space into several small square regions as shown in Figure 2. For the purpose of analyzing early exercise decisions, we must draw the trajectories of critical prices on the space S_1-S_2. In Figure 2, if the option's holding value is less than its early exercise value at point "a", while the holding value of the option is greater than its early exercise value at points "b" and "c", we can judge that the critical price trajectories should allocate between lines ab and ac. We then use linear interpolation to allocate two critical prices — one between "a" and "b", and the other between "a" and "c"[5]. Using the same procedures, we are able to trace the critical price trajectories for American options on the maximum or minimum of two risky assets.

[5] A numerical example for calculating the critical prices of the option is referred to Table 3.

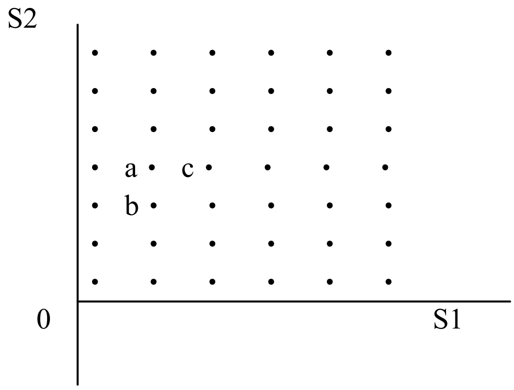

Figure 2 The Small Square Regions in two-dimension for Early Exercise Decisions

The calculations of critical price trajectories for other subintervals follow the same procedures and steps as mentioned above. The first step is to compute the critical price trajectories for one period away from the time to maturity of the options. We then move forward to compute the critical price trajectories for two periods away from the time to maturity of the options, and so on. Hence, the whole critical price trajectories can be done by simulation at successive steps proceeding backward in time. Once all the critical price trajectories of each subintervals are decided, we will know whether early exercise is optimal or not at any time interval.

3.3 Hedge ratios

Option traders are interested in prices as well as hedge parameters such as delta, gamma, etc. These "Greeks" are used to evaluate and manage the risks of their option books. Usually the hedge parameters are approximated by a numerical differentiation when closed-form solutions are not available. For example, the delta and gamma are approximated by

$$\delta = \frac{C(S+h) - C(S-h)}{2h} \tag{10}$$

where $C(S+h)$, $C(S)$, and $C(S-h)$ are the option values obtained from some numerical method (e.g. binomial tree method) with initial stock price $S+h$, S, and $S-h$, respectively.

When a Monte Carlo simulation is applied to estimate the hedge parameters, it is advised that the option values $C(S+h)$, $C(S)$, and $C(S-h)$ are calculated using the same paths. The reason behind this advice is follows. If the option values are calculated using different paths, it is very likely that one value (let's say $C(S+h)$) is overestimated and one value ($C(S-h)$) is underestimated. In this case the numerical delta from equation (10) will be overestimated a lot (especially when h is very small) and may be higher than 1. Therefore, we use the same paths to calculate the option values $C(S+h)$, $C(S)$, and $C(S-h)$, and substitute them into equation (10) to obtain the hedge parameters.

Pelsser and Vorst (1994) showed that contrary to intuition when working with numerical derivatives (where a small h is preferred), it is suggested that h should be large enough. Otherwise, the numerical deltas will be a piecewise function of the initial stock prices and hence produce nonsense gamma[6]. Following their suggestion, we set h equal to \$1% of S to avoid the piecewise problem.

IV. Numerical Results

4.1 The Computational Accuracy and Efficiency of the Extended GVW Method

We first carry out simulations to show the accuracy of the extended GVW approach for the case of plain vanilla American options. Table 1 shows that the extended GVW approach outperforms the GVW approach with 20 exercisable points using the CRR binomial method with 10,000 time steps as a benchmark. The pricing error of the extended GVW approach is on average less than 1 percent. Furthermore, we address the issue of the relative speed of estimations using the GVW approach and the extended GVW approach. For this experiment, we use a C^{++} programming language running on a Pentium II 233. The CPU time of the GVW approach with 20 early-exercise dates, 1,000 iterations to estimate the critical prices, and 200,000 iterations to estimate the put value, is about 181 seconds. On the other hand, the CPU time of the extended GVW approach with 200,000 iterations to estimate the put value is only 39 seconds. Hence, the extended GVW approach is 4.5 times faster than the GVW approach.

Table 1 American Put Option Prices using different methods

Parameters	(1) GVW method	(2) Extended GVW Method	(3) CRR Binomial Model
X = 90	4.9612	5.0737	4.9968
95	6.8572	6.9133	6.9148
100	9.1716	9.2250	9.2186
105	11.8223	11.8067	11.9069
110	14.8747	14.9809	14.9673
r = 6%	9.9196	9.8928	9.9450
8%	9.5215	9.6591	9.5709
10%	9.1716	9.2250	9.2188
12%	8.8204	8.8814	8.8864
14%	8.5116	8.5511	8.5721
σ = 0.5	11.8227	11.9490	11.9042
0.4	9.1716	9.2250	9.2188
0.3	6.5130	6.5600	6.5458
0.2	3.8813	3.9248	3.9185
0.1	1.4175	1.4937	1.4519

Note: In Table 1, S equals 100 and T equals 0.5 year. Columns 1 to 3 represents the American put option value using the Grant, Vora and Weeks method, the extended Grant, Vora and Weeks method, and Cox, Ross and Rubinstein (1979) binomial method with 10,000 time steps respectively.

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[6] Chung and Shackleton (2000) show that when the perturbation \$h\$ is arbitrarily small, the (numerical) probability of finishing in the money for an option with initial stock price S+h is the same as that for an option with initial stock price S-h. As a result, the numerical delta is a piecewise function of S. From their proof, it is straightforward to conclude that the piecewise problem is more severe when the number of paths is fewer for the Monte- Carlo simulation method.

We now demonstrate the computational accuracy and efficiency of the GVW and the extended GVW methods for the case of American options on the maximum or minimum of two risky assets. Table 2 shows that the extended GVW model on average outperforms the GVW model for valuing American put options on the minimum of two risky assets using Boyle (1988) trinomial method as a benchmark. We also illustrate the relative speed of estimations using the GVW approach and the extended GVW approach. For this experiment, we also use a C^{++} programming language running on a Pentium II 233. The CPU time of the GVW approach with 20 early-exercise dates, 1,000 iterations to estimate the critical prices, and 30,000 iterations to estimate the put value, is about 5,862 seconds. The CPU time, however, of the extended GVW approach with 30,000 iterations to estimate the put value is 19 seconds. Hence, the extended GVW approach is about 308 times faster than the GVW approach.

Table 2 American Put Option on the Minimum of Two assets

Strike Price	GVW Method	The Extended GVW Method	Boyle Trinomial Method (50 stages)
X = 35	1.397	1.389	1.423
40	3.808	3.862	3.892
45	7.566	7.728	7.689

Note: In Table 2, we assume that S_1=S_2=40, σ_1 =0.2, σ_2 =0.3, ρ =0.5,r =0.05,T =7 months. We use the values computed by Boyle (1988) model with 50 time steps as benchmarks.

4.2 The critical price trajectories

In this subsection we first show how to generate the critical price trajectories for the American call and put options on the maximum or minimum of two risky assets. We then use these critical price trajectories to compute the option values. For demonstration purpose, we only use the American call option on the maximum and American put option on Minimum of two stocks as examples.

From the previous section, we know that in order to value the American options on the maximum or minimum of two risky assets, we have to trace the critical price trajectories in a two-dimension space. Hence, when we divide the early exercise and holding regions, we have to fix one of the two risky assets, say S_1, at a certain price. We then initiate a set of serial simulations, each with a different initial price of S_2, over a certain range. Comparing the holding value with the option's early exercise value, we can obtain one point of the trajectories of the critical prices. Repeating the above processes, we can trace the whole critical trajectory for each sub-period.

Table 3 shows the case of American put options on the minimum of two-risky assets. When the option's time to maturity is 0.25 years, S_1=100 and $100 \le S_2 \le 120$, the immediate exercise value of the option is zero. However, the holding value of the option is greater than zero, and hence the option holders should keep this option alive. When S_1 equals 120 and S_2 equals 96, the early exercise value of the option equals 4, while the holding value is 5.3404. Therefore, the option holders should still keep this option alive. In the case of S_1=120, S_2=92, the early exercise value of the option equals 8, which is greater than the holding value, 7.9829. In this case, the option holder should early exercise the option. From the above demonstrations, we find that the critical price for S_2 will allocate between 96 and 92, given S_1=120. The exact critical price is 92.0503. Repeating the above algorithm and procedures on the dimension of (S_1, S_2), we can trace the critical price trajectories as shown in Figure 1.

Table 3 The Critical Price Sets for American Put Options on Minimum Of Two Risky Assets

S_1 \ S_2	120	116	112	108	104	100	96	92	88	84	80
120		0.3213	0.5835	1.0884	1.9722	3.3695	5.3923	8.0663	11.3050	14.9203	18.7697
116	0.2990	0.4260	0.6802	1.1736	2.0399	3.4157	5.4181	8.0771	11.3099	14.9223	18.7708
112	0.5527	0.6733	0.9055	1.3759	2.2090	3.5338	5.4948	8.1173	11.3260	14.9296	18.7745
108	1.0454	1.1512	1.3616	1.7794	2.5509	3.8002	5.6723	8.2240	11.3771	14.9490	18.7822
104	1.9249	2.0128	2.1889	2.5473	3.2118	4.3374	6.0614	8.4725	11.5093	15.0089	18.8025
100	3.3269	3.3900	3.5276	3.8091	4.3508	5.2936	6.7955	8.9538	**11.7943**	15.1475	18.8612
96	**5.3404**	5.3785	5.4706	5.6731	6.0749	6.8014	8.0122	9.8321	**12.3338**	**15.4420**	18.9887
92	**7.9829**	**8.0050**	**8.0540**	**8.1784**	**8.4505**	8.9476	9.8336	11.2407	13.3011	**16.0066**	19.2713
88	11.1812	**11.1916**	**11.2160**	**11.2762**	**11.4294**	**11.7424**	**12.3150**	13.2905	14.8243	17.0026	**19.8223**
84	14.7888	14.7913	14.8008	14.8282	14.8971	**15.0564**	**15.3836**	**15.9834**	**16.9967**	18.5692	**20.8065**
80	18.6375	18.6382	18.6400	18.6466	18.6744	18.7375	18.8862	**19.2035**	**19.7918**	20.7944	22.3735
76		22.5941	22.5944	22.5954	22.5996	22.6186	22.6736	22.8033	23.0925	**23.6489**	**24.6242**
72	26.5933	26.5933	26.5933	26.5933	26.5933	26.5962	26.6076	26.6512	26.7569	**27.0109**	**27.5249**
68	30.6021	30.6021	30.6021	30.6021	30.6021	30.6024	30.6035	30.6095	30.6407	30.7240	30.9404
Point of critical price trajectory	**(92.05,120)**	**(91.97,116)**	**(91.74,112)**	**(91.21,108)**	**(90.23,104)**	**(88.85,100)**	**(86.64,96)**	**(84.05,92)**	**(98.47,88)** **(80.69,88)**	**(92.05,84)** **(77.23,84)**	**(87.28,80)** **(73.73,80)**

Note: We assume that S1=S2=100; the strike price K=$100; the volatility of stock price $\sigma_1 = \sigma_2$ =0.2; the correlation coefficient of two stock prices, ρ =0.3; the risk-free interest rate r=0.05; the time to maturity of the option, T=0.25 year.

4.3 Sensitivity analysis

In this subsection we carry out simulations by changing parameters to detect how the key factors affect the values of options. The benchmark parameters are the following: The initial stock price S_1=S_2=100; the strike price K=$100; the volatility of stock price $\sigma_1 = \sigma_2 = 0.2$; the correlation coefficient of two stock prices $\rho = 0.3$; the risk-free interest rate r = 0.05; the time to maturity of the option T = 1 year.

(a) The change in the strike price on the option values

Like a plain vanilla put option, the lower the strike price is, the higher chance is that the option will be "in the money". Therefore, the value of the put option on the minimum of two risky assets will increase as the strike price increases. On the other hand, the values of call options on the maximum of two risky assets will decrease as the strike price increases. We summarize the results in Table 4.

Table 4 Sensitivity analysis: the case of changing strike prices

Stock Price	**Call Options on Maximum of Two Stocks**			**Put Options on Minimum of Two Stocks**		
	Strike Price			Strike Price		
	K=90	K=100	K=110	K=90	K=100	K=110
S1=80 ,S2=100	11.813793	6.269004	3.089185	11.407249	20.025911	29.624855
S1=90 ,S2=100	13.182898	7.233236	3.607354	6.593451	13.227290	21.666472
S1=100,S2=100	16.382200	9.570794	5.044669	4.153702	9.423715	16.722591
S1=110,S2=100	22.041126	13.974738	8.030676	3.064775	7.489765	14.007339
S1=120,S2=100	29.955490	20.780962	13.159774	2.631299	6.592614	12.667324

(b) The change in the stock volatility on the option values

Based on option pricing theory, we know that the higher the volatility of the underling assets is, the more chances there will be that the options will be in the money. This property still holds for the case of a call (put) option on the maximum (minimum) of two risky assets. Table 5 and Figures 3 and 4 show the results of the volatility change from $\sigma_1 = \sigma_2 = 0.1$ to $\sigma_1 = \sigma_2 = 0.3$.

Table 5 The Sensitivity Analysis: the case of Volatility Change

Stock Price	Call Options on Maximum of Two Stocks				
	Volatility of Stock Price				
	σ 1=σ 2=0.1	σ 1=σ 2=0.15	σ 1=σ 2=0.2	σ 1=σ 2=0.25	σ 1=σ 2=0.3
S1=80 ,S2=100	2.376900	4.155182	6.269004	8.385653	10.899750
S1=90 ,S2=100	2.492130	4.606137	7.233236	9.769871	12.594995
S1=100,S2=100	3.786014	6.563631	9.570794	12.525365	15.370221
S1=110,S2=100	9.692407	11.323543	13.974738	16.886507	19.552776
S1=120,S2=100	19.437464	19.489101	20.780962	22.786716	25.079649
Stock Price	Put Options on Minimum of Two Stocks				
	Volatility of Stock Price				
	σ 1=σ 2=0.1	σ 1=σ 2=0.15	σ 1=σ 2=0.2	σ 1=σ 2=0.25	σ 1=σ 2=0.3
S1=80 ,S2=100	19.572483	19.565594	20.025911	21.304179	23.128235
S1=90 ,S2=100	9.627274	10.968495	13.227290	15.707577	18.215881
S1=100,S2=100	3.765051	6.567232	9.423715	12.223455	14.988344
S1=110,S2=100	2.524401	4.880692	7.489765	10.210578	12.944098
S1=120,S2=100	2.380748	4.319213	6.592614	9.047388	11.626433

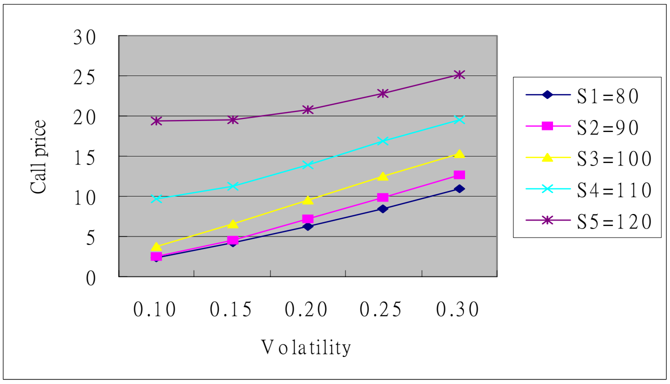

Figure 3 The volatility change in the case of call options on maximum of two risky assets

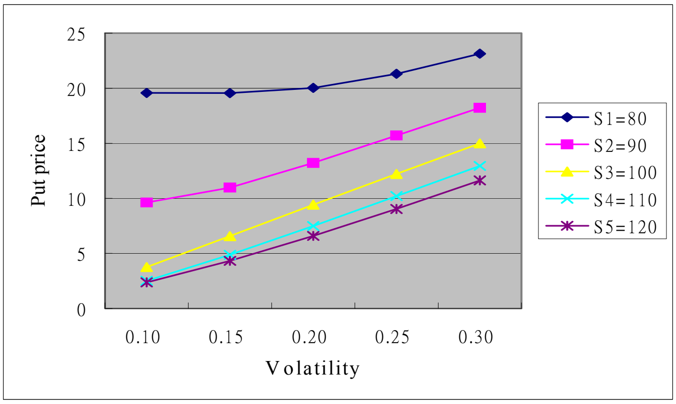

Figure 4 The volatility change in the case of put options on minimum of two risky assets

(c) The change in the correlation coefficient of the two risky asset prices on the option values

The higher the correlation coefficient is between the two risky asset prices, the higher there is of a possibility where the two risky asset prices move up or down at the same time. Hence, a high correlation coefficient will not increase the probability that makes options become in the money. On the contrary, the lower the correlation is between the two risky asset prices, the lower the possibility will be for which the two risky asset prices will not move up or down at the same time. Hence, a low correlation coefficient will increase the chances that make options become in the money. Therefore, the change in the correlation coefficients has an inverse relationship with the change in the values of options on call and put options on the maximum and minimum of two risky assets. The results are shown in Table 6 and Figures 5 and 6.

Table 6 The change of correlation

Stock Price	**Call Options on Maximum of Two Stocks**						
	Coefficient of Two Stocks						
	ρ=-1	ρ=-0.6	ρ=-0.3	ρ=0	ρ=0.3	ρ=0.6	ρ=1
S1=80 ,S2=100	6.552075	6.599467	6.514948	6.310575	6.269004	6.012285	5.8879005
S1=90 ,S2=100	8.279318	8.176158	7.842778	7.521178	7.233236	6.607903	5.888609
S1=100,S2=100	11.716550	11.338548	10.754653	10.231882	9.570794	8.593609	5.846547
S1=110,S2=100	17.051163	16.200921	15.557789	14.794249	13.974738	12.9685965	11.678880
S1=120,S2=100	23.886681	22.876541	22.164606	21.436956	20.780962	20.050289	19.855904
Stock Price	**Put Options on Minimum of Two Stocks**						
	Coefficient of Two Stocks						
	ρ=-1	ρ=-0.6	ρ=-0.3	ρ=0	ρ=0.3	ρ=0.6	ρ=1
S1=80 ,S2=100	22.818130	21.816484	21.141283	20.561605	20.025911	19.652977	19.635233
S1=90 ,S2=100	16.663141	15.615911	14.836497	14.060149	13.227290	12.365191	11.281821
S1=100,S2=100	11.994658	11.325033	10.729671	10.141745	9.423715	8.565649	5.667369
S1=110,S2=100	8.960818	8.656335	8.357699	7.968040	7.489765	6.899270	5.996993
S1=120,S2=100	7.092293	7.116772	6.946944	6.801280	6.592614	6.279063	6.025959

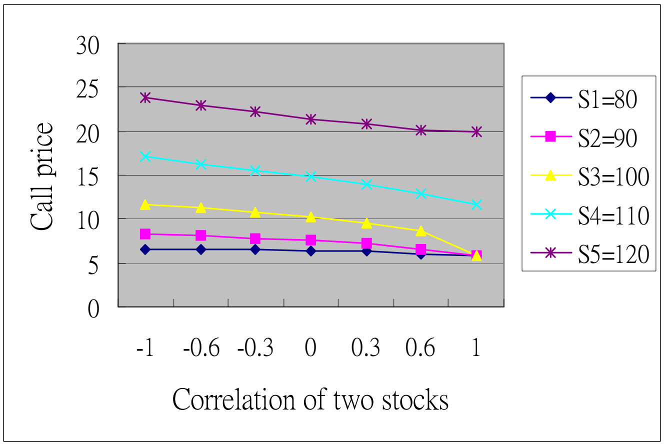

Figure 5 The change of correlation in the case of call option on maximum of two risky assets

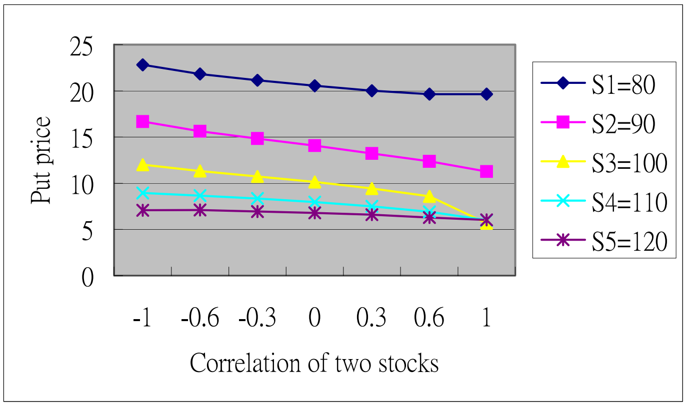

Figure 6 The change of correlation in the case of put option on minimum of two risky assets

4.4 Hedge ratios

Table 7 shows the hedge ratios using a Monte Carlo simulation approach. For each option, we generate 100 delta and gamma estimates. Each estimate are computed with 100000 sample paths. The errors of hedge ratios are on average less than 1% using the Pelsser and Vorst (1994) extended binomial tree with 10,000 time steps as a benchmark. These results indicate that the computational algorithm of hedge ratios proposed in this paper is reliable and applicable in practice.

Table 7 Hedge Ratios for American Put Options

Parameters	(1) GVW method	(2) Extended GVW method	(3) CRR Binomial Model
X=90	-0.256127	-0.264753	-0.261041
Absolute difference with (3)	1.8825%	1.4220%	
X=95	-0.317767	-0.313309	-0.331918
Absolute difference with (3)	4.2821%	5.5065%	
X=100	-0.395569	-0.400685	-0.407379
Absolute difference with (3)	2.8990%	1.6432%	
X=105	-0.472723	-0.475609	-0.485028
Absolute difference with (3)	2.5370%	1.9419%	
X=110	-0.537681	-0.540860	-0.563708
Absolute difference with (3)	4.6171%	4.0532%	
r=6%	-0.404591	-0.415726	-0.419194
Absolute difference with (3)	3.4836%	0.8273%	
r=8%	-0.404359	-0.409085	-0.412970
Absolute difference with (3)	2.0851%	0.9407%	
r=10%	-0.395569	-0.400685	-0.407379
Absolute difference with (3)	2.8990%	1.6432%	
r=12%	-0.386596	-0.404667	-0.402329
Absolute difference with (3)	3.9105%	0.5811%	
r=14%	-0.384013	-0.398943	-0.397752
Absolute difference with (3)	3.9628%	0.2994%	
σ=0.5	-0.389462	-0.392861	-0.400293
Absolute difference with (3)	2.7058%	1.8566%	
σ=0.4	-0.395569	-0.400685	-0.407379
Absolute difference with (3)	1.2898%	1.6704%	
σ=0.3	-0.403027	-0.400890	-0.411134
Absolute difference with (3)	1.9719%	2.4916%	
σ=0.2	-0.385502	-0.400365	-0.408098
Absolute difference with (3)	5.5369%	1.8949%	
σ=0.1	-0.359597	-0.362600	-0.389179
Absolute difference with (3)	7.6011%	6.8295%	

V. Conclusions

In this paper we first use the Geske and Johnson (1984) approach to improve the computational efficiency of the GVW approach for plain vanilla American options. We then extend the extended GVW approach to value American options on the maximum or minimum of two risky assets, whose prices follow a joint lognormal distribution. We demonstrate that the extended GVW approach is not only more computationally efficient than the GVW approach, but also can value American put options on one and two state variables to a high degree of accuracy.

We also carried out simulations to detect how the key parameters affect the values of maximum and minimum options. From the sensitivity analysis, the call (put) option value will decrease (increase) as the strike price K increases. As the property of plain vanilla options, the put and call option values increase when the volatility of the underlying assets increases. However, there exists a negative relationship between the correlation coefficient of the two underlying stocks and the option values.

We should finally point out that the approach developed in this paper has the potential to be extended to value options on the maximum or minimum of several (more than two assets) assets. It can also be applied to value basket options or basket warrants. Furthermore, the variance reduction techniques can be accompanied with our approach to improve the computational efficiency.

References

Barraquand, J. & D. Martineau, "Numerical Valuation of High Dimensional Multivariate American Securities," *Journal of Financial and Quantitative*, 30 (1995), pp.383-405.

Black, F & M.J. Scholes, "The Pricing of Options and Corporate Liability," *Journal of Political Economy*, 18 (May-June 1973), pp.637-659.

Boyle, P.P., "Options: A Monte Carlo Approach," Journal of Financial Economics, 4 (1977), pp.323-338.

Boyle, P.P., "A Lattice Framework for Option Pricing When There are Two State Variables," *Journal of Financial Quantitative Analysis*, 23 (1988), 1-12.

Boyle, P.P., M. Broadie, & P. Glasserman, "Monte Carlo Methods for Security Pricing," *Journal of Economic Dynamics and Control*, Vol. 21 (1997), 1367-1331.

Broadie, M. & P. Glasserman, "Pricing American-Style Securities Using Simulations," *Journal of Economic Dynamics and Control*, Vol. 21 (1997), 1323-1352.

Bunch, D. S. & H. Johnson, , "A Simple Numerically Efficient Valuation Method for American Puts Using a Modified Geske-Johnson Approach," *Journal of Finance*, Vol. 47 (1992), 809-816.

Chang, C.C. & S.L. Chung, & R. Stapleton, 2001, "Richardson Extrapolation Techniques for Pricing American-style Options," Proceedings of the 2001 Taiwanese Financial Assocation, June, Taipei, Tamkang University.

Chung, S.L. & M. Shackleton, 2000, The Binomial Black Scholes Model and Greeks, working paper, Lancaster University.

Cox, J.C., S.A. Ross & M. Rubinstein, " Option Pricing: A Simplified Approach," Journal of Financial Economics, 7, September (1979), 229-263.

Geske, R. & Johnson, H., "The American Put Valued Analytically," *Journal of Finance*, 39 (1984), December, pp.1511-1542.

Grant, D., G. Vora & D.E. Weeks, "Simulation and the Early-Exercise Option Problem," *Journal of Financial Engineering*, 5(1996), pp.211-227.

Hull J. C., "Options, Futures, and Other Derivative Securities," 3th edition, Prentice-Hall, 1995.

Johnson, H., "Options on the Maximum or Minimum of Several Assets," *Journal of Financial and Quantitative Analysis*, 22(1987), pp.277-283.

Pelsser, A., & T.C.F. Vorst, "The Binomial Model and the Greeks," *Journal of Derivatives*, Spring (1994), pp.45-49.

Stulz, R.M., "Options on the Minimum or the Maximum of Two Risky Assets," *Journal of Financial Economics,* 10(1982), pp.161-185.

Tan, K.S. & K. Vetzal, "Early Exercise Regions for Exotic Options," *Journal of Derivatives*, 3(Fall 1995), pp.42-56.

Tilly, J. A., "Valuing American Options in a Path Simulation Model," Transactions of the Society of Actuaries, 45(1993), pp.83-104.

強化蒙地卡羅模擬法之計算效率

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摘要

於 1993 年以前，只有少數論文探討如何利用蒙地卡羅模擬法來對美式選擇權訂價，此後即有許多學者企圖提出各式各樣的蒙地卡羅模擬法來對美式選擇權訂價。Grant-Vora-Weeks (1996)成功地發展出一種簡單的蒙地卡羅模擬法，其可以決定美式選擇權在各個時點的提早履約價值，進而可以計算出美式選擇權的價格。本文首先以 Geske-Johnson 的 Richardson 外插法，來增進 Grant-Vora-Weeks (1996)之蒙地卡羅美式選擇權訂價法之計算效率，並將此法擴展至最小值或最大值選擇權的訂價，同時本文亦執行敏感度分析，來探討一些重要模型參數之變動，如何影響最小值或最大值選擇權的價格。

關鍵字：蒙地卡羅模擬法、美式選擇權、最小值選擇權、最大值選擇權