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Exponentially Stabilizing Division Controllers for Dyadic Bilinear Systems

Min-Shin Chen, Yean-Ren Hwang, and Yu-Jeng Kuo

***Abstract*—It is difficult to asymptotically stabilize a dyadic bilinear system with only multiplicative control inputs when the open-loop dynamics are unstable. The previous approach of cascading a division controller with a constant-size dead zone can only stabilize but *not* asymptotically stabilize the system. This note proposes a new control design which cascades a division controller with a modified dead zone whose size is proportional to the modulus of the system state. It is shown that this new division controller can globally and exponentially stabilize any open-loop unstable dyadic bilinear system as long as it is controllable.**

***Index Terms*—Asymptotic stability, dead zone, division controller, dyadic bilinear system, exponential stability.**

I. INTRODUCTION

A division controller is one whose control input is a quotient of two state functions

$$u = \frac{\beta(x)}{\alpha(x)}. \tag{1}$$

Such a control structure can be found in the feedback linearization control for nonlinear systems [1], and in the control for dyadic bilinear systems [2]. In the division controller (1), if $\alpha(x) = 0$ at some singular point x, the control signal becomes infinitely large at x. In the case of feedback linearization control, the singularity problem arises when the nonlinear system has no well-defined relative degree [3]. In the case of dyadic bilinear system control [2], the singularity problem is avoided by cascading the division controller (1) with a dead zone

$$u = \begin{cases} \frac{\beta(x)}{\alpha(x)}, & |\alpha(x)| > \epsilon \\ 0, & |\alpha(x)| \le \epsilon \end{cases} \tag{2}$$

where $\epsilon > 0$ is the size of the dead zone. The use of a dead zone is first proposed in the control [4] of a dyadic bilinear system whose control input is both multiplicative and additive

$$\dot{x} = Ax + b(y + d_0)u, \qquad y = cx. \tag{3}$$

where $x \in R^n$ is the state vector, $u \in R$ is a single control input, $A \in R^{n\times n}$ is a constant matrix, b and c^T are constant vectors, and d_0 is a nonzero constant. The division controller (2) becomes

$$u = \begin{cases} -\frac{kx}{y+d_0}, & |y + d_0| > \epsilon \\ 0, & |y + d_0| \le \epsilon \end{cases} \tag{4}$$

where $\epsilon > 0$ is the size of the dead zone, and the state feedback gain $k \in R^{1\times n}$ is chosen such that $A - bk$ is a stable matrix. It is proved that asymptotic stability is achieved by the division controller (4) if the open-loop trajectories satisfy a geometric condition [4].

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However, when the control input of the dyadic bilinear system is multiplicative only ($d_0 = 0$ in (3)), achieving asymptotic stability becomes a very challenging problem especially when the open-loop dynamics of (3) is unstable. The previously mentioned division controller (4) can no longer achieve asymptotic stability for the following reason. When the control input is multiplicative only ($d_0 = 0$ in (3)), the origin $x = 0$ will fall inside the dead zone region defined by $|y| \le \epsilon$, where according to the control law (4), no control is applied to change the local stability of the origin. Hence, if the open-loop dynamics of (3) is unstable at the origin, it will remain locally unstable under the division controller (4).

The goal of this note is to find a new division controller to asymptotically stabilize a dyadic bilinear system with only multiplicative control inputs ($d_0 = 0$ in (3)). In a broader sense, the goal is to redesign the division controller (1) so as to ensure asymptotic convergence to the targeted equilibrium point even if this targeted equilibrium point is a singular point of (1). To achieve this goal for dyadic bilinear systems, this note proposes to cascade the division controller with a modified dead zone whose *size is proportional to the modulus of the system state*. It will be proved that the new design can ensure not only asymptotic stability but also exponential stability of the closed-loop dyadic bilinear system.

It is interesting to note that for the bilinear system (3) with $d_0 = 0$, a *quadratic state feedback control* [5]–[7] has been proposed to achieve asymptotic stability, and recently a *normalized* quadratic state feedback control [8] is suggested to achieve exponential stability. However, all these controls are applicable only if the open-loop system is (neutrally) stable. When the open-loop system is unstable, but can be rendered neutrally stable by a constant control, a switched controller is suggested in [9]. In this note, the proposed division controller can be applied to any open-loop-unstable system as long as it is controllable.

The remainder of this note is arranged as follows. Section II introduces the new division controller, which is cascaded with a state-dependent dead zone. Section III studies the stability property of the closed-loop system. Section IV proposes two more division controllers to ensure not only exponential stability, but also smoothness of the control signals. Finally, Section V gives the conclusions. Throughout this note, $\overline{\sigma}\{A\}$ and $\underline{\sigma}\{A\}$ denote respectively the maximum and minimum singular value of a matrix A.

II. New Division Controller

Consider a dyadic bilinear system with a multiplicative control input

$$\dot{x} = Ax + byu, \qquad y = cx \tag{5}$$

where all variables and matrices are as defined in (3), and $cb = 0$. Assume that the bilinear system (5) satisfies the controllability condition [1], which can be easily verified by the following theorem.

Theorem 1 [10]: The dyadic bilinear system (5) is controllable if and only if (A, b) is controllable and (A, c) observable.

This note proposes the following division controller for the controllable system (5):

$$u = \begin{cases} -\frac{1}{y}kx, & |y| > \epsilon\|x\| \\ 0, & |y| \le \epsilon\|x\| \end{cases} \tag{6}$$

where the division controller is cascaded with a dead zone whose size is proportional to $\|x\|$ with a sufficiently small proportional constant ϵ to ensure exponential stability of the closed-loop system. The controllability condition in Theorem 1 ensures arbitrary eigenvalue assignment [11] and stabilization of $A - bk$ by the state feedback gain k. Note that the proposed control is applicable even if the control magnitude is subject to a constraint

$$|u(t)| \le u_{\max}, \text{ with } \frac{\|k\|}{\epsilon} \le u_{\max}. \tag{7}$$

However, when the control magnitude constraint $u_{\max}$ is too tight, there may not exist control parameters k and ϵ that satisfy the second inquality in (7), especially when the open-loop system matrix A has far away right-half-plane eigenvalues, which require large state feedback gain k to make $A - bk$ stable.

The resultant closed-loop dynamics are given by

$$\dot{x} = \left[A - F_0\left(\frac{y}{\|x\|}\right)bk\right]x, \text{ where}$$
$$F_0\left(\frac{y}{\|x\|}\right) = \begin{cases} 1, & \frac{|y|}{\|x\|} > \epsilon \\ 0, & \frac{|y|}{\|x\|} \le \epsilon. \end{cases} \tag{8}$$

III. Stability Analysis

In this section, one will establish exponential stability for the closed-loop system (8). The analysis consists of two steps. In the first step, it will be shown that (8) is *asymptotically* stable. In the second step, it will further be shown that the system is in fact *exponentially* stable.

Definition: The state–space R^n is divided into two sets Ω^- and Ω^+ with a boundary $\Omega^0 \ (\subset \Omega^-)$

$$\begin{aligned} \Omega^- &= \{x \in R^n \,|\, |y| \le \epsilon\|x\|\} \\ \Omega^+ &= \{x \in R^n \,|\, |y| > \epsilon\|x\|\} \\ \Omega^0 &= \{x \in R^n \,|\, |y| = \epsilon\|x\|\}. \end{aligned}$$

Let $\{t_i\}$ be a nondecreasing time sequence, where t_{2i}'s denote time instants when the state $x(t)$ exits Ω^+ to enter Ω^-, and t_{2i+1}'s time instants when $x(t)$ exits Ω^- to enter Ω^+. The time durations staying in zone Ω^- and Ω^+ are given, respectively, by

$$\Delta_i^- = t_{2i+1} - t_{2i} \text{ and } \Delta_i^+ = t_{2i+2} - t_{2i+1}.$$

Hence, $x(t) \in \Omega^-$ when $t \in [t_{2i}, t_{2i+1}]$, and $x(t) \in \Omega^+$ when $t \in (t_{2i+1}, t_{2i+2})$.

The first lemma, whose proof can be found in [12, Prop. 1.4.1], shows that the closed-loop state can grow or decay only exponentially fast.

Lemma 1 [12]: The system state in (8) can grow or decay at most exponentially

$$\|x(\tau)\|e^{-q(t-\tau)} \le \|x(t)\| \le \|x(\tau)\|e^{q(t-\tau)} \qquad \forall\, t \,\&\, \tau \tag{9}$$

where $q = \|A\| + \|bk\|$.

The second lemma shows how the proportional constant ϵ affects the time duration the closed-loop state can stay inside Ω^-.

Lemma 2: Define a matrix $H(\Delta t)$ as

$$H(\Delta t) = \frac{1}{\Delta t}\int_0^{\Delta t} \frac{e^{A^T\tau}c^T ce^{A\tau}}{\|e^{A\tau}\|^2}d\tau \tag{10}$$

and denote $\underline{\sigma}\{H(\Delta t)\}$ the minimum singular value of $H(\Delta t)$.

I) If ϵ can be chosen such that

$$0 < \epsilon^2 < \underline{\sigma}\{H(\Delta t)\} \tag{11}$$

the closed-loop state will not stay inside Ω^- longer than Δt; that is, $\Delta_i^- \le \Delta t$.

II) If ϵ can be chosen such that

$$0 < \epsilon^2 < \sup_{0<\Delta t<\infty} \underline{\sigma}\{H(\Delta t)\} \tag{12}$$

the closed-loop state will not stay inside Ω^- forever. In other words, Ω^- is not an invariant set for the closed-loop system (8) under (12).

III) If ϵ approaches zero, the time duration the closed-loop state can stay inside Ω^- also approaches zero. In other words, $\lim_{\epsilon\to 0}\Delta_i^-(\epsilon) = 0$ for all i.

Proof: **Part I)** Assume the contrary; that is, the closed-loop state $x(t)$ stays inside Ω^- for a time span longer than Δt. Hence, one has $x(t) \in \Omega^-$, $t \in [t_k, t_k + \Delta t]$ for some t_k. Over this time interval, $u(t) = 0$ according to the control law (6) and, hence, $x(t) = e^{A(t-t_k)}x(t_k)$ and $y(t) = ce^{A(t-t_k)}x(t_k)$. By definition, one has $\epsilon \geq |y(t)|/\|x(t)\|$ inside Ω^-. Taking the square of this inequality, and integrating from t_k to $t_k + \Delta t$, one obtains

$$\epsilon^2 \geq \left(\frac{x(t_k)}{\|x(t_k)\|}\right)^T H(\Delta t)\left(\frac{x(t_k)}{\|x(t_k)\|}\right) \geq \underline{\sigma}\{H(\Delta t)\} > 0 \quad (13)$$

where $H(\cdot)$ is the matrix in (10), and one has used the inequality $\|e^{A\tau}x(t_k)\| \leq \|e^{A\tau}\| \cdot \|x(t_k)\|$ to derive the first inequality in (13). However, (13) contradicts the hypothesis—(11). Hence, one concludes that $x(t)$ can not stay inside Ω^- for a time span longer than Δt.

Observe that the matrix $H(\Delta t)$ in (10) is, in fact, the observability grammian matrix of the pair (A, c) with a scaler weighting $1/\|e^{A\tau}\|^2$. Since (A, c) is observable according to Theorem 1, the observability grammian matrix and hence $H(\Delta t)$ are positive definite matrices [11]. Therefore, the minimum singular value $\underline{\sigma}\{H(\Delta t)\}$ is bounded above from zero for any $\Delta t > 0$. This guarantees the existence of ϵ in (11) given any Δt.

Part II): If (12) holds, there exists some Δt^* such that $\epsilon^2 < \underline{\sigma}\{H(\Delta t^*)\}$. From part I) of the lemma, $x(t)$ cannot stay longer than Δt^* inside Ω^-, and hence can not stay forever inside Ω^-. Consequently, Ω^- is not an invariant set for the closed-loop system (8) under (12).

Part III): Note that $\lim_{\Delta t\to 0} H(\Delta t) = c^T c$ is a singular matrix, and hence $\underline{\sigma}\{H(0)\} = 0$. Since $\underline{\sigma}\{H(\Delta t)\}$ is greater than zero for any $\Delta t > 0$, and is equal to zero for $\Delta t = 0$, one concludes from (13) that Δt must approach zero as ϵ approaches zero. □

Lemma 3: If $cb = 0$ in (5), there exists a lower bound $m > 0$ of $\Delta_i^+(\epsilon)$ for sufficiently small ϵ; that is, $\lim_{\epsilon\to 0}\Delta_i^+(\epsilon) \geq m > 0$, for all i.

Proof: Assume the contrary; that is, there exists a subsequence i_k of i such that $\lim_{\epsilon\to 0}\Delta_{i_k}^+ = 0$ as i_k approaches infinity. From part III) of Lemma 2, one has both $\lim_{\epsilon\to 0}\Delta_{i_k}^-(\epsilon) = 0$ and $\lim_{\epsilon\to 0}\Delta_{i_k}^+(\epsilon) = 0$. This suggests that once the state enters Ω^+, it will immediately return to Ω^-, and *vice versa*. For this to take place, the sign of the inner product $\langle c, \dot{x}\rangle|_{x\in\Omega^+}$ must be different from that of $\langle c, \dot{x}\rangle|_{x\in\Omega^-}$ when evaluated at neighboring points in Ω^+ and Ω^-. However, since $cb = 0$ by hypothesis, $\langle c, \dot{x}\rangle|_{x\in\Omega^+} = c(A - bk)x = cAx - cb\cdot kx = cAx = \langle c, \dot{x}\rangle|_{x\in\Omega^-}$, a contradiction is reached. Hence, there exists a lower bound for $\lim_{\epsilon\to 0}\Delta_i^+(\epsilon)$. □

One can now establish the first stability property for the closed-loop system (8).

Theorem 2: The new division controller (6) globally and *asymptotically* stabilizes the controllable dyadic bilinear system (5) if ϵ is sufficiently small and satisfies the saturation condition (7).

Proof: One needs to discuss different cases.

Case A): The state stays in Ω^- forever after some finite time. This contradicts part II) of Lemma 2 under (12). Hence, this case will not take place if ϵ is sufficiently small.

Case B): The state stays in Ω^+ forever after some finite time. Since in Ω^+ the closed-loop dynamics is asymptotically stable by the choice of state feedback gain k, $x(t)$ will converge asymptotically to the origin in this case.

Case C): The state switches between Ω^- and Ω^+ indefinitely. Define a Lyapunov function $V(t) = x^T(t)Px(t)$ for the closed-loop system (8), where $P > 0$ is a positive–definite matrix in the Lyapunov equation

$$(A - bk)^T P + P(A - bk) = -Q, \qquad Q > 0. \quad (14)$$

For $t \in [t_{2i}, t_{2i+1}]$, $x(t) \in \Omega^-$, and the Lyapunov function satisfies

$$\dot{V}(\tau) \leq \pi V(\tau), \qquad \pi = \frac{\overline{\sigma}_H}{\underline{\sigma}_P} \quad H = A^T P + PA. \quad (15)$$

Integrating (15) yields

$$V(t_{2i+1}) \leq V(t_{2i})e^{\pi\Delta_i^-} \quad \Delta_i^- = t_{2i+1} - t_{2i}. \quad (16)$$

For $t \in [t_{2i+1}, t_{2i+2}]$, the system–state falls in the other region Ω^+, and the Lyapunov function satisfies

$$\dot{V}(\tau) \leq -\gamma V(\tau), \qquad \gamma = \frac{\underline{\sigma}_Q}{\overline{\sigma}_P} \quad (17)$$

where Q is from (14). Integrating (17) yields

$$V(t_{2i+2}) \leq V(t_{2i+1})e^{-\gamma\Delta_i^+} \quad \Delta_i^+ = t_{2i+2} - t_{2i+1}. \quad (18)$$

Combining (18) and (16) yields

$$V(t_{2i+2}) \leq V(t_{2i})e^{-\gamma\Delta_i^+ + \pi\Delta_i^-} \quad (19)$$

According to Lemma 3 and part III) of Lemma 2, given any small number $\zeta < \gamma m$, there exists an ϵ^* such that for all $\epsilon < \epsilon^*$, one has

$$-\gamma\Delta_i^+(\epsilon) + \pi\Delta_i^-(\epsilon) \leq -\gamma m + \zeta. \quad (20)$$

Substituting (20) into (19), one has

$$V(t_{2i+2}) \leq V(t_{2i})e^{-\gamma m + \zeta}, \qquad \gamma m > \zeta. \quad (21)$$

Since the time instants t_{2i}'s are not equally spaced, one concludes from (21) that $V(t_{2i})$ approaches zero only *asymptotically* but not *exponentially*. Finally, using Lemma 1, one can show that the continuous state $x(t)$ also approaches zero asymptotically in Case C).

After considering all the aforementioned cases, one concludes that the state trajectory $x(t)$ must converge to zero asymptotically, and hence the closed-loop system is asymptotically stable. □

To prove that the controlled system is in fact exponentially stable, observe that the closed-loop state equation $\dot{x} = f(x)$ in (8) is homogeneous of degree one; that is, $f(\lambda x) = \lambda f(x)$ for any real number $\lambda \neq 0$. One can, therefore, quote the following stability result from [13].

Lemma 4 [13]: If the state–space origin of the system $\dot{x} = f(x)$, $x \in R^n$, is asymptotically stable, and f is homogeneous of degree one, the system is globally exponentially stable.

An immediate consequence of Theorem 2 and Lemma 4 is that the proposed division control actually *exponentially* stabilizes the system.

Theorem 3: The new division controller (6) globally and *exponentially* stabilizes the controllable dyadic bilinear system (5) under the saturation condition (7).

To verify the new division controller design, one presents a simulation example.

Example: Consider an open-loop unstable dyadic system (5) with

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad c = [1 \quad 1]$$

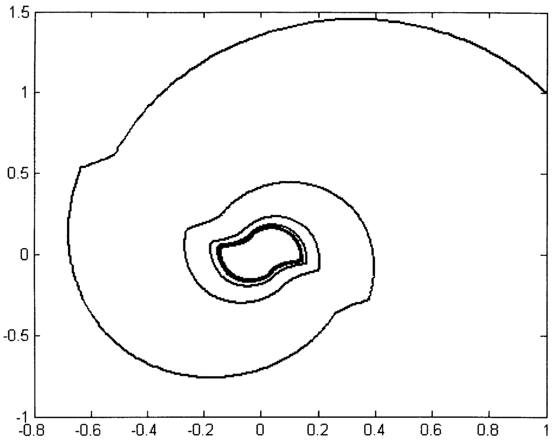

Fig. 1. Phase portrait with controller (4) $(d_0 = 0)$.

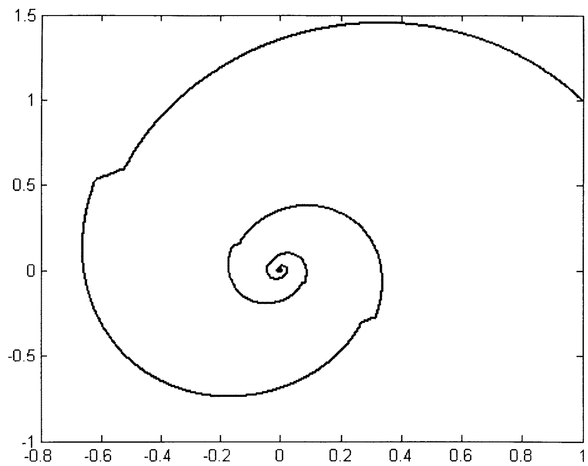

Fig. 2. Phase portrait with new controller (6).

and the initial condition $x^T(0) = [10, 10]$. For the purpose of comparison, one first tests the division controller (4) with $d_0 = 0$ in [4], in which the dead zone has a constant width $\epsilon = 0.1$, and the state feedback gain $k = [3.25 \quad 1.25]$ is chosen to place eigenvalues of $A - bk$ at $-0.5 \pm 2j$. The phase portrait in Fig. 1 shows that the controller (4) with $d_0 = 0$ drives the system state to approach a limit cycle. Then, one tests the proposed division controller (6) in this note, which uses the same design parameters as in (4) except that the size of dead zone is now proportional to the modulus of the state with $\epsilon = 0.1$. The phase portrait in Fig. 2 shows that the new controller (6) can now drive the system state to zero asymptotically (in fact exponentially).

IV. Continuous Division Controllers

Even though the *discontinuous* division controller (6) proposed in Section III successfully stabilizes the system, it has a disadvantage: the control generates discontinuous signals at $|y| = \epsilon\|x\|$ due to the discontinuity of the dead zone. Since discontinuous control signal is not acceptable in many practical applications, two *continuous* division controllers are suggested as follows to ensure continuity of the control signal at any time instant.

The first *continuous* division controller is given by

$$u_1 = \begin{cases} -\frac{1}{y}kx, & |y| > \epsilon\|x\| \\ -\frac{y}{\epsilon^2\|x\|^2}kx, & |y| \le \epsilon\|x\| \end{cases} \tag{22}$$

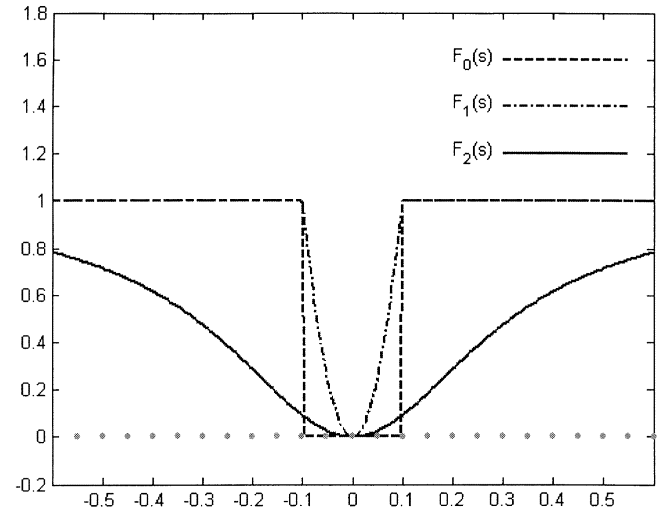

Fig. 3. Interpolation of dead zone $F_0(s)$ $(\epsilon = 0.1)$.

which results in a closed-loop dynamics

$$\dot{x} = \left[A - F_1\left(\frac{y}{\|x\|}\right)bk\right]x, \text{ where}$$
$$F_1\left(\frac{y}{\|x\|}\right) = \begin{cases} 1, & \frac{|y|}{\|x\|} > \epsilon \\ \frac{y^2}{\epsilon^2\|x\|^2}, & \frac{|y|}{\|x\|} \le \epsilon. \end{cases} \tag{23}$$

The second *continuous* division controller is given by

$$u_2 = -\frac{y}{y^2 + \epsilon^2\|x\|^2}kx \tag{24}$$

which results in a closed-loop dynamics

$$\dot{x} = \left[A - F_2\left(\frac{y}{\|x\|}\right)bk\right]x, \text{ where}$$
$$F_2\left(\frac{y}{\|x\|}\right) = \frac{y^2}{y^2 + \epsilon\|x\|^2}. \tag{25}$$

Notice that the control signal from the controller (22) is continuous at $|y| = \epsilon\|x\|$. However, the time derivatives of the control signal are still discontinuous at $|y| = \epsilon\|x\|$. This situation is improved in the second controller (24), which generates control signals that have continuous time derivatives up to any order at $|y| = \epsilon\|x\|$.

The same stabilizing property as in Theorem 3 can be established for the above two continuous division controllers. Instead of presenting tedious stability analysis for the continuous division controllers, one can use a simple graphical comparison of the $F_0(\cdot)$, $F_1(\cdot)$, and $F_2(\cdot)$ in (8), (23), and (25) to motivate the continuous control designs in (22) and (24). It is seen from Fig. 3 that $F_1(\cdot)$ and $F_2(\cdot)$ are simply continuous interpolation functions in replacement of the discontinuous dead zone function $F_0(\cdot)$, and the smaller ϵ is, the better $F_1(\cdot)$ and $F_2(\cdot)$ approximate $F_0(\cdot)$. Therefore, it is not surprising that the continuous division controllers (22) and (24), which use $F_1(\cdot)$ and $F_2(\cdot)$, respectively, to replace $F_0(\cdot)$, will have the exponentially stabilizing property if ϵ is small.

V. Conclusion

This note presents new division controller designs for a *dyadic* bilinear system with multiplicative control inputs only. Conventional division controller design uses a constant-width dead zone to avoid the singularity problem. This note proposes using a dead zone whose size is proportional to the modulus of the system state. Such a control design successfully achieves exponential stability under the controllability assumption of the bilinear system.

The future work is to extend the control design in this note to division controllers for more general nonlinear systems when the targeted equilibrium point is a singular point of the division controller. Such cases can be found in the feedback linearization control when the nonlinear system does not have a well-defined relative degree.

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A Result on Common Quadratic Lyapunov Functions

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***Abstract*—In this note, we define strong and weak common quadratic Lyapunov functions (CQLFs) for sets of linear time-invariant (LTI) systems. We show that the simultaneous existence of a weak CQLF of a special form, and the nonexistence of a strong CQLF, for a pair of LTI systems, is characterized by easily verifiable algebraic conditions. These conditions are found to play an important role in proving the existence of strong CQLFs for general LTI systems.**

***Index Terms*—Quadratic stability, stability theory, switched linear systems.**

I. INTRODUCTION

The existence or nonexistence of common quadratic Lyapunov functions (CQLFs) for two or more stable linear time-invariant (LTI) systems is closely connected to recent work on the design and stability of switching systems [1], [2]. In this context, numerous papers have appeared in the literature [2]–[6] in which sufficient conditions have been derived under which two stable dynamical systems

$$\Sigma_{A_i}: \dot{x} = A_i x, \qquad A_i \in \mathbb{R}^{n\times n}, \qquad i \in \{1, 2\}$$

have a CQLF. If the matrix $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$, simultaneously satisfies the Lyapunov equations $A_i^T P + PA_i = -Q_i$, $i \in \{1, 2\}$, where $Q_i > 0$, then $V(x) = x^T Px$ is said to be a strong CQLF for Σ_{A_1} and Σ_{A_2}. If $Q_i \geq 0$ for $i \in \{1, 2\}$ then $V(x)$ is said to be a weak CQLF. This technical note considers pairs of stable LTI systems for which a strong CQLF does not exist, but for which a weak CQLF exists where $-Q_1$ and $-Q_2$ are both negative semidefinite and of rank $n - 1$. We derive a result that can be used to determine necessary and sufficient conditions for the existence of a strong CQLF for certain classes of stable LTI systems.

II. MATHEMATICAL PRELIMINARIES

In this section, we present some results and definitions that are useful in proving the principal result of this note. Throughout, the following notation is adopted

$\mathbb{R}$ and $\mathbb{C}$	fields of real and complex numbers, respectively;
$\mathbb{R}^n$	n-dimensional real Euclidean space;
$\mathbb{R}^{n\times n}$	space of $n \times n$ matrices with real entries;
x_i	ith component of the vector x in $\mathbb{R}^n$;
a_{ij}	entry in the (i, j) position of the matrix A in $\mathbb{R}^{n\times n}$.

Where appropriate, the proofs of individual lemmas are presented in the Appendix.

i) ***Strong and weak common quadratic Lyapunov functions:*** Consider the set of LTI systems

$$\Sigma_{A_i}: \dot{x} = A_i x, \qquad i \in \{1, 2, \ldots, M\} \tag{1}$$

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