Natural Frequencies and Stability of a Spinning Disk Under Follower Edge Tractions

Jen-San Chen
Associate Professor,
Department of Mechanical Engineering,
National Taiwan University,
Taipei, Taiwan 107
Republic of China

The vibration and stability of a spinning disk under follower edge tractions are studied both numerically and analytically. The edge traction is circumferentially stationary in the space. When the compressive traction is uniform, natural frequencies of most of the non-reflected waves decrease, except some of the zero-nodal-circle modes with small number of nodal diameters in the low frequency range. When the spinning disk is under nonuniform traction in the form of cos kθ, where k is a nonzero integer, it is found that the eigenvalue only changes slightly under the edge traction if the natural frequency of interest is well separated from others. When two modes are almost degenerate, however, modal interaction (frequency loci veering or merging) occurs when the difference between the number of nodal diameters of these two modes is equal to ±k. Types of modal interaction vary as the radius ratio of the circular disk changes. Analytical methods for predicting how the eigenvalue changes and what type of modal interaction will occur are proposed and verified.

Introduction

Buckling analysis of a stationary circular disk subjected to general stationary in-plane loads is a classical problem and has been thoroughly investigated. On the other hand, the problem of a spinning disk under stationary in-plane edge loads attracts relatively little attention. This research topic arises naturally from the stability analysis of a spinning circular saw during cutting operation (Mote and Szymani, 1978). The main difference between this problem and the classical one with non-rotating disk and edge load is the complex phenomenon which occurs as a result of the relative motion between the spinning disk and the stationary edge tractions. Srinivasan and Ramamurti (1980) calculated the membrane stresses of a stationary disk subjected to a spinning, concentrated, in-plane load on the outer boundary. Carlin et al. (1975) first wrote the equation of motion for a non-rotating circular plate with asymmetric stresses due to a stationary concentrated normal edge load. In order to account for the effects of disk rotation, they then superposed the axisymmetric stresses due to the centrifugal force to the membrane stress field induced by the edge load. Radcliffe and Mote (1977) extended the work of Carlin et al. by considering a general concentrated edge load with both normal and tangential components. These two papers did not take into account the relative motion between the spinning disk and the stationary edge load. On the contrary, their concentrated edge load appears to be spinning along with the spinning disk.

In order to account for the relative motion between the spinning disk and the stationary concentrated edge load, Chen (1994) formulated the problem with respect to the stationary coordinate system and studied various stability properties of the coupled system. While Chen can explain why instability occurs when two natural frequencies merge together with the concept of energy consumption, he was unable to explain why and when the natural frequency loci tend to merge together or veer away, partly due to the mathematical complexity associated with the concentrated force.

Since a general edge load, including the special case of a concentrated force, can be expressed as a Fourier series expansion, it is desirable to study the effects of each Fourier component of the stationary edge load on the natural frequencies and stability of the spinning disk. The natural frequency variations and stability properties of the spinning disk under general edge loads may be considered as due to the combined effects of each Fourier component. Chen (1996) studied the stability of a spinning disk under stationary conservative edge tractions in the form of cos $k\theta$ at the outer rim, where $k = 0, 1, 2, \ldots$. He found that compressive uniform traction brings down the natural frequencies of all non-reflected waves and raises the natural frequencies of all reflected waves. When two modes are almost degenerate and the difference between the number of nodal diameters of these two modes is equal to $\pm k$, he observed that frequency loci veer away when both modes are non-reflected, and merge together when one of these two modes is a reflected wave. These rules are proved to be independent of the radius ratio of the annular disk.

In practice, the edge load can be either conservative or non-conservative, or even a combination of both. In the present paper we extend Chen's work to consider the case of a spinning disk under stationary follower tractions. The equation of motion was first presented. Numerical results for annular disks with various radius ratios are obtained by a computation scheme based on Fourier series expansion and Galerkin's method. The effects of the number k on the eigenvalues and modal interactions are particularly of interest. In order to study analytically the first order effect of the edge traction on the eigenvalues, we derive the derivative of the eigenvalue with respect to the magnitude of the in-plane traction. In order to study the complex phenomena of modal interactions, we then use a two-mode eigenfunction expansion theory to investigate the inherent mathematical structure and propose a formula to predict the occurrence of various types of modal interactions based on the eigensolutions of the freely spinning disk.

Equation of Motion

Figure 1 shows a circular disk, which is clamped at the inner radius $r = a$ and subjected to a distributed normal traction at the outer radius $r = b$. The form of the traction is assumed to be $\epsilon \cos k\theta$. The disk is rotating with constant speed Ω, while the edge traction is circumferentially stationary in the space. The magnitude of the edge traction remains the same when the disk vibrates laterally, while the direction is always tangent to the radial slope of the disk at $r = b$. In other words, the edge

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS. Manuscript received Aug. 1994; revised March 1995. Associate Technical Editor: L. A. Bergman.

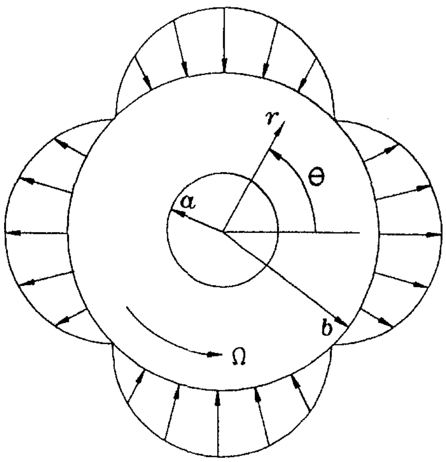

Fig. 1 A spinning disk subjected to stationary follower edge traction

traction is a follower load. The equation of motion of the system, in terms of the transverse displacement w and with respect to the stationary coordinate system (r, θ), is

$$\rho h\left(\frac{\partial^2 w}{\partial t^2} + 2\Omega \frac{\partial^2 w}{\partial t \partial \theta} + \Omega^2 \frac{\partial^2 w}{\partial \theta^2}\right) + D\nabla^4 w + \mathbf{L}w + \epsilon \hat{\mathbf{L}} w = 0 \quad (1)$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

The standard subscript notation for partial derivative is used here. The parameters ρ, h, E, and ν are the mass density, thickness, Young's modulus, and Poisson ratio of the disk, respectively. $\mathbf{L}$ is the membrane operator associated with the axisymmetrical stress field due to the centrifugal force, and $\hat{\mathbf{L}}$ is the membrane operator associated with the stress field due to the normal edge traction $\epsilon \cos k\theta$ at $r = b$,

$$\mathbf{L} = -\frac{h}{r}\left[\frac{\partial}{\partial r}\left(r\sigma_r \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\sigma_\theta \frac{\partial}{\partial \theta}\right)\right]$$

$$\hat{\mathbf{L}} = -\frac{h}{r}\left[\frac{\partial}{\partial r}\left(r\hat{\sigma}_r \frac{\partial}{\partial r} + \hat{\sigma}_{r\theta}\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\hat{\sigma}_{r\theta}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\sigma}_\theta \frac{\partial}{\partial \theta}\right)\right]$$

σ_r and σ_θ are due to the centrifugal force and are proportional to Ω^2. On the other hand, $\hat{\sigma}_r$, $\hat{\sigma}_{r\theta}$, and $\hat{\sigma}_\theta$ are the stress fields induced by the edge traction and can be calculated as

$$\hat{\sigma}_r = \epsilon \sigma_r^*(r)\cos k\theta$$

$$\hat{\sigma}_\theta = \epsilon \sigma_\theta^*(r)\cos k\theta \quad k = 0,1,2, \ldots$$

$$\hat{\sigma}_{r\theta} = \epsilon \sigma_{r\theta}^*(r)\sin k\theta$$

σ_r^*, σ_θ^*, and $\sigma_{r\theta}^*$ are functions of radial coordinate r only, and can be found readily in Coker and Filon (1957). It is noted that $\hat{\sigma}_r$ and $\hat{\sigma}_\theta$ are odd functions of θ, and $\hat{\sigma}_{r\theta}$ is an even function of θ. Furthermore, $\sigma_r^*(b) = 1$ and $\sigma_{r\theta}^*(b) = 0$ due to the in-plane traction boundary conditions on the outer rim.

The associated boundary conditions for Eq. (1) are

$$w = \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = a$$

and

$$\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2 \partial \theta^2}\right) + \frac{1-\nu}{r^2}\left(\frac{\partial^3 w}{\partial r \partial \theta^2} - \frac{\partial^2 w}{r\partial \theta^2}\right) = 0$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{\partial^2 w}{r\partial \theta^2}\right) = 0 \quad \text{at} \quad r = b.$$

For a freely spinning disk the eigenvalues λ_{mn} are purely imaginary and occur in complex conjugate pairs, i.e., $\lambda_{mn} = i\omega_{mn}$, where ω_{mn} is real. The eigenfunction corresponding to λ_{mn} is in general complex and assumes the form

$$w_{mn} = R_{mn}(r)e^{in\theta} \quad (2)$$

R_{mn} is a real-valued function of r. The eigenfunction corresponding to $\bar{\lambda}_{mn}$ is $\bar{w}_{mn}$, where overbar means complex conjugate. If we consider only the positive ω_{mn}, then w_{mn} in Eq. (2) with positive n is a backward traveling wave with n nodal diameters and m nodal circles, which is also denoted by $(m, n)_b$. Similarly, w_{mn} with negative n is a forward traveling wave $(m, -n)_f$. The critical speed Ω_c for the mode (m, n) is defined as the rotation speed at which ω_{mn} of the backward traveling wave $(m, n)_b$ becomes zero. For Ω greater than Ω_c, this mode is a forward traveling wave, and is termed a "reflected wave," denoted by $(m, -n)_r$.

Numerical Results

The nontrivial solution of Eq. (1) is assumed to be in the form of a Fourier sine and cosine series expansion. After substituting this expansion into Eq. (1), and by a finite element numerical analysis based on Galerkin's method with third order polynomial shape functions in the radial direction (Ono and Maeno, 1987), we can transform Eq. (1) into matrix algebraic equations and solve them with a generalized eigenvalue solver in EISPACK. From the viewpoints of both computing efficiency and accuracy, the number of elements in the radial direction and the maximum number of nodal diameters are chosen to be 6 and 4, respectively. The material properties of the disk used in the calculation are: $\rho = 7.84 \times 10^3$ kg/m^3, $E = 203 \times 10^9$ N/m^2, $\nu = 0.27$, $h = 1.02$ mm, and $b = 203.2$ mm.

Figures 2, 3, and 4 show the relations between the natural frequencies of a spinning disk with radius ratio 0.5 (i.e., $a = 101.6$ mm) under follower tractions and the rotation speed in both the low and high frequency ranges. The solid lines represent the case for $\epsilon = 4 \times 10^6$ N/m^2, and the dashed lines correspond to the case of a freely spinning disk (i.e., $\epsilon = 0$). Figure 2 shows the case when the spinning disk is under compressive uniform edge traction. It is found that in the low frequency range for the modes with zero nodal circle, the compressive uniform edge traction tends to bring down the natural frequencies of the non-reflected modes with more than 2 nodal diameters, but raise the natural frequencies of the non-reflected modes with two or less nodal diameters. The effect of the edge traction on the reflected waves are opposite to the effect on the non-reflected waves. In the high frequency range for the modes with one nodal circle, on the other hand, the compressive traction tends to bring down the natural frequencies of all non-reflected waves. It is also noted that the uniform traction does not induce any modal interaction.

Figures 3 and 4 show the cases when the nonuniform edge traction is in the form of $\epsilon \cos k\theta$, where $k = 3$ and 4, respectively. When the natural frequency of interest is well separated from others, it is found that the eigenvalue only changes slightly under nonuniform edge tractions. On the other hand, when two modes are almost degenerate, modal interaction may or may not occur. It is found that when the difference between the number of nodal diameters of these two modes is equal to $+k$ or $-k$, the frequency loci may merge together or veer away. Otherwise, modal interaction will not occur and the frequency loci simply cross over. In applying these rules the number of nodal diameters of a forward or a reflected wave is considered as *negative.* When modal interaction occurs in the low frequency range, it is found for this specific radius ratio that frequency loci veer away when both modes are non-reflected, and frequency loci merge together when one of these two modes is a reflected wave. For example, in Fig. 4 frequency loci merging occurs when $(0,3)_r$ meets $(0.1)_b$, crossing occurs when $(0,3)_b$ meets $(0,1)_b$, and veering occurs when $(0,3)_b$ meets $(0,1)_f$. While veering does not change the stability properties of the system, frequency merging renders the spinning disk unstable with one of the real parts of the eigenvalues becoming positive. When merging occurs, the degenerate frequency remains unchanged and merging occurs on both sides of the degenerate rotation speed in the natural frequency-rotation speed diagram.

When modal interaction occurs in the high frequency range, it is found for radius ratio 0.5 that veering occurs when both modes are non-reflected waves and have the same number of nodal circle. In the case when the two non-reflected waves have different number of nodal circle, frequency loci merge together. For example, in Fig. 4 merging occurs when mode $(0,3)_f$ meets $(1,1)_b$, and veering occurs when $(1,3)_b$ meets $(1,1)_f$.

Figures 5 and 6 show the cases for a different radius ratio $a/b = 0.1$ (i.e., $a = 20.32$ mm) under uniform edge traction $\epsilon = 4 \times 10^6$ N/m^2 and nonuniform traction $\epsilon \cos 4\theta$, respectively. Figure 5 shows that uniform compressive edge traction brings down the natural frequencies of most of the non-reflected waves, except the zero-nodal-circle modes with zero and one nodal diameter. It is noted that while the natural

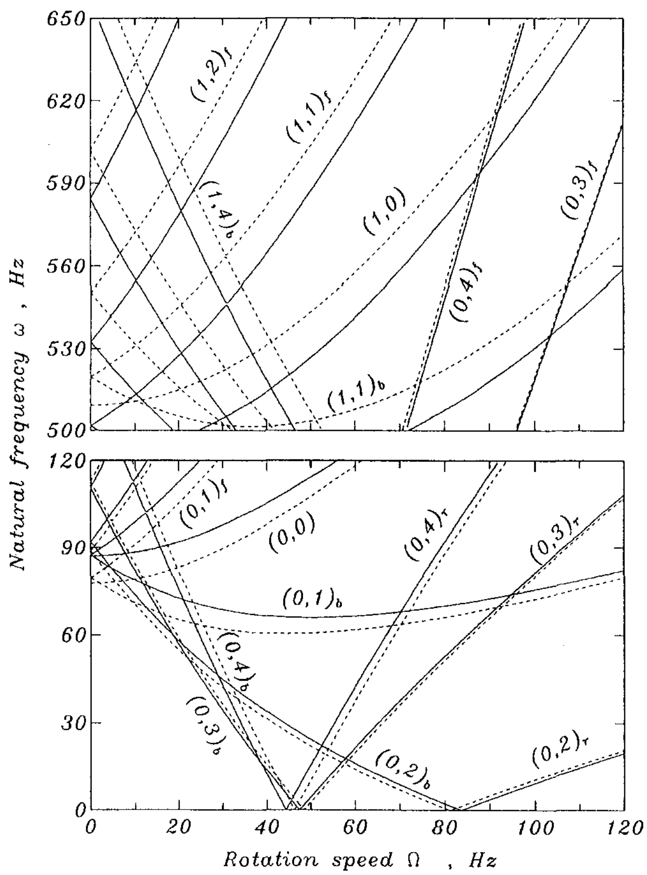

Fig. 2 Effects of uniform compressive follower traction on the eigenvalues of the spinning disk with radius ratio 0.5

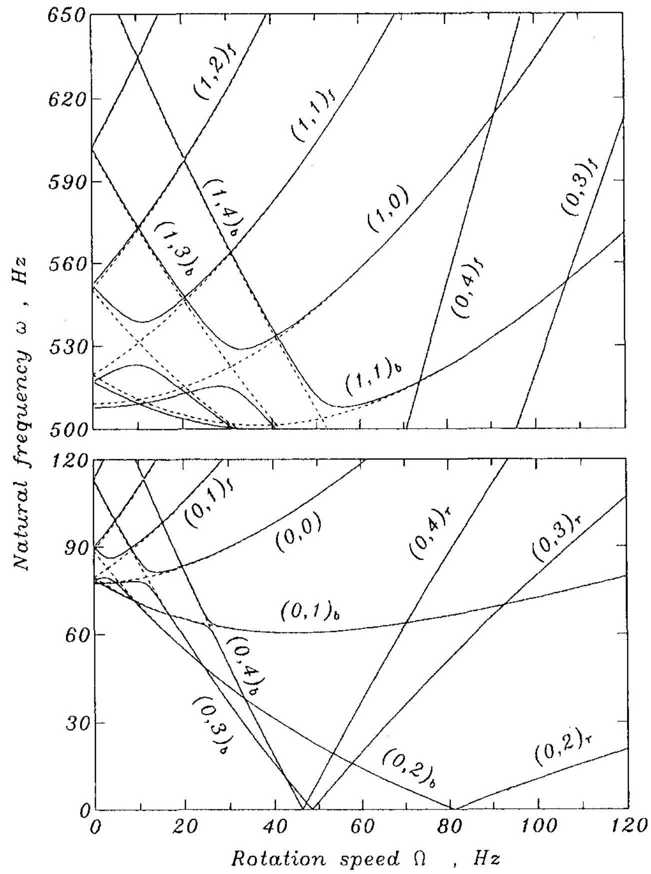

Fig. 3 Effects of follower edge traction $\epsilon \cos 3\theta$ on the eigenvalues of the spinning disk with radius ratio 0.5

frequencies of modes $(0,2)_f$ and $(0,2)_b$ increase under compressive edge traction for radius ratio 0.5, the natural frequencies of these two modes decrease for the radius ratio 0.1. Figure 6 shows that for a disk with radius ratio 0.1, merging occurs when $(1,4)_b$ wave meets $(1,0)$ mode. This phenomenon is different from the case with radius ratio 0.5, in which veering occurs whenever both modes are non-reflected and have the same number of nodal circles, as shown in Fig. 4.

Contrary to the case of conservative edge traction (Chen, 1996), it is difficult to draw any simple conclusion regarding the effect of follower edge traction on the eigenvalue changes and modal interactions based on the above observations. In particular, while the effect of the conservative edge traction is independent of the radius ratio, the effect of the follower edge traction differs as the radius ratio of the annular disk varies. In the following, we will derive analytically some formulae to predict the effects of the follower edge traction based on the eigensolutions of the freely spinning disk. In doing so, we can also examine the correctness of the finite element solutions by comparing with the eigenfunction expansion solutions.

Derivative of Eigenvalue

The derivative of the eigenvalue λ_{mn} of the spinning disk with respect to the magnitude of the distributed edge traction ϵ can be derived following a procedure described in Chen and Bogy (1992a). For the spinning disk under uniform traction, it is found that

$$\frac{\partial \lambda_{mn}}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{-ibR_{mn}(b)\,\dfrac{\partial R_{mn}(b)}{\partial r} + i\displaystyle\int_a^b \left[r\sigma_r^*\left(\frac{\partial R_{mn}}{\partial r}\right)^2 + \frac{n^2\sigma_\theta^*}{r} R_{mn}^2 \right] dr}{2\rho(\omega_{mn} \pm n\Omega)\displaystyle\int_a^b R_{mn}^2 r dr} \tag{3}$$

It is noted that the derivative is evaluated at the point $\epsilon = 0$. Since this derivative is purely imaginary, the uniform traction affects the natural frequencies of the spinning disk, but not the real part of the eigenvalue. It is also noted that the denominator is positive for a non-reflected wave, but is negative for a reflected wave (Chen and Bogy, 1992a). Furthermore, since σ_r^* and σ_θ^* are positive due to the uniform edge traction, the integral term in the numerator is positive. On the other hand, the first term in the numerator is always negative. Therefore, the sign of the numerator depends on the magnitude of these two terms. We can observe from Eq. (3) that the integral term in the numerator tends to dominate for those modes with large n, i.e., the number of nodal diameters. Therefore, the derivative (3) is a positive imaginary number for a non-reflected wave with large number of nodal diameters. Physically speaking, the natural frequency of a non-reflected wave with a large number of nodal diameters tends to increase when the magnitude of the uniform traction increases from zero to a positive number, i.e., a spinning disk under tensile edge traction. It is obvious that the natural frequency changes in the opposite way as the edge traction is compressive. The way uniform edge traction affects the natural frequency of a reflected wave is opposite to the way edge traction affects the non-reflected wave, because the sign of the numerator in Eq. (3) remains the same and the sign in the denominator changes. Equation (3) can be used to predict the eigenvalue changes as the uniform edge traction is applied, as long as the eigensolutions of the freely spinning disk are known.

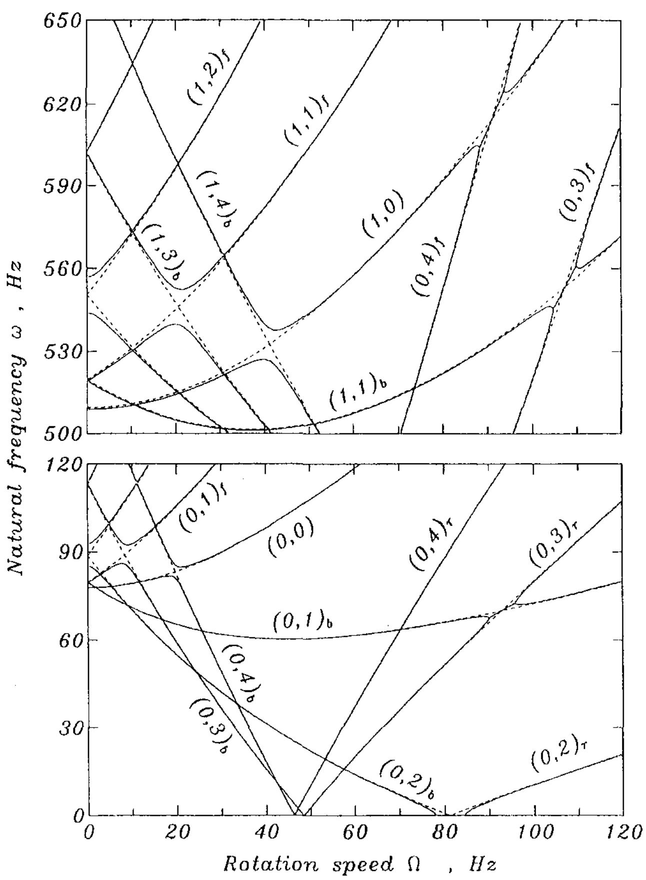

Fig. 4 Effects of follower edge traction ϵ cos 4θ on the eigenvalues of the spinning disk with radius ratio 0.5

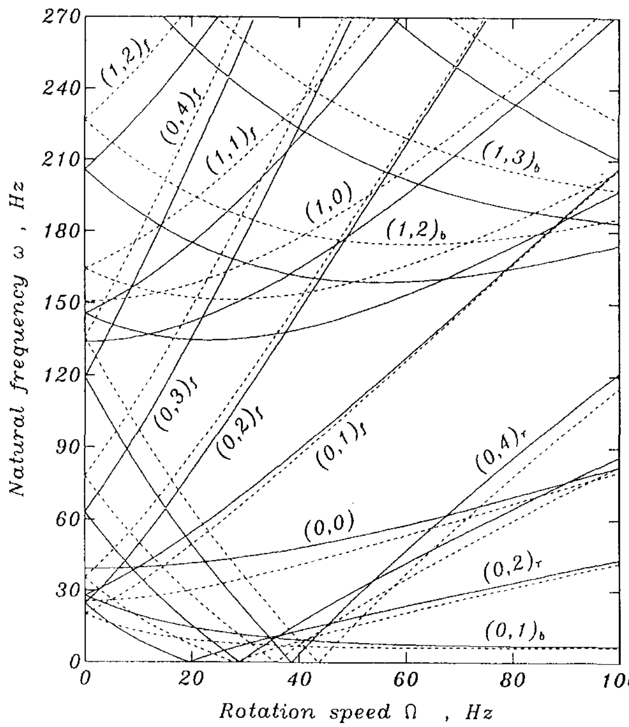

Fig. 5 Effects of uniform compressive edge traction on the eigenvalues of the spinning disk with radius ratio 0.1

When the spinning disk is under follower edge traction in the form $\epsilon \cos k\theta$ with $k \neq 0$, it is found that $\partial\lambda_{mn}/\partial\epsilon = 0$. This verifies the observation that nonuniform traction has no effect on the eigenvalues of the spinning disk, at least to the first order in the sense of Taylor series expansion. However, this conclusion is good only when the natural frequency of interest is well separated from others. When the natural frequencies of two modes are close together, strong modal interaction may occur, and first order approximation fails to predict the complex behavior of eigenvalue changes.

Modal Interactions

When two modes are almost degenerate, as observed from the finite element solutions, modal interaction may or may not happen when nonuniform traction is applied. In this section we follow a procedure described in Chen and Bogy (1992b) based on a two-mode eigenfunction expansion theory to investigate the complex mathematical structure associated with various types of modal interactions.

Assume for the freely spinning disk that the two neighboring modes of interest are w_{mn} and w_{pq} with natural frequencies ω_{mn} and ω_{pq} respectively. For the two-mode approximation, the eigenvalue λ of the spinning disk under distributed edge traction

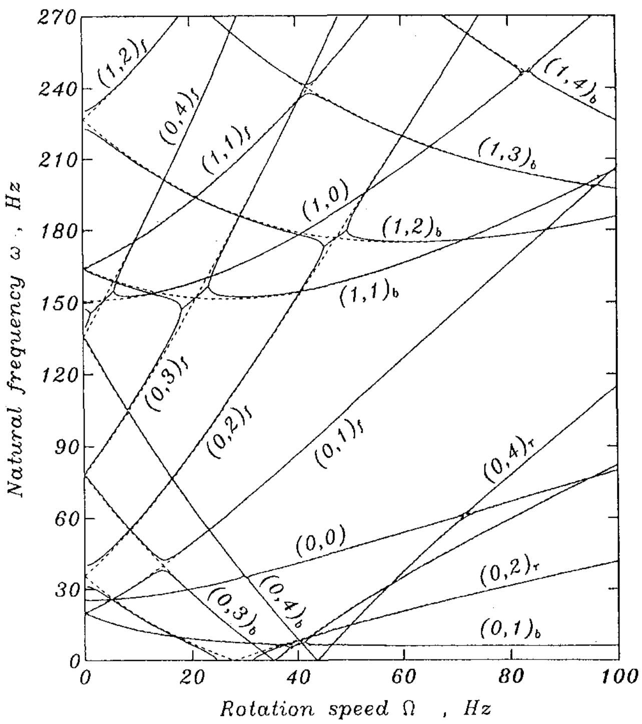

Fig. 6 Effects of follower edge traction ϵ cos 4θ on the eigenvalues of the spinning disk with radius ratio 0.1

ϵ cos $k\theta$ can be obtained by solving the following quadratic equation,

$$\lambda^2(\alpha_0 + \alpha_2\epsilon^2) + \lambda(\beta_0 + \beta_2\epsilon^2) + \gamma_0 + \gamma_2\epsilon^2 = 0 \quad (4)$$

In the case when $n - q \neq \pm k$, α_2, β_2, and γ_2 are zero, and α_0, β_0, and γ_0 are

$$\alpha_0 = 16\pi^2\, \omega_{mn}\omega_{pq}(\omega_{mn} + n\Omega) \times (\omega_{pq} + q\Omega) \int_a^b R_{mn}^2 r dr \int_a^b R_{pq}^2 r dr$$

$$\beta_0 = -i(\omega_{mn} + \omega_{pq})\alpha_0$$

$$\gamma_0 = -\,\omega_{mn}\omega_{pq}\alpha_0$$

The solution λ of Eq. (4) can then be solved as $i\omega_{mn}$ and $i\omega_{pq}$. Therefore, the eigenvalues of these two modes do not change when the edge traction ϵ cos $k\theta$ is applied, and no modal interaction is induced.

On the other hand, when $n - q = \pm k$ the coefficients α_0, β_0, and γ_0 remain the same, while α_2, β_2, and γ_2 become

$$\alpha_2 = -\frac{\pi^2}{\rho^2}\left[A - bR_{mn}(b)\frac{\partial R_{pq}(b)}{\partial r}\right]\left[A - bR_{pq}(b)\frac{\partial R_{mn}(b)}{\partial r}\right]$$

$$\beta_2 = -2i(\omega_{mn} + \omega_{pq})\alpha_2 \quad (5)$$

$$\gamma_2 = -(\omega_{mn} + \omega_{pq})^2\,\alpha_2$$

where

$$A = \int_a^b \left[r\sigma_r^* \frac{\partial R_{mn}}{\partial r}\frac{\partial R_{pq}}{\partial r} + q\left(n\frac{\sigma_\theta^*}{r} - \frac{\partial \sigma_{r\theta}^*}{\partial r}\right) R_{mn}R_{pq} - (n+q)\,\sigma_{r\theta}^* R_{mn}\frac{\partial R_{pq}}{\partial r}\right) \Bigg] dr$$

It is noted that the solution of Eq. (4) is independent of the sign of ϵ and can be written in terms of the parameter $\eta = \epsilon^2$ as

$$\lambda_\pm = \frac{-(\beta_0 + \beta_2\eta) \pm [4\omega_{mn}\omega_{pq}\alpha_0\alpha_2(\eta - \eta^*)]^{1/2}}{2(\alpha_0 \pm \alpha_2\eta)}$$

where

$$\eta^* = \frac{(\omega_{mn} - \omega_{pq})^2\alpha_0}{4\omega_{mn}\omega_{pq}\alpha_2} \quad (6)$$

It is noted that α_0, α_2, γ_0, and γ_2 are real, while β_0 and β_2 are purely imaginary. In the case when both modes are non-reflected, α_0 is positive (Chen and Bogy, 1992b). When one of these two modes is a reflected wave and the other is non-reflected, α_0 is negative. On the other hand, there is no simple rule to determine the sign of α_2 in general. However, as long as the eigensolutions of the freely spinning disk are known, we can always determine the sign of α_2 by Eq. (5). When α_0 and α_2 are of the same sign, η^* is positive. Consequently, $\lambda_\pm$ are two different purely imaginary numbers when $\eta < \eta^*$, and become two complex numbers with the same imaginary parts when $\eta > \eta^*$. Therefore, frequency loci in the natural frequency-rotation speed diagram merge together. On the other hand, when α_0 and α_2 are of the opposite sign, η^* is negative and veering occurs.

In Fig. 7 we fix the rotation speed Ω at 108 Hz, just a little higher than the degenerate rotation speed of modes $(1,1)_b$ and $(0,3)_f$, and change the edge traction magnitude ϵ from 0 to 2.5 $\times$ 10^6 N/m^2. The dashed lines represent the eigenvalues obtained by solving the quadratic equation (4), while the solid lines represent the finite element solutions. It can be seen that the two-mode eigenfunction expansion exhibits all the important characteristics of the finite element solution in the neighborhood of the degenerate frequency. η^* in Eq. (6) corresponds to the traction magnitude ϵ in Fig. 7 at which the two dashed frequency

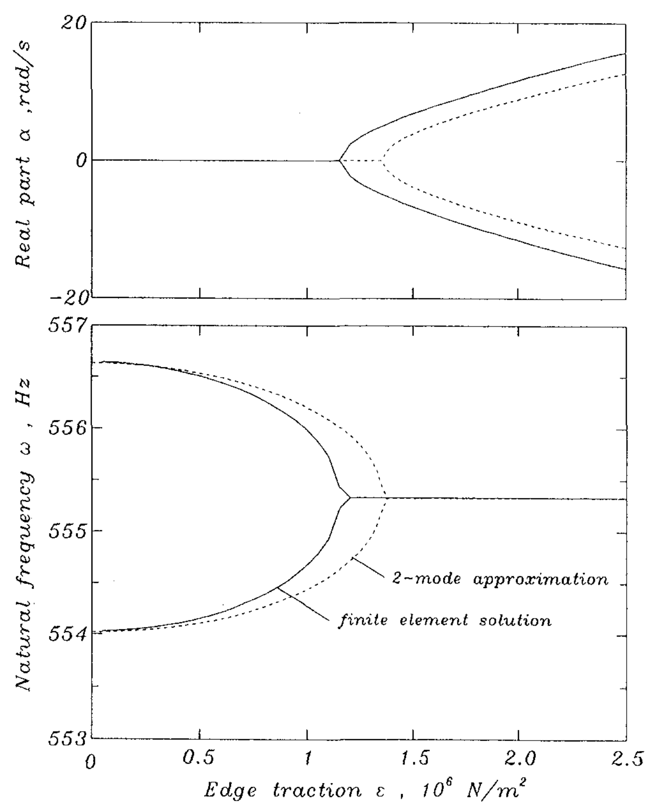

Fig. 7 Comparison between two-mode approximation (dashed lines) and finite element solution (solid lines) at Ω = 108 *Hz*

loci start to merge and their real parts start to split away. It is noted that the two-mode approximation will approach the finite element solution as the fixed rotation speed is chosen closer to the degenerate rotation speed.

In the case when a backward wave $(m,n)_b$ meets its complex conjugate around the critical speed Ω_c, $w_{mn} = \bar{w}_{pq}$, and $n - q = 2n = k$. Equation (4) can then be reduced to

$$\lambda^2 = \frac{-\alpha_0\omega_{mn}^2}{\alpha_0 + \alpha_2\eta}$$

For this case α_2 is always negative and α_0 is always positive. Consequently, for fixed rotation speed close to Ω_c, $\lambda_\pm$ are purely imaginary until η exceeds $\sqrt{-(\alpha_0/\alpha_2)}$. For $\eta > \sqrt{-(\alpha_0/\alpha_2)}$, $\lambda_\pm$ become two real numbers with identical absolute value but different signs. Therefore, we can predict that divergence instability will be induced before and after the critical speed, as we have observed in the finite element solutions.

The above analyses can readily be extended to the case when the edge traction contains more than one Fourier component, for instance, $\epsilon(\cos k_1\theta + \cos k_2\theta)$, where $k_1 \neq k_2$. In doing so we can see that when $n - q = \pm k_1$, only the stress field induced by component $\cos k_1\theta$ affects the coefficients α_2, β_2, and γ_2 in Eqs. (5). In other words, the eigenvalues of the two-mode expansion are independent of the traction component $\cos k_2\theta$. This analysis shows that the effect of each Fourier component on the modal interactions is additive.

Conclusions

The natural frequencies and stability of a spinning disk under stationary follower edge tractions in the form of $\cos k\theta$ are considered in this paper. This is a follow up of our previous work on conservative edge tractions. Unlike the observations and conclusions we have made for conservative loads, no general rules regarding the effects of follower edge tractions on eigenvalues and modal interactions can be drawn, because the phenomenon changes as the radius ratio and the material properties of the annular disk change. However, as long as the eigensolutions of the freely spinning disk are known, we can predict the behavior of eigenvalue changes and modal interactions of the loaded disk by Eq. (3) and by determining the sign of α_2 in Eq. (5). In considering the natural frequency variations and stability properties of a spinning disk under arbitrary follower edge traction, we can always expand the general loading in a Fourier series. After understanding the effect of each Fourier component, we may predict the dynamic response of the loaded spinning disk by considering the effect of the general traction as the combined effects of its individual Fourier components.

Acknowledgments

The results presented here were obtained in the course of research supported by a grant from the National Science Council of the Republic of China. The author thanks Mr. Jhi-Lu Jhu for his assistance in drawing most of the illustrations.

References

Carlin, J. F., Appl, F. C., Bridwell, H. C., and Dubois, R. P., 1975, "Effects of Tensioning on Buckling and Vibration of Circular Saw Blades," ASME *Journal of Engineering for Industry,* Vol. 2, pp. 37–48.

Chen, J.-S., 1994, "Stability Analysis of a Spinning Elastic Disk Under a Stationary Concentrated Edge Load," ASME *Journal of Applied Mechanics,* Vol. 61(4), pp. 788–792.

Chen, J.-S., 1996, "Vibration and Stability of a Spinning Disk Under Stationary Distributed Edge Loads," ASME *Journal of Applied Mechanics,* Vol. 63, pp. 439–444.

Chen, J.-S., and Bogy, D. B., 1992a, "Effects of Load Parameters on the Natural Frequencies and Stability of a Flexible Spinning Disk with a Stationary Load System," ASME *Journal of Applied Mechanics,* Vol. 59(2), Part 2, pp. 230–235.

Chen, J.-S., and Bogy, D. B., 1992b, "Mathematical Structure of Modal Interactions in a Spinning Disk-Stationary Load System," ASME *Journal of Applied Mechanics,* Vol. 59(2), Part 1, pp. 390–397.

Coker, E. G., and Filon, L. N. G., 1957, *A Treatise on Photo-Elasticity,* Cambridge University Press, London.

Mote, C. D. Jr., and Szymani, R., 1978, "Circular Saw Vibration Research," *The Shock and Vibration Digest,* Vol. 10(6), pp. 15–30.

Ono, K., and Maeno, T., 1987, "Theoretical and Experimental Investigation on Dynamic Characteristics of a 3.5-Inch Flexible Disk Due to a Point Contact Head," *Tribology and Mechanics of Magnetic Storage Systems,* Vol. 3, SP.21 (STLE), pp. 144–151.

Radcliffe, C. J., and Mote, C. D., Jr., 1977, "Stability of Stationary and Rotating Discs Under Edge Load," *International Journal of Mechanical Sciences,* Vol. 19, pp. 567–574.

Srinivasan, V., and Ramamurti, V., 1980, "Dynamic Response of an Annular Disk to a Moving Concentrated, In-Plane Edge Load," *Journal of Sound and Vibration,* Vol. 72(2), pp. 251–262.