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Exact Critical Loads for a Pinned Half-Sine Arch Under End Couples

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In this note we show that for a pinned half-sine arch under end couples snap-through buckling will occur unsymmetrically if the initial height of the shallow arch is greater than $6.5466r$, where r is the radius of gyration of the cross section. The closed-form expression for the critical couple can be obtained analytically. [DOI: 10.1115/1.1827244]

1 Introduction

The prediction of snap-through buckling of a shallow arch is a classical problem in applied mechanics. Fung and Kaplan [1] derived the exact critical loads for a pinned half-sine arch under sinusoidal loading. For other load distributions such as uniform pressure and a concentrated force at the midpoint, the critical loads can be obtained by summing a few terms of a rapidly converging Fourier series. For a complete review of the previous works of arch stability, the readers are referred to the two books by Simitses [2,3]. In all these previous works, the external loads causing snap-through buckling are lateral forces. In this note we consider the case when the sinusoidal arch is under couples at both ends.

2 Equilibrium Equation

We consider a pinned shallow arch with equal and opposite moments M^* applied to the two ends. This model finds applica-

tion in an electromechanical switching device with a curved beam coated with piezoelectric films on the top and bottom surfaces. The end moment is proportional to the actuating voltage. The equilibrium equation of the loaded arch can be written as

$$EI(y - y_0)_{,xxxx} - p^* y_{,xx} + M^*[-\delta'(x) + \delta'(x-L)] = 0 \quad (1)$$

p^* is the axial force

$$p^* = \frac{AE}{2L} \int_0^L (y_{,x}^2 - y_{0,x}^2) dx \quad (2)$$

E , A , and I are Young's modulus, area, and moment of inertia of the cross section. L is the distance between the two pinned ends. δ' is the derivative of the Dirac delta function with respect to x . $y_0(x)$ is the initial shape of the arch. Equations (1) and (2) can be nondimensionalized to the forms

$$(u - u_0)_{,\xi\xi\xi\xi} - pu_{,\xi\xi} + \frac{\pi}{4} M[-\delta'(\xi) + \delta'(\xi - \pi)] = 0 \quad (3)$$

$$p = \frac{1}{2\pi} \int_0^\pi (u_{,\xi}^2 - u_{0,\xi}^2) d\xi \quad (4)$$

where

$$u = \frac{y}{r}, \quad u_0 = \frac{y_0}{r}, \quad \xi = \frac{\pi x}{L}, \quad p = \frac{p^* L^2}{\pi^2 EI}, \quad M = \frac{4M^* L^2}{\pi^5 EI r} \quad (5)$$

r is the radius of gyration of the cross section. The initial shape of the arch is assumed to be in the form

$$u_0(\xi) = h \sin \xi \quad (6)$$

It is assumed that the shape of the deformed arch can be expanded as

$$u(\xi) = \sum_{n=1}^{\infty} \alpha_n \sin n\xi \quad (7)$$

After expanding the derivative of the Dirac delta function δ' as a Fourier sine series and substituting Eqs. (6) and (7) into Eqs. (3) and (4) we obtain the equations for α_n

$$n^4 \alpha_n + n^2 p \alpha_n + q_n = 0, \quad n = 1, 2, 3, \dots \quad (8)$$

where

$$p = \frac{1}{4} \sum_{k=1}^{\infty} k^2 \alpha_k^2 - \frac{h^2}{4}, \quad (9)$$

$$q_1 = M - h, \quad (10)$$

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$$q_n = 0, \quad n = 2, 4, 6, \dots, \quad (11)$$

$$q_n = nM, \quad n = 3, 5, 7, \dots \quad (12)$$

3 Equilibrium Configurations

Equation (8) represents a set of an infinite number of coupled nonlinear equations for an infinite number of coordinates α_n . There are two types of solutions.

3.1 Symmetrical Solution. $\alpha_{2i} = 0$, where $i = 1, 2, 3, \dots$. The equations in Eq. (8) with even number of n are satisfied automatically because of condition (11). The remaining coordinates α_{2i+1} can be related to α_1 by a simple deduction procedure from Eq. (8) as

$$\alpha_{2i+1} = \frac{-\alpha_1 q_{2i+1}}{(2i+1)^2 [4i(i+1)\alpha_1 - q_1]}, \quad i = 1, 2, 3, \dots \quad (13)$$

After substituting Eq. (13) into Eq. (9), and substituting the resulting p into Eq. (8) for $n=1$, we obtain the following equation for α_1

$$f_1(\alpha_1) + \sum_{i=1}^{\infty} \left\{ \frac{q_{2i+1}^2 \alpha_1^3}{4(2i+1)^2 [4i(i+1)\alpha_1 - q_1]^2} \right\} = 0 \quad (14)$$

$$\alpha_{2j} = \pm \frac{1}{2j} \sqrt{h^2 - 16j^2 - \frac{q_1^2}{(4j^2 - 1)^2} - \sum_{i=1}^{\infty} \frac{q_{2i+1}^2}{(2i+1)^2 [4j^2 - (2i+1)^2]^2}} \quad (18)$$

This pair of solutions are denoted as $P_{1(2j)}^{\pm}$.

4 Snap-Through Buckling

By inspecting Eq. (14) for the root locus of α_1 we can show that for smaller h the root α_1 corresponding to P_0 will merge with $\alpha_1(P_1^+)$ as M increases from zero. For larger h , on the other hand, $\alpha_1(P_0)$ will merge with $\alpha_1(P_{12}^{\pm})$ instead, which is known explicitly from Eq. (17) as

$$\alpha_1(P_{12}^{\pm}) = \frac{q_1}{3} \quad (19)$$

There then exists a special h , denoted by \bar{h} , at which $\alpha_1(P_0)$ will merge with both $\alpha_1(P_1^+)$ and $\alpha_1(P_{12}^{\pm})$ simultaneously. This situation occurs when Eq. (14) admits a double root, which requires the derivative of Eq. (14) with respect to α_1 to vanish

$$4 \frac{df_1}{d\alpha_1} + \sum_{i=1}^{\infty} \left\{ \frac{q_n^2 \alpha_1^2}{(2i+1)^2 [4i(i+1)\alpha_1 - q_1]^2} \left[3 - \frac{8i(i+1)\alpha_1}{4i(i+1)\alpha_1 - q_1} \right] \right\} = 0 \quad (20)$$

Therefore, if h is smaller than \bar{h} , the arch will snap symmetrically. On the other hand, if h is greater than \bar{h} , the arch will snap unsymmetrically. This note intends to present the exact critical moment for the latter case. After replacing α_1 in Eqs. (14) and (20) by $q_1/3$, both equations can be rearranged further into the forms

$$(1 + 9\kappa_1)M^2 - 2hM + 144 - 8h^2 = 0 \quad (21)$$

$$(1 + 3\kappa_2)M^2 - 2hM + 12 - 2h^2 = 0 \quad (22)$$

where

$$f_1(\alpha_1) = \alpha_1 + \frac{\alpha_1}{4} (\alpha_1^2 - h^2) + q_1 \quad (15)$$

In the special case when $M=0$, the three solutions of Eq. (14) are denoted by P_0 , P_1^+ , and P_1^- [4], respectively. P_0 represents the original shape, Eq. (6). P_1^- is another stable configuration on the other side. P_1^+ is an unstable position between P_0 and P_1^- .

3.2 Unsymmetrical Solution. $\alpha_{2j} \neq 0$ for some j , and all other $\alpha_{2i} = 0$, $i = 1, 2, 3, \dots, i \neq j$. This type of solution involves odd number of n in Eq. (7), plus one additional harmonic with $n = 2j$. For this type of solution we can solve for p from the $2j$ th equation of Eq. (8) as

$$p = -4j^2 \quad (16)$$

After substituting Eq. (16) into the $(2i+1)$ -th equation in Eq. (8) we can solve for α_{2i+1} exactly as

$$\alpha_{2i+1} = \frac{q_{2i+1}}{(2i+1)^2 [4j^2 - (2i+1)^2]} \quad i = 0, 1, 2, \dots \quad (17)$$

After substituting Eqs. (16) and (17) into Eq. (9) we can solve for α_{2j} as

where

$$\kappa_1 = \sum_{i=1}^{\infty} \frac{1}{(2i-1)^2 (2i+3)^2} = \frac{9\pi^2 - 64}{576} \quad (23)$$

$$\kappa_2 = \sum_{i=1}^{\infty} \frac{1}{(2i-1)^3 (2i+3)^3} = \frac{51\pi^2 - 512}{1536} \quad (24)$$

After eliminating M from Eqs. (21) and (22) we can solve for this special h as $\bar{h} = 6.5466$.

For the easy use in practical design procedure, we may summarize the conclusion in terms of physical parameters as follows. If the initial height h^* at the midpoint of the sinusoidal arch under end couples is greater than $6.5466r$, the shallow arch will snap unsymmetrically, and the critical couples can be found exactly as

$$M_{cr}^{*\pm} = \frac{\pi^3 EI}{9L^2} [16h^* \pm 2\sqrt{(64 + 72\pi^2)h^{*2} - 1296r^2\pi^2}] \quad (25)$$

M_{cr}^{*+} corresponds to the critical couple which will snap the arch from position P_0 to P_1^- . On the other hand, M_{cr}^{*-} , which is always negative, corresponds to the critical couple which will allow the arch to snap back from position P_1^- to P_0 .

References

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