

$\sigma_{1,i}, \sigma_{2,i}, \sigma_{3,j}, p_i, q_i, r_j$, and ζ_j^* , $i = 1, \dots, n, j = 2, \dots, n$, the compact set Π can be kept arbitrarily small. Thus, the errors in the controlled closed-loop system can be made arbitrarily small.

REFERENCES

- [1] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. Hoboken, NJ: Wiley Interscience, 1995.
- [2] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Systematic design of adaptive controller for feedback linearizable system," *IEEE Trans. Autom. Control*, vol. 36, no. 11, pp. 1241–1253, 1991.
- [3] D. Swaroop, J. K. Hedrick, P. P. Yip, and J. C. Gerdes, "Dynamic surface control for a class of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 10, pp. 1893–1899, 2000.
- [4] P. P. Yip and J. K. Hedrick, "Adaptive dynamic surface control: A simplified algorithm for adaptive backstepping control of nonlinear systems," *Int. J. Control*, vol. 71, no. 5, pp. 959–979, 1998.
- [5] D. Wang and J. Huang, "Neural network-based adaptive dynamic surface control for a class of uncertain nonlinear systems in strict-feedback form," *IEEE Trans. Neural Netw.*, vol. 16, no. 1, pp. 195–202, 2005.
- [6] Z. J. Yang, K. Miyazaki, S. Kane, and K. Wada, "Robust position control of a magnetic levitation system via dynamic surface control technique," *IEEE Trans. Ind. Electron.*, vol. 51, no. 1, pp. 26–34, 2004.
- [7] S. Duraiswamy and G. T. C. Chiu, "Nonlinear adaptive nonsmooth dynamic surface control of electro-hydraulic systems," in *Proc. Amer. Control Conf.*, 2003, pp. 3287–3292.
- [8] S. K. Nguang, "Robust stabilization of a class of time-delay nonlinear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 4, pp. 756–762, 2000.
- [9] S. S. Ge, F. Hong, and T. H. Lee, "Robust adaptive control of nonlinear systems with unknown time delays," *Automatica*, vol. 41, pp. 1181–1190, 2005.
- [10] S. S. Ge, F. Hong, and T. H. Lee, "Adaptive neural network control of nonlinear systems with unknown time delays," *IEEE Trans. Autom. Control*, vol. 48, no. 11, pp. 2004–2010, 2003.
- [11] F. Hong, S. S. Ge, and T. H. Lee, "Practical adaptive neural control of nonlinear systems with unknown time delays," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 35, no. 4, pp. 849–854, 2005.
- [12] D. W. C. Ho, J. Li, and Y. Niu, "Adaptive neural control for a class of nonlinearly parametric time-delay systems," *IEEE Trans. Neural Netw.*, vol. 16, no. 3, pp. 625–635, 2005.
- [13] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 1996.
- [14] S.-L. Niculescu, *Delay Effects on Stability: A Robust Control Approach*. New York: Springer-Verlag, 2001.
- [15] P. A. Ioannou and P. V. Kokotovic, *Adaptive Systems with Reduced Models*. New York: Springer-Verlag, 1983.
- [16] L. X. Wang, *A Course in Fuzzy Systems and Control*. Upper Saddle River, NJ: Prentice-Hall, 1997.
- [17] K. S. Narendra and A. M. Annaswamy, "A new adaptive law for robust adaptation without persistent excitation," *IEEE Trans. Autom. Control*, vol. 32, no. 2, pp. 134–145, 1987.

Robust Nonlinear Observer for Lipschitz Nonlinear Systems Subject to Disturbances

Min-Shin Chen and Chi-Che Chen

Abstract—This note proposes a robust nonlinear observer for systems with Lipschitz nonlinearity. The proposed nonlinear observer, whose linear part adopts the linear LTR observer design technique, has two important advantages over previous designs. First, the new observer does not impose the small-Lipschitz-constant condition on the system nonlinearity, nor other structural conditions on the system dynamics as in the existing observer designs. Second, it is robust in the sense that its state estimation error decays to almost zero even in the face of large external disturbances.

Index Terms—Disturbance, Lipschitz condition, nonlinear observer, nonlinear system, robust observer.

I. INTRODUCTION

State estimation of nonlinear systems has been a research topic for decades. A survey paper on this topic can be found in [1]. In recent literature, there are mainly four different approaches to the nonlinear observer design. In the first approach [2]–[4], the nonlinear system is transformed, via a nonlinear state transformation, into a so-called observer normal form. This special normal form admits the design of a nonlinear observer that yields a completely linear error dynamics. Linear observer theory can then be applied to guarantee stability. However, the conditions to obtain completely linearizable error dynamics are strict, and finding the required nonlinear state transformation is nontrivial.

In the second approach, the system dynamics is split into a linear part and a nonlinear part. The linear part is assumed to be observable from the system output, and the nonlinear part is locally or globally Lipschitz. The idea is first suggested in [5], and then further pursued in [6]–[8]. The design can be easily carried out by solving a Lyapunov equation [5] or a Riccati equation [7]. Reduce-order nonlinear observers of this type are introduced in [9] and [10]. Unfortunately, this approach suffers from a severe limitation that the Lipschitz constant of the nonlinearity has to be small.

There are some other approaches to the nonlinear observer design that do not impose the small-Lipschitz-constant condition. These approaches include the variable structure observer design [11]–[13] and the high-gain observer design [14], [15]. However, their application is limited due to strict conditions on the system dynamics. For example, in the variable structure observer design [11]–[13], the transfer function from the nonlinearity to the system output is limited to be strictly positive real (SPR). In the high-gain observer design [14], the system dynamics must be in a semitriangular form. In [15], one must assume nonsingularity of a nonlinear transformation matrix, and the transformed system must have an input matrix that is also in a triangular form.

To overcome the limitations of existing observer designs, this note proposes a new nonlinear observer that imposes less restrictions, and hence has a larger application domain. The new observer design utilizes the linear LRT observer design technique [16], [17] to enhance the

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The authors are with the Department of Mechanical Engineering, National Taiwan University, Taipei 106, Taiwan, Republic of China (e-mail: mschen@ntu.edu.tw).

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robustness of nonlinear observer. The resultant nonlinear observer has two important advantages over conventional observers. The first advantage is that the new observer does not impose the small-Lipschitz-constant condition as in [5]–[10], nor the SPR condition as in [11]–[13], nor the structural conditions as in [14] and [15]. The only assumption required by the new observer is that the transfer function from the nonlinearity to the system output be minimum-phase [meaning that the triple (A, G_1, C) in (1) has only stable zeros]. The second advantage of the proposed observer is that its performance is robust with respect to external disturbances. The robustness issue of nonlinear observer against disturbance is rarely studied in the literature. Previous nonlinear observer designs [18]–[20] ensure only bounded-input bounded-state stability, while the new observer in this note ensures *practical* stability, meaning that the new observer in this note can achieve almost zero estimation error in the face of large disturbances.

The remainder of this note is arranged as follows. Section II presents the new nonlinear observer design for nonlinear systems that have an arbitrarily large Lipschitz constant. Section III further modifies the nonlinear observer so that its performance is robust with respect to external disturbances. Finally, Section IV gives the conclusions.

II. NONLINEAR OBSERVER

Consider the state estimation problem for a nonlinear system

$$\begin{aligned}\dot{x} &= Ax + Bu + G_1 f(x) \\ y &= Cx\end{aligned}\quad (1)$$

where $x \in R^n$ is the unknown system state, $u \in R^m$ is the control input, $y \in R^p$ is the measured system output, and $f(x) \in R^q$ is a nonlinearity satisfying the Lipschitz condition

$$\|f(x) - f(\hat{x})\| \leq \gamma \|x - \hat{x}\| \quad (2)$$

for some Lipschitz constant $\gamma > 0$. It is assumed that

$$\dim(y) \geq \dim(f(x)). \quad (3)$$

Without loss of generality it is assumed in the sequel that $\dim(y) = \dim(f(x))$; the case $\dim(y) > \dim(f(x))$ can be treated similarly by artificially expanding the dimension of $f(x)$. Further, it is assumed that (A, C) is observable, (A, G_1) controllable, and the square system $(A + \alpha I, G_1, C)$ has only stable zeros (minimum-phase), where $\alpha > 0$ is a design parameter to be specified later. Notice that one does not require the boundedness assumption of the control input u .

The proposed nonlinear observer for (1) is as follows:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) + G_1 f(\hat{x}), \quad L = QC^T \quad (4)$$

where the output injection gain $L \in R^{n \times p}$ is designed as in the linear LTR observer [16], [17], with Q a positive definite matrix solved from the LQ Riccati equation

$$Q(A + \alpha I)^T + (A + \alpha I)Q - QC^T CQ + \pi G_1 G_1^T = 0, \quad \pi > 0, \quad \alpha > 0. \quad (5)$$

The existence and positive definiteness of the solution Q of the LQ Riccati equation (5) is always guaranteed [21], [22] for all design parameters $\pi > 0$ and $\alpha > 0$ if (A, C) is observable and (A, G_1) controllable. Note that in [7], [8], and [10], their nonlinear observers use an H_∞ Riccati equation [21] in the following form:

$$QA^T + AQ - Q(C^T C - \gamma^2)Q + I = 0$$

which contains the Lipschitz constant γ . In order for this H_∞ Riccati equation to have a positive definite solution matrix, the Lipschitz constant γ is constrained to be small. In contrast, this note uses the LQ Riccati equation (5), which is γ independent.

Note that the solution matrix $Q \in R^{n \times n}$ depends on the design parameter π in (5). The relationship between $Q(\pi)$ and π satisfies the following theorem.

Theorem 1 [16], [23]: If the system $(A + \alpha I, G_1, C)$ is minimum-phase, C and G_1 are full rank, and $\text{rank}(C^T) = \text{rank}(G_1)$, the solution $Q(\pi)$ of the observer Riccati equation (5) satisfies

$$\lim_{\pi \rightarrow \infty} \frac{Q(\pi)}{\pi} = 0.$$

The state estimation error $\tilde{x} = x - \hat{x}$ resulting from the above nonlinear observer satisfies

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G_1(f(x) - f(\hat{x})). \quad (6)$$

The goal of this section is to prove that even for large Lipschitz constant in (2), the error dynamics (6) is globally exponentially stable if π in (5) is sufficiently large. This is the purpose of the theorem shown here.

Theorem 2: Consider the nonlinear system (1) with C and G_1 both full rank, $\text{rank}(C^T) = \text{rank}(G_1)$, and $(A + \alpha I, G_1, C)$ minimum-phase. For whatever large Lipschitz constant γ in (2), if the design parameter $\pi > 0$ in the Riccati equation (5) is sufficiently large, the state estimate \hat{x} of the nonlinear observer (4) approaches the true state x exponentially fast.

Proof: Define a Lyapunov function $V = \tilde{x}^T Q^{-1} \tilde{x}$, where $Q > 0$ is from the observer Riccati equation (5), and \tilde{x} as in (6). The change rate of V along the trajectory (6) satisfies

$$\begin{aligned}\dot{V} &\leq -2\alpha V - \|\tilde{y}\|^2 - \pi \left\| G_1^T Q^{-1} \tilde{x} \right\|^2 \\ &\quad + 2 \|f(x) - f(\hat{x})\| \cdot \left\| G_1^T Q^{-1} \tilde{x} \right\| \\ &\leq -2\alpha V - \|\tilde{y}\|^2 - \pi \left\| G_1^T Q^{-1} \tilde{x} \right\|^2 \\ &\quad + 2\gamma \|\tilde{x}\| \cdot \left\| G_1^T Q^{-1} \tilde{x} \right\|\end{aligned}$$

where $\tilde{y} = C\tilde{x}$ and the Lipschitz condition (2) was used to obtain the second inequality. Note that the maximum of the last two terms in the last equation occurs when $\|G_1^T Q^{-1} \tilde{x}\| = \gamma \|\tilde{x}\|/\pi$, with the maximum value being $\gamma^2 \|\tilde{x}\|^2/\pi$. Hence

$$\dot{V} \leq -2\alpha V - \|\tilde{y}\|^2 + \frac{\gamma^2 \|\tilde{x}\|^2}{\pi} \leq -2\alpha V + \frac{\gamma^2 \|\tilde{x}\|^2}{\pi}.$$

Using

$$V \geq \underline{\sigma}(Q^{-1}) \|\tilde{x}\|^2 = \|\tilde{x}\|^2 / \bar{\sigma}(Q) \quad (7)$$

one obtains

$$\dot{V} \leq - \left[2\alpha - \gamma^2 \frac{\bar{\sigma}(Q)}{\pi} \right] V. \quad (8)$$

According to Theorem 1, $\bar{\sigma}(Q)/\pi$ approaches zero as the observer design parameter π approaches infinity. Hence, given any Lipschitz constant γ , there always exists a sufficiently large design parameter π such that the number in the square bracket in (8) is positive. This implies that $V(t)$ and, hence, $\tilde{x}(t)$ decay to zero exponentially. One thus proves that the error dynamics (6) is globally exponentially stable. End of proof.

Remark 1: The proposed nonlinear observer (4) has two design parameters: π and α in (5). As explained in the proof of Theorem 2, $\pi > 0$ must be sufficiently large to ensure the observer stability, and $\alpha > 0$ is to specify the estimation error convergence rate.

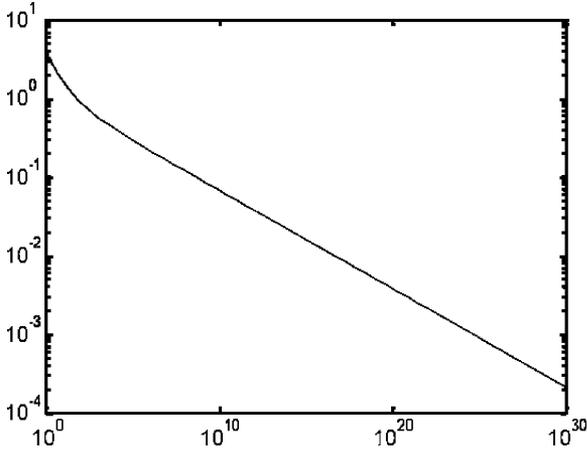


Fig. 1. $\sigma(Q)/\pi$ versus π .

Remark 2: Equation (8) mandates that the observer design parameter π must satisfy

$$\frac{\bar{\sigma}(Q(\pi))}{\pi} < \frac{2\alpha}{\gamma^2} \quad (9)$$

to ensure the observer stability. If the Lipschitz constant γ is known, and α has been chosen, one can determine a stabilizing π by referring to a figure that depicts the value of $\bar{\sigma}(Q)/\pi$ versus π . Fig. 1 shows such a figure for the system $(A, [G_1, G_2], C)$ in the simulation example in the next section.

Remark 3: Note that if the observer design parameter π is chosen too large, the proposed observer becomes high-gain. A disadvantage of high-gain observers is the peaking phenomenon [24], in which the estimated state $\hat{x}(t)$ peaks to extremely large values during the very initial period of the observation process. One way to relieve the peaking phenomenon is to stepwisely schedule the design parameter $\pi(t) = \pi_i, t \in [t_k, t_{k+1}]$, where π_i jumps within a finite time from 1 to the designed value requested by (9). See the simulation example in the next section.

III. ROBUST NONLINEAR OBSERVER

In practical applications, it may frequently happen that the nonlinear system (1) is subject to unknown external disturbances. It is therefore important to construct the nonlinear observer such that its performance is robust against external disturbances, and this will be the subject of this section.

Consider a nonlinear system subject to bounded disturbances

$$\begin{aligned} \dot{x} &= Ax + Bu + G_1 f(x) + G_2 d \\ y &= Cx \end{aligned} \quad (10)$$

where all symbols are as defined in (1), the nonlinearity $f(x)$ satisfies the Lipschitz condition (2), and $d \in R^r$ represents a bounded external disturbance

$$\|d(t)\| \leq D. \quad (11)$$

Note that $G_1 \neq G_2$; that is, the disturbance d enters the system equation in a direction different from that of the nonlinearity $f(x)$. Further, it is assumed that (A, C) is observable, and $(A, [G_1, G_2])$ controllable.

The ‘‘robust’’ nonlinear observer design is more difficult than the nonlinear observer design treated in the previous section. However, it

will be shown that if the number of output sensors is sufficient in the sense that

$$\dim(y) \geq \dim(f(x)) + \dim(d) \quad (12)$$

then the design of a robust nonlinear observer is achievable. For simplicity, it will be assumed in the sequel that $\dim(y) = \dim(f(x)) + \dim(d)$. The case $\dim(y) > \dim(f(x)) + \dim(d)$ can be treated similarly by artificially expanding the dimension of d or $f(x)$.

A robust nonlinear observer for the system (10) that can cope with both the nonlinearity and the disturbance is proposed as follows:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) + G_1 f(\hat{x}) \quad (13)$$

where the output injection gain $L = QC^T \in R^{n \times p}$, but the positive definite matrix $Q \in R^{n \times n}$ is now obtained from a *modified* Riccati equation

$$Q(A + \alpha I)^T + (A + \alpha I)Q - QC^T CQ + \pi[G_1, G_2][G_1, G_2]^T = 0, \quad \pi > 0, \quad \alpha > 0. \quad (14)$$

Note that the existence of a positive definite solution matrix $Q > 0$ is guaranteed [21], [22] when (A, C) is observable, and $(A, [G_1, G_2])$ controllable.

The following theorem shows that the above robust nonlinear observer (13) guarantees that the estimation error $x - \hat{x}$ decays to almost zero even in the face of large disturbance d .

Theorem 3: Consider the nonlinear system (10) subject to a bounded disturbance d , and the assumptions that C and $[G_1, G_2]$ both full rank, $\text{rank}(C^T) = \text{rank}([G_1, G_2])$, and $(A + \alpha I, [G_1, G_2], C)$ minimum-phase. For whatever large Lipschitz constant γ , and whatever large disturbance d , the robust nonlinear observer (13) guarantees that the state estimation error $\tilde{x} = x - \hat{x}$ will eventually approach a small residual set containing the origin, with the size of the residual set approaching zero as the design parameter π in (14) approaches infinity.

Proof: From the assumptions $\text{rank}(C^T) = \text{rank}([G_1, G_2])$ and $(A + \alpha I, [G_1, G_2], C)$ minimum-phase, one can quote Theorem 1 to conclude that the solution $Q(\pi)$ of the modified Riccati equation (14) still satisfies

$$\lim_{\pi \rightarrow \infty} \frac{Q(\pi)}{\pi} = 0. \quad (15)$$

With the unknown disturbance d , the state estimation error $\tilde{x} = x - \hat{x}$ resulting from the proposed robust nonlinear observer (13) obeys the dynamics

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + [G_1, G_2] \begin{bmatrix} f(x) - f(\hat{x}) \\ d \end{bmatrix} \quad (16)$$

where the last term satisfies the bound

$$\begin{aligned} \left\| \begin{bmatrix} f(x) - f(\hat{x}) \\ d \end{bmatrix} \right\| &\leq \|f(x) - f(\hat{x})\| + \|d\| \\ &\leq \gamma \|\tilde{x}\| + D \end{aligned} \quad (17)$$

in which one has used (2) and (11) to obtain the second inequality.

Choose a Lyapunov function $V = \tilde{x}^T Q^{-1} \tilde{x}$, where Q is from (14). Checking the change rate of V along the error dynamics (16) gives

$$\dot{V} \leq -2\alpha V - \|\tilde{y}\|^2 - \pi \|G^T Q^{-1} \tilde{x}\|^2 + 2(\gamma \|\tilde{x}\| + D) \|G^T Q^{-1} \tilde{x}\|$$

where $G = [G_1, G_2]$ and (17) is used to derive the inequality. Note that the maximum of the last two terms in the above equation occurs

when $\|G^T Q^{-1} \hat{x}\| = (\gamma \|\hat{x}\| + D)/\pi$, with the maximum value being $(\gamma \|\hat{x}\| + D)^2/\pi$. Hence

$$\begin{aligned} \dot{V} &\leq -2\alpha V - \|\tilde{y}\|^2 + \frac{(\gamma \|\hat{x}\| + D)^2}{\pi} \\ &\leq -\alpha V - \alpha \left(V - \frac{(\gamma \|\hat{x}\| + D)^2}{\alpha\pi} \right). \end{aligned}$$

From the last inequality, one can say that $\dot{V} < 0$ as long as $V > (\gamma \|\hat{x}\| + D)^2/(\alpha\pi)$. Therefore, asymptotically one has

$$\lim_{t \rightarrow \infty} V(t) \leq (\gamma \|\hat{x}\| + D)^2/(\alpha\pi)$$

which implies, using (7),

$$\lim_{t \rightarrow \infty} \|\hat{x}(t)\| \leq \sqrt{\frac{\bar{\sigma}(Q)}{\alpha\pi}} (\gamma \|\hat{x}\| + D).$$

Rearranging the equation gives

$$\lim_{t \rightarrow \infty} \left(1 - \gamma \sqrt{\frac{\bar{\sigma}(Q)}{\alpha\pi}} \right) \|\hat{x}(t)\| \leq D \sqrt{\frac{\bar{\sigma}(Q)}{\alpha\pi}}.$$

The property (15) ensures that given any Lipschitz constant γ , there always exists a sufficiently large observer design parameter π such that the number in the parenthesis in the above equation is positive. Hence, one can write

$$\lim_{t \rightarrow \infty} \|\hat{x}(t)\| \leq \frac{D \sqrt{\frac{\bar{\sigma}(Q)}{\alpha\pi}}}{1 - \gamma \sqrt{\frac{\bar{\sigma}(Q)}{\alpha\pi}}}. \quad (18)$$

The above inequality together with the property (15) imply that given any bounded Lipschitz constant γ and disturbance upper bound D , $\|\hat{x}\|$ will eventually become arbitrarily small as long as π is sufficiently large. End of proof.

The last theorem in this note given below will discuss the special case when the disturbance d enters the state equation (10) in the same direction as the nonlinearity $f(x)$; that is, the case $G_1 = G_2$. For this special case, the requirement on the sensor number can be relaxed from (12) to (3); hence, one needs only $\dim(y) \geq \dim(f(x))$. The following theorem shows that when $G_1 = G_2$ in the nonlinear system (10), the nonlinear observer (4) in the previous section, which uses the observer Riccati equation (5), can effectively cope with the disturbance.

Theorem 4: Consider the nonlinear system (10) subject to a bounded disturbance d , and the assumptions that $G_1 = G_2$, C and G_1 both full rank, $\text{rank}(C^T) = \text{rank}(G_1)$, and $(A + \alpha I, G_1, C)$ minimum-phase. For whatever large Lipschitz constant γ , and whatever large disturbance d , the nonlinear observer (4) in the previous section guarantees that the state estimation error \hat{x} will eventually approach a small residual set containing the origin, with the size of the residual set approaching zero as the design parameter π in (5) approaches infinity.

Proof: The proof is similar to that of Theorem 3, and is omitted.

Remark 4: Theorem 4 shows that the nonlinear observer (4) and (5) in the previous section is robust against disturbance that comes into the system equation in the same direction as the nonlinearity. However, Theorem 3 shows that when the disturbance comes into the system equation in a direction different from that of the nonlinearity, the Riccati equation in the nonlinear observer must be modified according to (14) so that the disturbance's effect on the observer can be reduced.

Remark 5: One should be aware that high gain design (using large design parameter π in the observer Riccati equation and hence large observer feedback gain) does not always guarantee robustness of the observer if the linear part of the nonlinear system is not minimum-phase. Hence, both Theorems 3 and 4 request the minimum-phase condition.

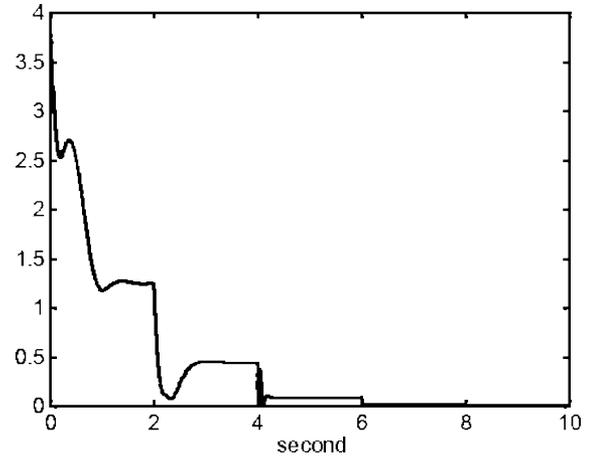


Fig. 2. State estimation error $\|\hat{x}\|$ versus time.

Example: Consider a flexible joint robot [25], whose governing equation is as in (10) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}, \quad B = G_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

the control $u(t) = 0$, the disturbance $d(t) = 2$, and nonlinearity $f(x) = -10 \sin(x_1)$. Note that the nonlinearity $f(x)$ is globally Lipschitz, and the large Lipschitz constant $\lambda = 10$ implies a long or heavy robot arm. The initial conditions are $x^T(0) = [1, 1, 1, 1]$ for the robot, and $\hat{x}^T(0) = [-1, -1, -1, -1]$ for the observer (13). The modified Riccati (14) has design parameters $\alpha = 1$, and $\pi(t)$ scheduled stepwisely from 1 to 10^{12} within 8 s according to the formula $\pi(t) = 10^{3k}$, $t \in [2k, 2k + 2)$, $k = 0, 1, 2, 3, 4$.

Fig. 2 shows the state estimation error $\|\hat{x}(t)\|$ versus time. The estimation error decays to almost zero ($\|\hat{x}(t)\| \approx 10^{-3}$), confirming that the proposed robust nonlinear observer design is successful in the face of disturbances and large-Lipschitz-constant nonlinearity. Note that if one uses $\pi = 10^{12}$ right from the beginning of observation, there will be a substantial peaking phenomenon. The proposed stepwise scheduling of $\pi(t)$ has successfully avoided the peaking phenomenon.

IV. CONCLUSION

This note proposes a new robust nonlinear observer for systems subject to Lipschitz nonlinearity and bounded external disturbances. The linear part of the proposed observer, like the linear LTR observer, is based on an LQ Riccati equation design. Robustness of the proposed observer is achieved by asymptotic tuning of the design parameter in the Riccati equation. It is proved that the proposed nonlinear observer ensures asymptotically an almost zero estimation error in the face of large Lipschitz nonlinearity and large external disturbances.

REFERENCES

- [1] E. A. Misawa and J. K. Hedrick, "Nonlinear observers—A state-of-the-art survey," *J. Dyn. Syst., Measur., Control*, vol. 111, pp. 344–352, 1989.
- [2] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Syst. Control Lett.*, vol. 3, pp. 47–52, 1983.
- [3] D. Bestle and M. Zeitz, "Canonical form observer design for nonlinear time-variable systems," *Int. J. Control*, vol. 38, pp. 419–431, 1983.

- [4] A. J. Krener and W. Respondek, "Nonlinear observer with linearizable error dynamics," *SIAM J. Control Optim.*, vol. 23, pp. 197–216, 1985.
- [5] F. E. Thau, "Observing the state of nonlinear dynamic systems," *Int. J. Control*, vol. 17, pp. 471–479, 1973.
- [6] S. R. Kou, D. L. Elliott, and T. J. Tarn, "Exponential observers for nonlinear dynamic systems," *Inf. Control*, vol. 29, pp. 204–216, 1975.
- [7] S. Raghavan, "Observers and compensators for nonlinear systems, with application to flexible joint robots," Ph.D. dissertation, Univ. of Calif., Berkeley, 1992.
- [8] R. Rajamani, "Observers for Lipschitz nonlinear systems," *IEEE Trans. Autom. Control*, vol. 43, pp. 397–401, 1998.
- [9] G. Besancon and H. Hammouri, "On uniform observation of nonuniformly observable systems," *Syst. Control Lett.*, vol. 29, pp. 9–19, 1996.
- [10] F. Zhu and Z. Han, "A note on observers for Lipschitz nonlinear systems," *IEEE Trans. Autom. Control*, vol. 47, pp. 1751–1754, 2002.
- [11] B. L. Walcott and S. H. Zak, "State observation of nonlinear uncertain dynamical systems," *IEEE Trans. Autom. Control*, vol. 32, pp. 166–170, 1987.
- [12] S. H. Zak, "On the stabilization and observation of nonlinear/uncertain dynamical systems," *IEEE Trans. Autom. Control*, vol. 35, pp. 604–607, 1990.
- [13] D. M. Dawson, Z. Qu, and J. C. Carroll, "On state observation and output feedback problems for nonlinear uncertain dynamic systems," *Syst. Control Lett.*, vol. 18, pp. 217–222, 1992.
- [14] K. K. Busawon and M. Saif, "A state observer for nonlinear systems," *IEEE Trans. Autom. Control*, vol. 44, pp. 2098–2103, 1999.
- [15] J. P. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems, applications to bioreactors," *IEEE Trans. Automat. Control*, vol. 37, pp. 875–880, 1992.
- [16] J. C. Doyle and G. Stein, "Robustness with observers," *IEEE Trans. Autom. Control*, vol. AC-24, pp. 607–611, 1979.
- [17] G. Stein and M. Athans, "The LQG/LTR procedure for multivariable feedback control design," *IEEE Trans. Autom. Control*, vol. 32, pp. 105–113, 1987.
- [18] M. Arcak and P. Kokotovic, "Nonlinear observers: A circle criterion design and robustness analysis," *Automatica*, vol. 37, pp. 885–892, 2003.
- [19] H. Shim, J. H. Seo, and A. R. Teel, "Nonlinear observer design via passivation of error dynamics," *Automatica*, vol. 39, pp. 1923–1930, 2001.
- [20] A. Alessandri, "Observer design for nonlinear systems by using input-to-state stability," in *Proc. IEEE Conf. Decision Control*, 2004, pp. 3892–3897.
- [21] A. Saberi, P. Sannuti, and B. M. Chen, *H₂ Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995, ch. 4, Theorem 4.1.2.
- [22] F. L. Lewis, *Applied Optimal Control and Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1992, ch. 3, Theorem 2.
- [23] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley, 1972, ch. 3, Theorem 3.14.
- [24] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 2000, ch. 14.5.
- [25] R. Marino and P. Tomei, *Nonlinear Control Design*. Upper Saddle River, NJ: Prentice-Hall, 1995, p. 27.

Parameterization of Suboptimal Solutions of the Nehari Problem for Infinite-Dimensional Systems

Kenji Kashima, *Member, IEEE*, Yutaka Yamamoto, *Fellow, IEEE*, and Hitay Özbay, *Senior Member, IEEE*

Abstract—The Nehari problem plays an important role in H^∞ control theory. It is well known that H^∞ control problem can be reduced to solving this problem. This note gives a parameterization of all suboptimal solutions of the Nehari problem for a class of infinite-dimensional systems. Many earlier solutions of this problem are seen to be special cases of this new parameterization. It is also shown that for finite impulse response systems this parameterization takes a particularly simple form.

Index Terms—Delay systems, H^∞ -control, infinite-dimensional systems, Nehari problem.

I. INTRODUCTION

It is well known that many interesting H^∞ control problems can be transformed to the so-called one-block problem; see for example, [1]–[3] and references therein. The one-block problem can be seen as a model matching problem where stable approximation(s), in the sense of L^∞ , of a given unstable system is sought. This is precisely the Nehari problem which can be stated as follows: Given $F \in L^\infty$, find all $\phi \in H^\infty$ such that

$$\|F + \phi\|_{L^\infty} < 1. \quad (1)$$

(We have chosen to write $F + \phi$ in place of $F - \phi$, which is more conventional, for the convenience of later developments.) Nehari's theorem states that a solution $\phi \in H^\infty$ satisfying (1) exists if and only if $\|\Gamma_F\| < 1$, where Γ_F is the Hankel operator associated with symbol F ; see Section II and [2]. In this note, we assume that $F \in L^\infty$ with $\|\Gamma_F\| < 1$ is given and we derive a parameterization of solutions $\phi \in H^\infty$ of (1). We approach this problem from an operator theoretic viewpoint.

The Nehari problem has been studied in the control community for various classes of F , and many different solution techniques have been developed, depending on the assumptions of F . In the finite-dimensional case where F is rational, the solution can be obtained easily by solving Lyapunov equations derived from a state space realization of F . However, for the infinite-dimensional case where F is irrational, state space approaches require solutions of operator equations (instead of matrix Lyapunov equations), see, e.g., [4].

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K. Kashima is with the Graduate School of Information Science and Engineering, Tokyo Institute of Technology, Tokyo 152-8552, Japan (e-mail: kashima@mei.titech.ac.jp; kashima@ieee.org).

Y. Yamamoto is with the Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (e-mail: yy@i.kyoto-u.ac.jp).

H. Özbay is with the Department of Electrical and Electronics Engineering, Bilkent University, Bilkent, Ankara TR-06800, Turkey (e-mail: hitay@bilkent.edu.tr).

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