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TECHNICAL NOTE

A new model reference adaptive control for linear time-varying systems

Min-Shin Chen*,† and Jia-Ming Wu

Department of Mechanical Engineering, National Taiwan University, Taipei 106, Taiwan, R.O.C.

SUMMARY

Recent results on the adaptive control of linear time-varying systems have considered mostly the case in which the range or rate of parameter variations is small. In this paper, a new state feed-back model reference adaptive control is developed for systems with bounded *arbitrary* parameter variations. The important feature of the proposed adaptive control is an *uncertainty* estimation algorithm, which guarantees almost zero tracking error. Note that the conventional *parameter* estimation algorithm in the adaptive control guarantees only bounded tracking error. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: linear time-varying system; model reference adaptive control; tracking control; uncertainty estimation

1. INTRODUCTION

The early study of adaptive control addressed only the control problem for a linear system with unknown constant parameters. It is only recently that the adaptive control of linear systems with unknown *time-varying* parameters has received intense attention since time variation in systems parameters is frequently encountered in practical situations. In the research of time-varying adaptive control, only limited classes of parameter variations are considered. These include the finite jump parameter variation [1], the periodical parameter variation [2], and the parameter variations that decay to zero exponentially [3]. For the class of general parameter variations. Anderson and Johnstone [4] show that the conventional adaptive control is capable of tolerating small parameter variations if the persistence excitation condition [5] is met. Later, it is further shown that the conventional adaptive control can still operate satisfactorily without the persistence excitation condition if the time variation of system parameters is 'slow' in some sense [6–8].

* Correspondence to: M.S. Chen, Department of Mechanical Engineering, National Taiwan University, Taipei 106, Taiwan, R.O.C.
† E-mail: mschen@ccms.ntu.edu.tw

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One result that does not restrict the parameter variations to be small or slow is obtained by Annaswamy and Narendra [9]. They show that when the system state is accessible for measurement, a model reference adaptive control based on the conventional *parameter* estimation algorithm can guarantee *boundedness* of all signals in the adaptive system.

In this paper, the same problem as in Reference [9] is addressed. However, the conventional constant parameter estimation algorithm is replaced by an *uncertainty* estimation algorithm. When this uncertainty estimation algorithm is combined with the state feedback model reference control, it guarantees not only boundedness of all signals, but also *almost* perfect tracking of the system outputs.

2. PROBLEM FORMULATION

Consider a multivariable linear time-varying system

$$\dot{x}(t) = (A(t) + \Delta A(t))x(t) + Bu(t), \quad x(0) = x_0 \tag{1}$$

$$y(t) = Cx(t)$$

where $x(t) \in R^n$ is the system state, $y(t) \in R^p$ the system output, $u(t) \in R^m$ the control input, and matrices $A(t)$, B, C represent the nominal model for the system, and $\Delta A(t)$ the system uncertainty satisfying

$$\| \Delta \dot{A} \| \leq \alpha_1, \quad \| \Delta A \| \leq \alpha_0 \tag{2}$$

for two finite positive constants α_0 and α_1.

The control objective is to find an adaptive state feedback control $u(t)$ such that the behaviour of system (1) will resemble that of a time-*invariant* reference model

$$\dot{x}_m(t) = A_m x_m(t) + Br(t), \quad x_m(0) = x_{m0} \tag{3}$$

$$y_m(t) = Cx_m(t)$$

where $x_m(t) \in R^n$ is the reference state, $y_m(t) \in R^p$ the reference output, $r(t) \in R^m$ the reference input, and A_m a *stable* constant matrix. As in Reference [9], it is assumed that there exists a time-varying state feedback gain matrix $K(t)$ such that

$$A(t) + \Delta A(t) + BK(t) = A_m \tag{4}$$

This equation quarantees that a model reference state feedback control exists, and a formula for the feedback gain matrix is given by

$$K(t) = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}(A_m - A(t) - \Delta A(t)) \tag{5}$$

if the uncertain matrix $\Delta A(t)$ is precisely known.

In Section 3, an estimation algorithm will be proposed to estimate the unknown $\Delta A(t)$. In Section 4, it will be shown that the *certainty equivalence principle* holds for the proposed estimate when it is substituted into Equation (5). Note that this is to be achieved in the face of possibly large or fast time-varying uncertainty $\Delta A(t)$; in other words, α_0 and α_1 in Equation (2) can be arbitrarily large.

3. UNCERTAINTY ESTIMATION ALGORITHM

The estimation of the time-varying uncertain matrix $\Delta A(t)$ in Equation (1) consists of two steps. In the first step, an estimation algorithm will be developed to estimate $\Delta A(t)x(t)$, which is a potentially unbounded signal since it is state-dependent. Then, in the second step, an estimate of $\Delta A(t)$ will be given based on the estimation results of $\Delta A(t)x(t)$.

For the estimation of $\Delta A(t)x(t)$ in Equation (1), consider the following uncertainty estimation algorithm:

$$\dot{z}(t) = A(t)z(t) + Bu(t) + v(t), \quad z(0) = z_0 \in R^n \tag{6}$$

$$v(t) = \sigma e(t) + (\rho_1 \| x(t) \| + \rho_0)e(t) \tag{7}$$

$$e(t) = x(t) - z(t) \tag{8}$$

where $A(t)$, B, and $u(t)$ are as in Equation (1), σ is an arbitrary positive constant, and ρ_1, ρ_0 are two positive estimation gains to be specified.

Theorem 1

The following results are ensured for the uncertainty estimation algorithm (6)–(8):

(I) The estimation error $e(t)$ in Equation (8) becomes arbitrarily small within a finite time in the sense that given any small constant $\varepsilon_1 > 0$, there exist sufficiently large estimation gains ρ_1 and ρ_0 in Equation (7), and a finite time constant T_1, such that

$$\| e(t) \| \leq \varepsilon_1, \quad \forall t > T_1 \tag{9}$$

(II) If the system control input $u(t)$ is linearly bounded by the system state, i.e.

$$\| u(t) \| \leq \beta_1 \| x(t) \| + \beta_0 \tag{10}$$

for some positive numbers β_1 and β_0, then $v(t)$ in Equation (7) asymptotically becomes an estimate of the time-varying uncertainty $\Delta A(t)x(t)$ in the sense that given any small constant $\varepsilon_2 > 0$, there exist sufficiently large estimation gains ρ_1 and ρ_0, and a finite time constant T_2, such that

$$\| v(t) - \Delta A(t)x(t) \| \leq \varepsilon_2, \quad \forall t > T_2 \tag{11}$$

Proof. (I) According to Equations (1), (6)–(8), one has

$$\dot{e} = -\sigma e - (\rho_1 \|x\| + \rho_0)e + \Delta A x \tag{12}$$

Define

$$N_1 \triangleq \left\{ e: \|e\| \le \varepsilon_1, \varepsilon_1 \triangleq \sup_{\forall x} \frac{\alpha_0 \|x\|}{\rho_1 \|x\| + \rho_0} = \frac{\alpha_0}{\rho_1} \right\} \tag{13}$$

Choose a Lyapunov function candidate $V_1 = 1/2 \|e\|^2$. Calculating the time derivative of V_1 along the trajectory (12) yields

$$\begin{aligned} \dot{V}_1 &= e^{\mathrm{T}} \dot{e} \\ &= e^{\mathrm{T}}(-\sigma e - (\rho_1 \|x\| + \rho_0)e + \Delta A x) \\ &\le -\sigma \|e\|^2 - \|e\|(\rho_1 \|x\| + \rho_0)\left(\|e\| - \frac{\alpha_0 \|x\|}{\rho_1 \|x\| + \rho_0} \right) \\ &\le -2\sigma V_1 \le -\sigma \varepsilon_1^2, \quad \forall e \notin N_1 \end{aligned} \tag{14}$$

where the first inequality results from assumption (2). Without loss of generality, it is assumed that initially $\|e\| > \varepsilon_1$. Integrating inequality (14) from $t = 0$ to T_1, where T_1 *is the first time when* $\|e\|$ *becomes smaller than or equal to* ε_1, gives

$$V_1(T_1) - V_1(0) = \int_0^{T_1} \dot{V}_1(t)\,\mathrm{d}t \le -\sigma \varepsilon_1^2 T_1$$

Rearranging the equation

$$V_1(T_1) \le V_1(0) - \sigma \varepsilon_1^2 T_1$$

and noting that $V_1(T_1)$ must be positive, one obtains

$$T_1 \le V_1(0)/(\sigma \varepsilon_1^2) < \infty$$

From the definition of T_1, the above equation implies that $\|e\|$ must become smaller than or equal to ε_1 within a finite time (T_1).

Further note that if after T_1, $\|e\|$ becomes larger than ε_1 again, one has $\dot{V}_1 < 0$ according to Equation (14), and hence $\mathrm{d}/\mathrm{d}t \|e\| < 0$ since $V_1 = \|e\|^2/2$. In other words, once the value of $\|e\|$ becomes larger than ε_1, it will immediately shrink back to a value that is smaller than or equal to ε_1. Therefore, it is concluded that $\|e\|$ will never exceed ε_1 after its first entry into N_1 at T_1. This completes the proof for Equation (9).

Also note from the definition of ε_1 in Equation (13) that ε_1 can be made arbitrarily small by the choice of a sufficiently large ρ_1.

(II) When the control satisfies assumption (10), it follows from Equations (1) and (2) that

$$\|\dot{x}\| \leq \xi_1 \|x\| + \xi_0$$

for some finite positive number ξ_1 and ξ_2. Furthermore, the following can be deduced:

$$\left|\frac{\mathrm{d}\|x\|}{\mathrm{d}t}\right| \leq \|\dot{x}\| \leq \xi_1 \|x\| + \xi_0 \tag{15}$$

Define

$$\begin{aligned} N_2 \triangleq \{\dot{e}\colon \|\dot{e}\| \leq \varepsilon_2, \varepsilon_2 \triangleq \sup_{\forall x} \Bigg\{ & \frac{(\rho_1\varepsilon_1\xi_1 + \alpha_1 + \alpha_0\xi_1)\|x\| + (\rho_1\varepsilon_1\xi_0 + \alpha_0\xi_0)}{\rho_1\|x\| + \rho_0} \\ & = \max\left(\varepsilon_1\xi_1 + \frac{\alpha_1 + \alpha_0\xi_1}{\rho_1}, \quad \varepsilon_1\xi_0\frac{\rho_1}{\rho_0} + \frac{\alpha_0\xi_0}{\rho_0}\right)\Bigg\} \end{aligned} \tag{16}$$

Choose another Lyapunov function candidate $V_2 = 1/2\|\dot{e}\|^2$. Calculating the time derivative of V_2 yields

$$\begin{aligned} \dot{V}_2 &= \dot{e}^{\mathrm{T}}\ddot{e} \\ &= \dot{e}^{\mathrm{T}}\left(-\sigma\dot{e} - (\rho_1\|x\| + \rho_0)\dot{e} - \rho_1\frac{\mathrm{d}\|x\|}{\mathrm{d}t}e + \Delta\dot{A}x + \Delta A\dot{x}\right) \end{aligned}$$

It follows form Equation (9) in Part (I), and inequalities (2) and (15) that, for all $t > T_1$,

$$\begin{aligned} \dot{V}_2 &\leq -\sigma\|\dot{e}\|^2 - (\rho_1\|x\| + \rho_0)\|\dot{e}\|^2 + \rho_1\varepsilon_1(\xi_1\|x\| + \xi_0)\|\dot{e}\| + \alpha_1\|x\|\cdot\|\dot{e}\| + \alpha_0(\xi_1\|x\| + \xi_0)\|\dot{e}\| \\ &\leq -\sigma\|\dot{e}\|^2 - (\rho_1\|x\| + \rho_0)\|\dot{e}\|\left\{\|\dot{e}\| - \frac{(\rho_1\varepsilon_1\xi_1 + \alpha_1 + \alpha_0\xi_1)\|x\| + (\rho_1\varepsilon_1\xi_0 + \alpha_0\xi_0)}{\rho_1\|x\| + \rho_0}\right\} \\ &\leq -2\sigma V_2 \leq -\sigma\varepsilon_2^2, \quad \forall \dot{e} \notin N_2 \end{aligned}$$

Using the same argument as in the proof for Part (I), one can show from the last inequality that $\dot{e}$ will become trapped in N_2 within a finite time; in other words, $\|\Delta Ax - v\|$ will eventually be bounded by ε_2. This proved Equation (11).

Recall from Part (I) that ε_1 can be made arbitrarily small if ρ_1 is sufficiently large; hence, it follows from Equation (16) that ε_2 can also be made arbitrarily small if ρ_1 and ρ_0 are sufficiently large. □

In the second step of estimation, an estimate of $\Delta A(t)$ will be given based on the estimate, $v(t)$ in Equation (7), of $\Delta A(t)x(t)$ obtained in Theorem 1.

3.1. *Time-varying parameter estimate*

$$\Delta \hat{A}(t) = \frac{v(t)x^{\mathrm{T}}(t)}{\| x(t) \|^2}, \quad \forall x(t) \neq 0 \tag{17}$$

By direct substitution of Equation (17) into Equation (11) in Theorem 1, an important property of the proposed estimate is derived.

Corollary

The estimate $\Delta \hat{A}(t)$ in Equation (17) satisfies

$$\| \Delta \hat{A}(t)x(t) - \Delta A(t)x(t) \| \leq \varepsilon_2, \quad \forall t > T_2 \tag{18}$$

where ε_2 and T_2 are as in Part (II) of Theorem 1.

It is assumed that in the tracking process the reference input $r(t)$ is non-zero, which keeps injecting energy into system (1) so that it is generally true that $x(t) \neq 0$ in Equation (17). However, even if the control objective is to regulate the system state $x(t)$ to zero ($r(t) = 0$), a second version of the proposed adaptive control law (see Equation (21) below) enables one to avoid calculating $\Delta A(t)$ in Equation (17) and hence avoids possible division by zero.

4. ADAPTIVE CONTROL LAW

In this section, an adaptive tracking control for the uncertain system (1) is presented. When the uncertain system matrix $\Delta A(t)$ is known exactly, the desired model reference state feedback control is given by (see [9])

$$u(t) = K(t)x(t) + r(t)$$

where $r(t)$ is the reference input in Equation (3) and $K(t)$ is from Equation (5):

$$K(t) = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}(A_m - A(t) - \Delta A(t))$$

When the uncertain system matrix $\Delta A(t)$ is *unknown*, the adaptive version of the above control becomes

$$u(t) = \hat{K}(t)x(t) + r(t) \tag{19}$$

$$\hat{K}(t) = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}(A_m - A(t) - \Delta \hat{A}(t)) \tag{20}$$

in which $\Delta \hat{A}(t)$ is the estimate (17) obtained in the previous section. From Equations (17), (19) and (20), the adaptive control can be re-written as

$$u(t) = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}(A_m - A(t))x(t) - (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}v(t) + r(t) \tag{21}$$

where $v(t)$ is the feedback term in the uncertainty estimation algorithm (6), representing an adaptive estimate of $\Delta A(t)x(t)$ according to Part II of Theorem 1. Note that in Equation (21) calculating the estimated $\Delta \hat{A}(t)$ in Equation (17) is avoided.

The following theorem demonstrates that the adaptive control (19) and (20) can drive the system state $x(t)$ arbitrarily close to the reference state $x_m(t)$; thus, achieving *almost* perfect tracking.

Theorem 2

Consider the system (1) and the adaptive control (19) and (20). Given any small number $\varepsilon > 0$, there exist sufficiently large estimation gains ρ_1 and ρ_2 in the uncertainty estimation algorithm (7), and a finite time constant T, such that

$$\|x(t) - x_m(t)\| \leq \varepsilon, \quad \forall t > T$$

Proof. Substituting control (19) and (20) into system (1), one obtains

$$\dot{x} = (A + \Delta A)x + B(\hat{K}x + r)$$

Adding and subtracting BKx, K as in Equation (5), into the right-hand side of the equation gives

$$\begin{aligned}\dot{x} &= (A + \Delta A + BK)x + B(\hat{K} - K)x + Br \\ &= A_m x + B(\hat{K} - K)x + Br \end{aligned} \tag{22}$$

where the second equality is due to Equation (4). Define

$$e \triangleq x - x_m$$

and subtract Equation (3) from Equation (22) to yield

$$\begin{aligned}\dot{e} &= A_m e + B(\hat{K} - K)x \\ &= A_m e + B(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}(\Delta A - \Delta \hat{A})x \end{aligned} \tag{23}$$

where Equations (5) and (20) have been used to obtain the second equality.

Recall from Equation (9) in Part (I) of Theorem 1 that e in Equation (7) is bounded; hence, the feedback term v in Equation (7) is linearly bounded by x, i.e.

$$\|v\| \leq m_1\|x\| + m_0, \quad t > T_1$$

for some positive m_1 and m_2. With this result and boundedness of $A(t)$ and $r(t)$, it is concluded that assumption (10) is satisfied by the proposed adaptive control u in Equation (21). Therefore, one can legally invoke Part (II) of Theorem 1 and Corollary (18) in the following proof.

Since A_m is stable, there exists a positive definite constant matrix Q satisfying the Lyapunov equation [10]

$$A_m^{\mathrm{T}} Q + Q A_m = -I$$

Choose a Lyapunov function candidate $V = e^{\mathrm{T}} Q e$ for system (23). Note that

$$\lambda_1 \| e \|^2 \leq V \leq \lambda_2 \| e \|^2$$

where λ_1 and λ_2 are the minimum and maximum eigenvalues of Q. Calculating the time derivative of V gives

$$\begin{aligned} \dot{V} &= -\| e \|^2 + 2e^{\mathrm{T}} Q B (B^{\mathrm{T}} B)^{-1} B^{\mathrm{T}} (\Delta A - \Delta \hat{A}) x \\ &\leq -\| e \|^2 + 2\lambda_2 \varepsilon_2 \| e \| \\ &\leq -\lambda_2^{-1} V + 2\varepsilon_2 \lambda_2 \lambda_1^{-0.5} \sqrt{V} \end{aligned}$$

where the first inequality results from Equation (18). Since ε_2 can be made arbitrarily small by sufficiently large estimation gains according to Part II of Theorem 1, the same argument as in the proof of Theorem 1 shows that V, and hence e, will become arbitrarily small within a finite time. This completes the proof of the theorem.

5. SIMULATION EXAMPLE

Consider system (1) with

$$A(t) = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix}, \quad \Delta A(t) = \begin{bmatrix} 0 & 0 \\ \Delta a_1(t) & \Delta a_2(t) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0]$$

in which the uncertain parameters $\Delta a_2(t) = 50 \cos(2\pi\sqrt{t})$ varies non-periodically, and $\Delta a_1(t)$ periodically with a period $T = 4\,\mathrm{s}$

$$\Delta a_1(t) = \begin{cases} 3, & 0 \leq t < 2 \\ -3, & 2 \leq t < 4 \end{cases}$$

The system matrix for the reference model is specified as

$$A_m = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

and the reference input $r(t) = 4$. The initial conditions for both the system and the reference model are at the origin of the state space. One can verify that for this system, there does exist a feedback gain matrix $K(t)$ such that assumption (4) is satisfied.

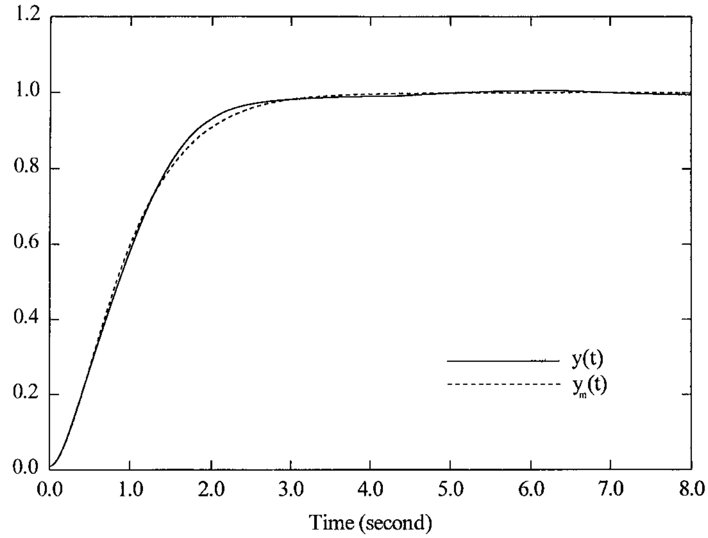

Figure 1. System output of new adaptive control.

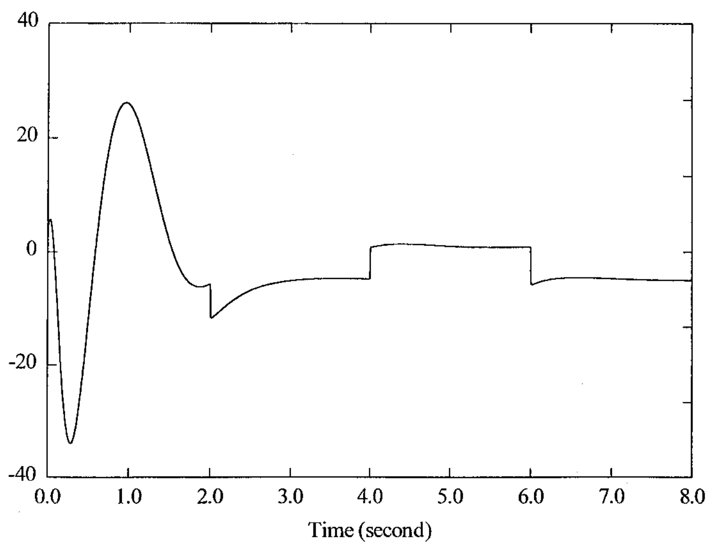

Figure 2. Control input of new adaptive control.

The proposed adaptive control is simulated for the above system using the software MatLab with the following design parameters in Equation (7) $\rho_1 = 50$, $\rho_0 = 500$, and $\sigma = 5$. The initial condition of the uncertainty estimator $z(0)$ is chosen the same as the system initial condition $x(0)$. The solid line in Figure 1 shows the controlled system output, and the dashed line the reference output. It is seen that the two almost coincide with each other. Figure 2 shows the time history of

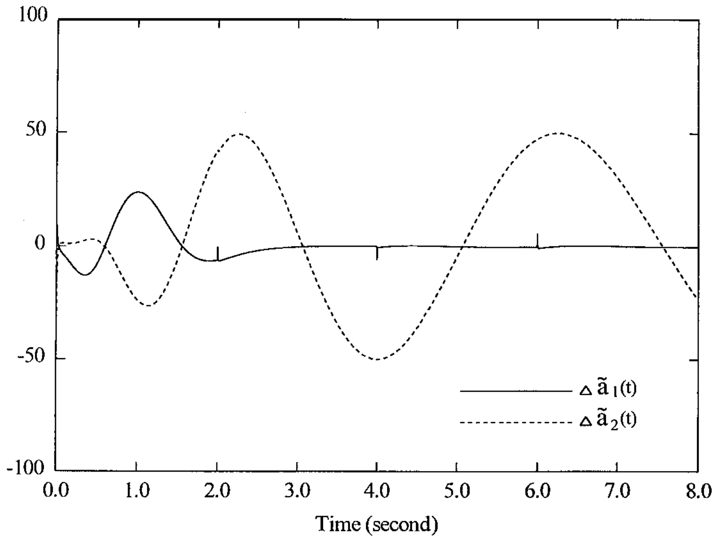

Figure 3. Parameter estimation errors.

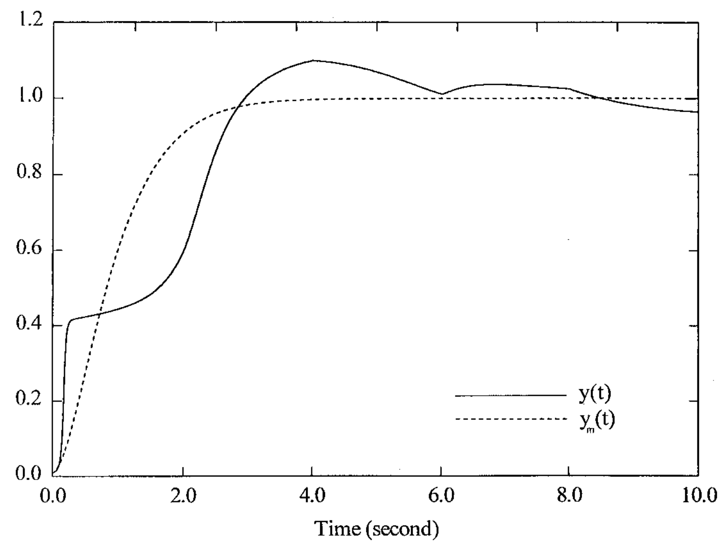

Figure 4. System output of conventional adaptive control.

the adaptive control input, and Figure 3 the estimation errors $\Delta a_1(t) - \Delta \hat{a}_1(t)$ and $\Delta a_2(t) - \Delta \hat{a}_2(t)$, obtained by the proposed estimation algorithm. Note carefully that the proposed adaptive control can respond almost *instantaneously* to the finite jump of system parameter $\Delta a_1(t)$, so that almost perfect tracking can be maintained all the time. Such tracking performance is not attainable by the conventional constant parameter estimation algorithm. For comparison, the same system is simulated using the conventional adaptive control in Reference [9]. Figure 4 shows the tracking performance, which is much worse than in Figure 1, indicating that the adaptive control with constant parameter estimation algorithms responds poorly in the face of fast time-varying parameters.

6. CONCLUSIONS

A new model reference adaptive control is proposed in this paper for a class of linear time-varying systems. The unique feature of the proposed adaptive control is an uncertainty estimation algorithm in replacement of the conventional constant parameter estimation algorithm. Though the new estimation algorithm not necessarily guarantees the convergence of parameter estimation errors, it does ensure that the system output can track closely the reference output in the face of fast or large time-variations in unknown system parameters. Research is now being conducted in extension of the result in this paper to the control design for more general linear time-varying systems, which are not constrained by Equation (4).

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