Decentralized Control of a Class of Large-Scale Systems by Uncertainty Estimator

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A new decentralized controller is proposed for a group of subsystems subject to unknown interconnections and external disturbances. Under the assumption that the interconnections and disturbances satisfy a certain structural condition, the new controller suppresses the disturbances and intercoupling effects completely, making the overall system behave as a decoupled system. In this new control design, each local controller uses an uncertainty estimator proposed by Chen (1990) for estimation of the interconnections and disturbances, and then cancels these undesirable inputs directly. The major advantages of the proposed controller are that the interconnections need not satisfy the so-called "conical" condition, and there is no need for a priori information on the magnitudes of interconnections and disturbances.

1 Introduction

In controlling a large-scale system, we often use decentralized controllers due to the infeasibility of communication between subsystems or to avoid the complexity of centralized controllers. Traditional fixed-gain decentralized controllers proposed by Siljak (1978) can perform adequately in the presence of small interconnections between subsystems. However, when the size of interconnections is large, performance as well as stability can be destroyed. Later study shows that a particular class of large-scale systems, whose interconnections satisfy a certain structural condition, can always be decentralizedly stabilized even for large interconnections in the case of linear state interconnections by Sezer and Siljak (1981) and conical output interconnections by Huseyin et al. (1982). The underlying approach used by them is to synthesize each local controller to make the loop gains of the interconnected systems small so that stability can be retained in the presence of interconnections. Hence, the larger the interconnections are, the smaller the loop gains should be. A drawback of such a design approach is that small loop gains make system performance susceptible to external disturbances and system parameter variations. To remedy the situation, Gavel and Sijak (1989) suggest the use of decentralized adaptive control [4] to account for possible parameter variations. The proposed adaptive scheme can stabilize the overall system regardless of the size of interconnections when there is no external disturbance. Another approach suggested by Youcef-Toumi and Fuhlbrigge (1989) is to apply the concept of Time Delay Control. With this approach, the unknown interconnections and disturbances are directly estimated through time delay, and the control actions are modified based on the estimation results. No explicit constraints are imposed on the type of interconnections; however, the interconnections and disturbances are assumed to be slowly time-varying, and evaluation of the time derivative of the system state is required in the estimation process.

In this paper we adopt an approach similar to the Time Delay Control. However, in the estimation of system interconnections and external disturbances, we use the uncertainty estimator proposed by Chen and Tomizuka (1989) and Chen (1990). As a result, we need not assume that the interconnections and disturbances are slowly time-varying. Our new controller differs from previously developed decentralized controllers in three aspects: (1) the proposed controller completely suppresses the effects of external disturbances, (2) the interconnections need not be "conical" as required by previous authors; in other words, the magnitude of the interconnections need not be bounded by the norm of the overall system state, (3) the magnitudes of interconnections and disturbances need not be known a priori. We mention that although we consider only the regulation problem, a decentralized tracking controller can be easily constructed through a slight modification of the regulation controller presented in this paper.

In the sequel, a matrix is said to be stable if all of its eigenvalues are in the open left-half plane. $\overline{\lambda}_p$ and $\underline{\lambda}_p$ denote, respectively, the maximum and minimum eigenvalues of a matrix P. $\mathbf{N}$ stands for the set $\{1, 2, \ldots, N\}$, where $N>0$ is an integer, and B_r is the set $\{x \in \mathbf{R}^n: \|x\| < r\}$.

2 Problem Formulation

Consider a system S, which is composed of N subsystems S_i, $i \in \mathbf{N}$. Each subsystem is described by

$$S_i: \quad \dot{x}_i = A_i x_i + B_i u_i + P_i w_i + Q_i d_i \tag{1}$$

where $x_i \in \mathbf{R}^{n_i}$ and $u_i \in \mathbf{R}$ are the state and input of S_i, $d_i \in \mathbf{R}^{h_i}$ is

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an external disturbance, and $w_i \in \mathbf{R}^{m_i}$ is the unknown interconnections between S_i and other subsystems, which is of the form

$$w_i = f_i(x_1, x_2, \ldots, x_N) \tag{2}$$

The overall system is then described by

$$S: \quad \dot{x} = Ax + Bu + Pw + Qd \tag{3}$$

where

$$A = \mathrm{diag}\{A_1, A_2, \ldots, A_N\}$$

$$B = \mathrm{diag}\{B_1, B_2, \ldots, B_N\}$$

$$P = \mathrm{diag}\{P_1, P_2, \ldots, P_N\}$$

$$Q = \mathrm{diag}\{Q_1, Q_2, \ldots, Q_N\}$$

$x = (x_1^T, x_2^T, \ldots, x_N^T)^T$, $u = (u_1^T, u_2^T, \ldots, u_N^T)^T$, $w = (w_1^T, w_2^T, \ldots, w_N^T)^T$ and $d = (d_1^T, d_2^T, \ldots, d_N^T)^T$.

The objective is to develop a decentralized controller for each subsystem S_i, which achieves regulation of the system state x_i in the presence of the external disturbance d_i and the unknown interconnection w_i. We make the following assumptions on every subsystem:

(A1) A_i is a stable matrix. (If A_i is not a stable matrix, we can always use state feedback control to stabilize the uninterconnected system by assuming the controllability of (A_i, B_i).)

(A2) There exist $p_i \in \mathbf{R}^{1\times m_i}$ and $q_i \in \mathbf{R}^{1\times h_i}$ such that

$$P_i = B_i p_i \quad \text{and} \quad Q_i = B_i q_i$$

(A3) The external disturbance d_i satisfies

$$|d_i(t)| < D_{i0} \quad \text{and} \quad /\dot{d}_i(t)/ < D_{i1} \quad \text{for all} \quad t > 0$$

(A4) The interconnection $f_i(x)$ in Eq. (2) is such that for any $r_0 > 0$, there exists a finite time $T(r_0) > 0$ so that given any disturbances specified in (A3), if $x(0) = x_0 \in B_{r0}$, the solution of the open-loop system $x(t)$ can grow at most exponentially with exponent $\zeta > 0$ for all $t \in [0, T]$. Hence,

$$\text{if} \quad x_0 \in B_{r0} \Rightarrow x(t) \in B_r \subset B_{r_p} \quad \forall t \in [0, T]$$

where $r = r_0 e^{\zeta T}$, $r_p = (\bar{\lambda}_p / \underline{\lambda}_p)^{1/2} r + \kappa$ with κ a small positive number, and P is the solution of the Lyapunov Equation

$$A^T P + PA = -I \tag{4}$$

with the matrix A as given in Eq. (3). In particular, we assume that f_i and its first derivative are continuous; hence, we have the following upper bounds:

$$\|f_i\| < L_{r_p} \qquad \forall x \in B_{r_p} \tag{5a}$$

$$\|\partial f_i / \partial x\| < M_{r_p} \quad \forall x \in B_{r_p} \tag{5b}$$

Combining the structural condition (A2) with Eq. (1), we can rewrite the subsystem dynamics as

$$S_i: \quad \dot{x}_i = A_i x_i + B_i(u_i + e_i) \tag{6}$$

where $e_i = p_i w_i + q_i d_i$ is an equivalent uncertainty viewed from the input channel. Similarly, Eq. (3) is rewritten as

$$S: \quad \dot{x} = Ax + B(u + e) \tag{7}$$

where $e = (e_1^T, e_2^T, \ldots, e_N^T)^T$.

Using Assumptions (A1)–(A4) and Eq. (7) with $u = 0$, we show in Appendix A that if $x_0 \in B_{r0}$, the equivalent uncertainty e_i, for all $i \in N$, satisfies

$$|\dot{e}_i| \le \eta_{i1} |e_i| + \eta_{i0}, \quad \forall x \in B_{r_p} \tag{8}$$

for some positive constants η_{i1} and η_{i0}. Eq. (8) implies that if $x_0 \in B_{r0}$ and there is no control ($u_i = 0$), the equivalent uncertainty e_i can grow at most exponentially with exponent η_{i1} before x exits the ball B_{r_p}.

3 Uncertainty Estimator

We now follow the procedure developed by Chen (1990) to set up an uncertainty estimator for on-line estimation of the equivalent uncertainty e_i in Eq. (6). For each subsystem S_i, define an output

$$y_i = C_i x_i \quad \text{with} \quad C_i B_i = 1 \tag{9}$$

Taking the time derivative of y_i yields

$$\dot{y}_i = C_i A_i x_i + u_i + e_i \tag{10}$$

Before estimating e_i, we need first obtain an upper bound of its magnitude. Construct the following uncertainty-bound estimator based on Eq. (10):

$$\dot{z}_{i1} = C_i A_i x_i + u_i + (\rho_i - \lambda_0)/\lambda_1 \tag{11a}$$

$$\dot{z}_{i2} = C_i A_i x_i + u_i - (\rho_i - \lambda_0)/\lambda_1 \tag{11b}$$

where λ_1 and λ_0 are two positive constants with $\lambda_1 > 1$, and ρ_i, which is an estimate of a magnitude upper bound of e_i, is governed by

$$\dot{\rho}_i = \nu_i(\mu_1 \rho_i + \mu_0), \quad \rho_i(0) \ge \lambda_0 \tag{11c}$$

$$\nu_i = \begin{cases} 1 & \dot{\epsilon}_{i1}\dot{\epsilon}_{i2} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{11d}$$

with $\epsilon_{i1} = y_i - z_{i1}$, $\epsilon_{i2} = y_i - z_{i2}$, and μ_1 and μ_0 two positive constants.

Lemma 1: If the uncertainty-bound estimator (11) is applied to the free system (1) ($u_i = 0$ for all $i \in N$) subject to Assumptions (A1)–(A4), given any $T_0 > 0$, there exist sufficiently large μ_1 and μ_0 in Eq. (11c) such that

$$\rho_i(t) \ge \lambda_1 |e_i|_t + \lambda_0 \ge \lambda_1 |e_i(t)| + \lambda_0 \quad \text{for all} \quad t \ge T_0$$

where $|e_i|_t \equiv \sup_{T_0 \le \tau \le t} |e_i(\tau)|$.

Proof: See Appendix B.

Remark: To avoid z_{i1} and z_{i2} blowing up to infinity, we can reset them to zero when their magnitudes exceed a certain prescribed bound. This reset action will not affect the result of Lemma 1.

Finally, we construct the following uncertainty estimator:

$$\dot{\chi}_i = C_i A_i x_i + u_i + \nu_i \tag{12a}$$

$$\nu_i = \sigma s_i + \rho_i g(s_i) \tag{12b}$$

$$s_i = y_i - \chi_i \tag{12c}$$

where σ is an arbitrary positive constant, ρ_i is given by the uncertainty-bound estimator (11), $g(\bullet) \in C^1$ can be any monotonically increasing odd function ranging from -1 to $+1$. The variable ν_i in Eq. (12b) then becomes an estimate of the uncertainty e_i in Eq. (10). This result is shown in Lemma 2.

Lemma 2: If the uncertainty estimator (12) is applied to the free system (1) ($u_i = 0$ for all $i \in N$) subject to Assumptions (A1)–(A4), we obtain the following results:

(I) There exists a finite time T_1 ($> T_0$, where T_0 is as given in Lemma 1) such that

$$s_i(t) \in M_{i1}, \quad \forall t \ge T_1$$

where $M_{i1} = \{s_i: \; |s_i| \le r_{i1}, \; r_{i1} = g^{-1}(1/\lambda_1 + \gamma_i)$, $\gamma_1 > 0$ is an arbitrarily small number satisfying $\gamma_1 < 1 - 1/\lambda_1\}$

(II) There exists a finite time T_2 ($> T_1$) such that

$$\dot{s}_i(t) \, (= \nu_i - e_i) \in M_{i2}, \quad \forall t \ge T_2$$

where $M_{i2} = \{\dot{s}_i: \; |\dot{s}_i| \le r_{i2}, \; r_{i2} = \mu_1 \dfrac{\pi_{i1}}{\pi_{i2}} + \max\left(\dfrac{\eta_1}{\pi_{i2}\lambda_1}, \dfrac{\eta_0}{\pi_{i2}\lambda_0} + \dfrac{\mu_0 \pi_{i1}}{\lambda_0 \pi_{i2}}\right) + \gamma_2$, $\pi_{i1} = \sup_{s_i \in M_{i1}} /g(s_i)/$, $\pi_{i2} = \inf_{s_i \in M_{i1}} g'(s_i)$ and $\gamma_2 > 0$ is an arbitrarily small number$\}$

(III) $T_1 - T_0$ and $T_2 - T_1$ can be made arbitrarily small if we use sufficiently large σ in Eq. (12*b*), and λ_1, λ_0 in Eq. (11).

Proof: See Appendix C.

Remark: It is clear from the definition of M_{i1} that we can choose λ_1 in Eq. (11) sufficiently large so that r_{i1} is arbitrarily small. Consequently, r_{i2} in M_{i2} is also small if λ_1 and λ_0 are large enough since π_{i1}/π_{i2} is in the order of r_{i1}. For example,

$$\frac{\pi_{i1}}{\pi_{i2}} = \frac{r_{i1}(r_{i1}+\delta)}{\delta} \quad \text{when} \quad g(s) = \frac{s}{|s|+\delta}$$

Hence, Part II of Lemma 2 implies that given an arbitrarily small r_{i2} there always exist large enough σ, λ_1 and λ_0 such that the estimation error $|e_i - \nu_i|$ becomes bounded by r_{i2}.

4 Decentralized Controller

Having obtained an estimate of the equivalent uncertainty e_i, the decentralized control law is given by

$$u_i = -\nu_i \quad \text{for all} \quad t \geq T_2 \tag{13}$$

where ν_i is given by the uncertainty estimator (12), and T_2 is as given in Lemma 2.

Theorem: Consider the system (1) subject to Assumptions (A1)-(A4). Given any initial condition $x_0 \epsilon B_{r0}$, if the parameters μ_1, μ_0, λ_1, and λ_0 in the uncertainty-bound estimator (11) and σ in the uncertainty estimator (12) are chosen sufficiently large so that $T_2 < T$, where T is as given in Assumption (4), and T_2 as in Lemma 2, and the control law (13) is applied to the system (1), the overall system state x converges to a small neighborhood of the origin in the state space, and all the signals in the controller remain bounded.

Proof: See Appendix D.

Remark: The control law (13) stipulates that we know T_2; that is, we know when the estimator (12) achieves estimation of e_i. Unfortunately, we have only access to the signal e_i, implying that we can determine only T_1 instead of T_2. However, since Lemma 2 says that we can choose λ_1, λ_0 in Eq. (11) and σ in Eq. (12*b*) sufficiently large so that T_2 is very close to T_1, in real implementation we can switch on the control law (13) at $t = T_1$. Actually, experience gained from simulation studies shows that T_2 coincides almost with T_1 even for small values of λ_1, λ_0, and σ.

Example: A four-dimensional system is simulated to verify the proposed decentralized control law.

$$S_1: \quad \dot{x}_1 = x_2$$

$$\dot{x}_2 = -24x_1 - 10x_2 + u_1 + w_1 + d_1$$

$$S_2: \quad \dot{x}_3 = x_4$$

$$\dot{x}_4 = -6x_3 - 4x_4 + u_2 + w_2 + d_2$$

where

$$w_1 = 20x_1x_4, \quad d_1 = \sin(t)$$

$$w_2 = 5x_2x_3, \quad d_2 = 20$$

Figure 1 shows performance of the uncontrolled system, where the magnitudes of x_1 and x_3 grow in an almost linear manner. Actually, all the state variables explode to infinity at about 3.8 seconds. We apply the control law (13) with the following parameters

$$\lambda_1 = 10 \quad \lambda_0 = 6.25 \quad \sigma = 1$$

$$\mu_1 = 10 \quad \mu_0 = 10 \quad g(s) = s/(/s/+1)$$

In implementation of Eq. (11*d*), we use the approximation

$$\text{sign}(\dot{\epsilon}_{i1}(t)) \approx \text{sign}(\epsilon_{i1}(t) - \epsilon_{i1}(t-\Delta))$$

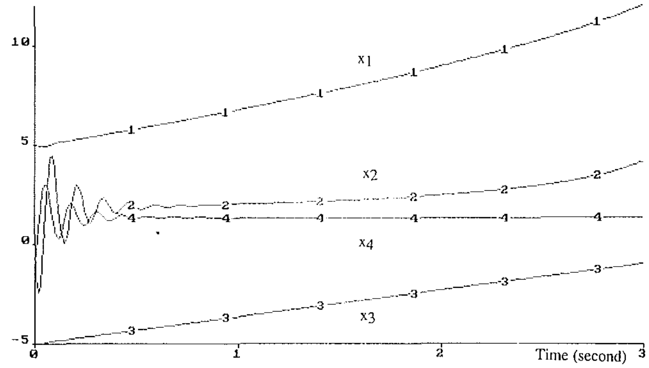

Fig. 1 Open-loop state response of the interconnected system

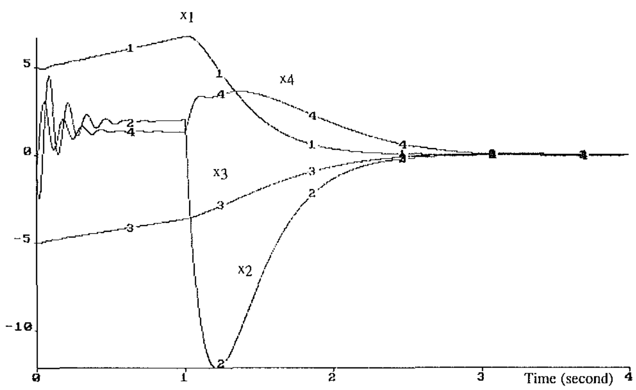

Fig. 2 State response of the Decentralized System

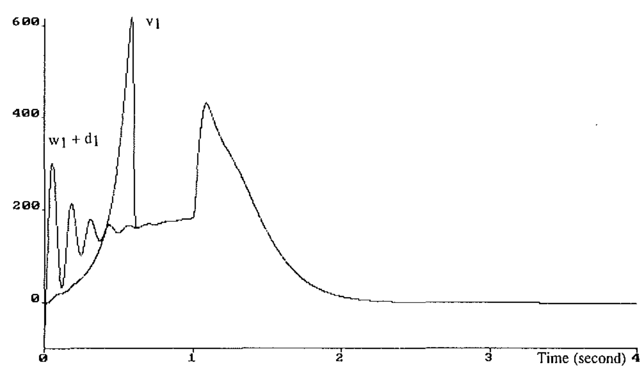

Fig. 3 Uncertainty $w_1 + d_1$ and estimated uncertainty ν_1

to determine the sign of ϵ_{i1}, where $\Delta = 0.01$ second, and sign($\dot{\epsilon}_{i2}(t)$) is determined similarly. The control law is switched on at $t = 1$ second although estimation of the uncertainties is achieved at about 0.7 second. The convergence of the system state to zero is verified by Fig. 2. Figures 3 and 4 show the estimates of uncertainties $w_1 + d_1$ and $w_2 + d_2$, respectively.

5 Conclusions

We apply the uncertainty estimator proposed by Chen (1990) to synthesize a decentralized controller for a group of subsystems subject to unknown interconnections and external disturbances. The new controller estimates and cancels the unknown interconnections and the disturbances so that the overall system behaves as a decoupled system with no external disturbances. The advantages of the new controller are that the interconnections may be a nonlinear function of the system state (not necessarily satisfying the conical condition), and the

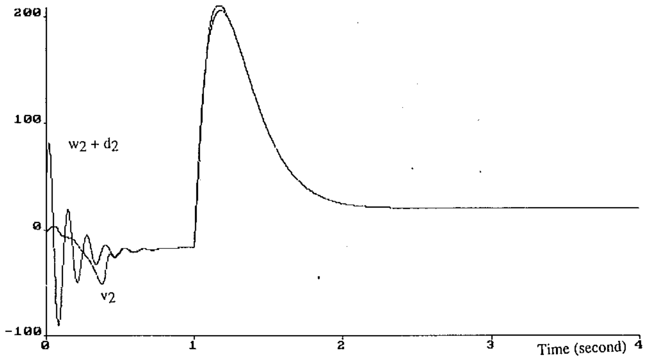

Fig. 4 Uncertainty $w_2 + d_2$ and estimated uncertainty v_1

magnitudes of the interconnections and disturbances can be of any bounded unknown size.

In this paper we proved the existence of large enough controller parameters which ensure the convergence of the system state; however, we did not explicitly show, given an initial condition, what the minimum values of these controller parameters are. One possible solution to this problem is to devise an adaptive version of the uncertainty (-bound) estimator. Nevertheless, we mention that overly large parameters in the uncertainty (-bound) estimators will not deteriorate the system performance in our case since large parameters affect the uncertainty estimate only during the transient period of the estimation process. Once the estimate has successfully tracked the uncertainty, the magnitudes of those parameters are no longer reflected in the estimation results. Since our control is not switched on until the uncertainty estimation is achieved, we do not need to worry about the system performance when using large parameters in the estimators. However, we have to mention that the above comment is true only when the measurement noise is negligible. When there is significant measurement noise, the problem of using large parameters in the proposed uncertainty estimator must be considered due to the weak robustness of the estimator to stochastic noise. How to make the estimator immune to the measurement noise is now under investigation.

References

Chen, M. S., 1990, "Estimation of System Uncertainties," *Proc. of the 1990 American Control Conference*, San Diego, Calif., pp. 2020–2024.

Chen, M. S., and Tomizuka, M., 1989, "Disturbance Estimator and Its Application in Estimation of System Output Derivatives," *Proc. of the 28th IEEE Conference on Decision and Control*, Tampa, Fla., pp. 452–457.

Gavel, D. T., and Siljak, D. D., 1989, "Decentralized Adaptive Control: Structural Conditions for Stability," IEEE Trans., A.C., Vol. 34, Apr., pp. 413–426.

Huseyin, O., Sezer, M. E., and Siljak, D. D., 1982, "Robust Decentralized Control Using Output Feedback," *IEE Proc.*, Vol. 129-D, No. 6, pp. 310–314.

Narendra, K. S., Annaswamy, A. M., and Singh, R. P., 1985, "A General Approach to the Stability Analysis of Adaptive Systems," *Int. J. Contr.*, Vol. 41, pp. 193–216.

Sezer, M. E., and Siljak, D. D., 1981, "On Decentralized Stabilization and Structure of Linear Large-Scale Systems," *Automatica*, Vol. 17, pp.641–644.

Siljak, D. D., 1978, *Large-Scale Dynamic Systems: Stability and Structure*, North-Holland, New York.

Youcef-Toumi, K., and Fuhlbrigge, T. A., 1989, "Application of a Decentralized Time Delay Controller to Robot Manipulators," *Proc. of the Robotic Conference*, pp. 1786–1791.

APPENDIX A

Derivation of Eq. (8): From the definition of e_i and Assumptions (A3) and (A4), we obtain

$$|e_i| \leq \pi_i \|w_i\| + \theta_i \|d_i\|$$
$$\leq \pi_i L_{r_p} + \theta_i D_{i0} \equiv e_i \quad \forall x \in B_{r_p} \qquad \text{(A.1)}$$

where $\pi_i = \|p_i\|$, $\theta_i = \|q_i\|$, and L_{r_p}, D_{i0} are as given, respectively, in Assumptions (A3) and (A4).

Applying Lemma 3 in Narendra et al. (1985) to Eq. (7) with $u=0$ yields

$$\|x(t)\| \leq C_i \|e\|_t + C_2, \text{ where } C_1, C_2 \in \mathbf{R}^+ \qquad \text{(A.2)}$$

Taking the norm of Eq. (7), and using (A.2), we obtain

$$\|\dot{x}(t)\| \leq C_3 \|e\|_t + C_4$$

where $C_3 = \|A\| C_1 + \|B\|$ and $C_4 = \|A\| C_2$. We now take the derivative of w_i:

$$\|\dot{w}_i\| \leq \|\partial f_i / \partial x\| \; \|\dot{x}\|$$
$$\leq M_{r_p} C_3 \|e\|_t + M_{r_p} C_4 \quad \forall x \in B_{r_p}$$

Hence,

$$|\dot{e}_i| = |p_i \dot{w}_i + q_i \dot{d}_i|$$
$$\leq \pi_i \|\dot{w}_i\| + \theta_i D_{i1}$$
$$\leq \pi_i M_{r_p} C_3 \|e\|_t + \pi_i M_{r_p} C_4 + \theta_i D_{i1} \quad \forall x \in B_{r_p}$$

Finally, noting that $\|e\|_t \leq \sum_i |e_i|_t$, and using (A.1), we obtain

$$|\dot{e}_i| \leq \eta_{i1} |e_i| + \eta_{i0} \quad \forall x \in B_{r_p}$$

where $\eta_{i1} = \pi_i M_{r_p} C_3$ and $\eta_{i0} = \pi_i M_{r_p} C_3 \sum_{j \neq i} e_j + \pi_i M_{r_p} C_4 + \theta_i D_{i1}$.

APPENDIX B

Proof of Lemma 1: From Eq. (8), we know that $e_i(t)$ can grow at most exponentially. In fact, we have

$$|e_i(t)| \leq (|e_i(0)| + \eta_{i0}/\eta_{i1}) e^{\eta_{i1} t} \quad \forall x \in B_{r_p}$$

Subtracting Eqs. (11*a*) and (11*b*) from Eq. (10), respectively, yields

$$\dot{\epsilon}_{i1} = -(\rho_i - \lambda_0)/\lambda_1 + e_i \text{ and } \dot{\epsilon}_{i2} = (\rho_i - \lambda_0)/\lambda_1 + e_i$$

$\dot{\epsilon}_{i1} \dot{\epsilon}_{i2} > 0$ implies that $\rho_i(t) < \lambda_1 |e_i(t)| + \lambda_0$; thus, $\rho_i(t)$ grows exponentially with exponent μ_1 by the choice of ν in Eq. (11*d*). If we choose $\mu_{i1} > \eta_{i1}$, given any initial conditions $e_i(0)$ and $\rho_i(0) > \lambda_0$, there exists a finite time T_0 such that $\rho_i(t) \geq \lambda_1 |e_i(t)| + \lambda_0$ for all $t \geq T_0$. Since $\rho_i(t)$ is a monotonically increasing function of time, we have

$$\rho_i(t) \geq \lambda_1 |e_i|_t + \lambda_0 \text{ for all } t \geq T_0. \qquad \text{(B.1)}$$

where $|e_i|_t \equiv \sup_{T_0 \leq \tau \leq t} |e_i(\tau)|$.

APPENDIX C

Proof of Lemma 2: Part (I) Define $V_1 = 1/2 s_i^2$. Let T_0 be as given in Lemma 1, then

$$\dot{V}_1 = s_i(-\sigma s_i - \rho_i g(s_i) + e_i) \quad \forall t \geq T_0.$$
$$\leq -\sigma s_i^2 - \rho_i |s_i| g(|s_i|) + |s_i| \, |e_i|$$
$$\leq -\sigma s_i^2 - (\lambda_1 |e_i| + \lambda_0) |s_i| (g(|s_i|) - |e_i| / (\lambda_1 |e_i| + \lambda_0))$$
$$\leq -\sigma s_i^2 - (\lambda_1 |e_i| + \lambda_0) |s_i| (g(|s_i|) - 1/\lambda_1)$$
$$\leq -\sigma r_{i1}^2 - (\lambda_1 |e_i| + \lambda_0) r_{i1} \gamma_1 \quad \forall s_i \notin M_{i1}$$
$$\text{(C.1)}$$

where we have used the result of Lemma 1 in the second inequality. Since $V_1(t) \geq 0$, we conclude that s_i must enter M_{i1} within a finite period of time, say T_1, and stays inside M_{i1} thereafter. Furthermore, from Eq. (C.1) we know that $T_1 - T_0$ can be made arbitrarily small if λ_1, λ_0 and σ are chosen sufficiently large.

Part (II) Define $V_2 = 1/2\dot{s}_i^2$. Then

$$\dot{V}_2 = \dot{s}_i(-\sigma\dot{s}_i - \nu\mu_1\rho_i g(s_i) - \nu\mu_0 g(s_i) - \rho_i g'(s_i)\dot{s}_i + \dot{e}_i)$$
$$\leq -\sigma\dot{s}_i^2 - \rho_i\dot{s}_i^2 g'(s_i) + |\dot{s}_i|\mu_1\rho_i|g(s_i)| + |\dot{s}_i|\mu_0|g(s_i)| + |\dot{s}_i|\,|\dot{e}_i|$$
$$\leq -\sigma\dot{s}_i^2 - \rho_i\dot{s}_i^2 g'(s_i) + |\dot{s}_i|\mu_1\rho_i|g(s_i)| + |\dot{s}_i|\mu_0|g(s_i)| + |\dot{s}_i|(\eta_1|\dot{e}_i|_t + \eta_0) \quad \text{(C1)}$$

Since $s_i(t)\epsilon M_{i1}$ after $t \geq T_1$, defining $\pi_{i1} = \sup_{s_i\epsilon M_{i1}} |g(s_i)|$ and $\pi_{i2} = \inf_{s_i\epsilon M_{i1}} g'(s_i)$, we obtain

$$\dot{V}_2 \leq -\sigma\dot{s}_i^2 - \rho_i\dot{s}_i^2\pi_{i2} + |\dot{s}_i|\mu_1\rho_i\pi_{i1} + |\dot{s}_i|\mu_0\pi_{i1} + \eta_1|\dot{s}_i|\,|e_i|_t + \eta_0|\dot{s}_i| \quad \forall t \geq T_1$$

$$\times\left(|\dot{s}_i| - \mu_1\frac{\pi_{i1}}{\pi_{i2}} - \frac{\eta_1|e_i|_t + \eta_0 + \mu_0\pi_{i1}}{\pi_{i2}\lambda_1|e_i|_t + \pi_{i2}\lambda_0}\right)$$
$$\leq -\sigma\dot{s}_i^2 - (\lambda_1|e_i|_t + \lambda_0)|\dot{s}_i|\pi_{i2} \times\left(|\dot{s}_i| - \mu_1\frac{\pi_{i1}}{\pi_{i2}} - \max\left(\frac{\eta_1}{\pi_{i2}\lambda_1}, \frac{\eta_0}{\pi_{i2}\lambda_0} + \frac{\mu_0\pi_{i1}}{\lambda_0\pi_{i2}}\right)\right)$$
$$\leq -\sigma r_{i2}^2 - (\lambda_1|e_i|_t + \lambda_0)r_{i2}\pi_{i2}\gamma_2 \quad \forall \dot{s}_i \notin M_{i2} \quad \text{(C.2)}$$

where we have used the result of Lemma 1 again in the second inequality. From the last inequality we conclude that $\dot{s}_i$ must enter M_{i2} within a finite period of time, say T_2, and stays inside M_{i2} thereafter. Furthermore, from Eq. (C.2) we know that $T_2 - T_1$ can be made arbitrarily small if λ_1, λ_0, and σ are chosen sufficiently large.

APPENDIX D

Proof of the Theorem. We first make an observation that if Eq. (8) remains true for the same parameters η_{i1} and η_{i0} before and after $t = T_2$, we retain the results of Lemma 1 and Lemma 2 for all $t \geq T_2$. These results then justify the Theorem. Hence, to prove the Theorem, all we need to show is that Eq. (8) remains true after $t = T_2$. A careful examination of the derivation of Eq. (8) in Appendix A shows that in fact all we need to do is to verify that under the condition $|u_i + e_i| \leq r_{i2}$, $x(t)$ remains in B_{r_p} for all $t \geq T_2$.

If $|u_i + e_i| \leq r_{i2}$ for all $i\epsilon N$, then

$$\|u + e\| \leq \sum_i |u_i + e_i| \leq \sum_i r_{i2}. \quad \text{(D.1)}$$

Since $T_2 < T$, Assumption (A4) ensures that

$$\|x(T_2)\| < r. \quad \text{(D.2)}$$

Writing down the solution of Eq. (7), and using Eq. (D.1), we have

$$\|x(t)\| = \left\| e^{A(t-T_2)}x(T_2) + \int_{T_2}^{t} e^{A(t-\tau)}B(u+e)d\tau \right\| \quad \forall t \geq T_2$$
$$\leq \|e^{A(t-T_2)}x(T_2)\| + \left\| \int_{T_2}^{t} e^{A(t-\tau)}Bd\tau \right\| \sum_i r_{i2}$$

It is easy to verify that the first term is smaller than $(\bar{\lambda}_p/\underline{\lambda}_p)^{1/2} r$, with the subscript P denoting the solution of Eq. (4) and r as given by Eq. (D.2), and the second term can be made smaller than any small positive number κ since A is a stable matrix and r_{i2} can be made arbitrarily small by Lemma 2. Finally, we obtain

$$\|x(t)\| \leq (\bar{\lambda}_p/\underline{\lambda}_p)^{1/2} r + \kappa = r_p$$

Hence, $x(t)\epsilon Br_p$ for all $t \geq T_2$. This concludes our proof.