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Application of the Least Squares Algorithm to the Observer Design for Linear Time-Varying Systems

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***Abstract*—In this paper, it is shown that the least squares algorithm with covariance reset, which is originally developed for the purpose of constant parameter identification, can be effectively applied to the observer design for a general linear time-varying system. The new observer successfully avoids many of the disadvantages of other time-varying observers, such as slow convergence rate, heavy computation load, high amplification of measurement noise, and the inapplicability to systems with *time-varying* observability indexes or *discontinuous* parameter variations.**

***Index Terms*— Least squares algorithm, linear time-varying system, observer, persistent excitation, uniform observability.**

I. Introduction

Since the development of the Kalman–Bucy filter [1], there have been several different approaches reported in the literature to the observer design for linear *time-varying* systems. The first approach utilizes a matrix differential Riccati equation [2], [3] for the observer design. In the time invariant case, this Riccati equation approach can exercise control over the observer convergence rate by properly selecting the covariance matrices of state and output noises (see [4, Ch. 6.1] for a detailed discussion). However, in the time-varying case, the link between the covariance matrices in the Riccati equation and the observer convergence rate becomes too complicated to be analyzed. Hence, it is still an open question as to how a time-varying Riccati equation should be designed so that a desired convergence rate can be achieved. The second approach [5] to the time-varying observer design utilizes a weighted observability grammian. Although the observer convergence rate can be effectively controlled in this design, the amount of calculation required is huge since *two* matrix differential equations need to be computed online. The third approach [6], [9] is the dual of the pole-placement control [7] for linear time-varying systems. Such an approach requires high-order time derivatives of the time-varying system parameters, which are difficult to measure due to the ever present nature of measurement noise. In addition, this approach restricts the observability indexes [8] to remain constant.

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In this paper, a new approach to the observer design is presented for a linear time-varying system. The new approach is based on the least squares algorithm with covariance reset [10], which was originally developed for the purpose of constant parameter identification. Since the least squares algorithm can produce arbitrarily fast convergence rate [11], one can then take advantage of this fact for the observer design. The resultant new observer has the following advantages: 1) the convergence rate of the observer can be effectively controlled by the design parameters; 2) no time-derivatives of the system parameters are required, consequently, the new observer can be applied to systems with *discontinuous* parameters; 3) the computation of the proposed observer feedback gain requires solving only *one* matrix differential equation; and 4) the observability indexes of the system are allowed to vary with time.

II. Problem Formulation

Consider a multivariable linear time-varying system

$$\dot{x} = A(t)x + B(t)u(t), \qquad x(0) = x_0, \qquad y = C(t)x \tag{1}$$

where $x(t) \in R^n$ is the system state vector, $u(t) \in R^m$ the control input, and $y(t) \in R^p$ the system output. The system matrix $A(t) \in R^{n\times n}$, the input matrix $B(t) \in R^{n\times m}$, and the output matrix $C(t) \in R^{p\times n}$ are time-varying matrices whose elements are bounded and (piecewise) continuous functions of time. It is assumed that if the open-loop system (1) is unstable, its state divergence rate is *uniformly* bounded in the sense that given any time span T, there exists a constant $\mu > 0$ such that the state transition matrix $\Phi(t,\tau) \in R^{n\times n}$ of the open-loop system (1) satisfies, for all integer k

$$\|\Phi(t, kT)\| \le \mu, \qquad \forall t \in [kT, (k+1)T] \tag{2}$$

where $\|\cdot\|$ denotes the maximum singular value of a matrix.

The objective of this paper is to reconstruct the system state $x(t)$ given only the measurement of the system output $y(t)$ under the assumption that the system (1) is uniformly observable in the following sense.

Definition 1 [5]: The pair $(A(t), C(t))$ is uniformly observable if there exist α_1 and $\alpha_2 \in R^+$ such that

$$\alpha_1 I \le W_o(k) \le \alpha_2 I, \qquad k = 0, 1, 2, \cdots \tag{3}$$

where $W_o(k) \in R^{n \times n}$ is the observability grammian defined by

$$W_o(k) \triangleq \int_{kT-T}^{kT} \Phi^T(\tau, kT-T) C^T(\tau) C(\tau) \Phi(\tau, kT-T)\, d\tau \tag{4}$$

in which $\Phi(t, \tau)$ is the open-loop state transition matrix of (1).

In this paper, the observer design will be based on the "least squares algorithm with covariance reset" developed in the parameter identification problem. A brief review of the least squares algorithm will be given below. Let $z(t) \in R^n$ represent the parameter error between the true parameter $\theta \in R^n$ and the estimated parameter $\hat{\theta}(t)$. If $\hat{\theta}(t)$ is updated based on the least squares algorithm with covariance reset, the governing equation of $z(t)$ is given by

$$\dot{z}(t) = -\gamma P(t) w(t) w^T(t) z(t), \qquad \forall t > 0 \tag{5}$$

$$\dot{P}(t) = -\gamma P(t) w(t) w^T(t) P(t), \qquad P(kT^+) = p_0 I_{n\times n} > 0, \quad \forall t \in [kT, (k+1)T) \tag{6}$$

where the least squares gain γ and the reset initialization value p_0 can be any positive constants, and $w(t) \in R^{n \times p}$ is called the "regressor." A well-known sufficient condition [12] on the exponential stability of (5) is that the regressor $w(t)$ be "persistently exciting" as defined below.

Definition 2 [12]: The regressor $w(t)$ is persistently exciting if there exists some time span T and positive constants η_1 and η_2 such that

$$\eta_1 I \le \int_{kT-T}^{kT} w(\tau) w^T(\tau)\, d\tau \le \eta_2 I, \qquad \forall k. \tag{7}$$

III. Observer Design

Consider the following observer for system (1):

$$\dot{\hat{x}} = A(t)\hat{x} + B(t)u + L(t)(y - C(t)\hat{x}), \qquad \hat{x}(0) = \hat{x}_0 \tag{8}$$

where $\hat{x}(t) \in R^n$ is an estimate of the system state $x(t)$, and $L(t) \in R^{n \times p}$ is the observer feedback gain to be determined so that $\hat{x}(t)$ approaches $x(t)$ exponentially. Denote the state estimation error by $\tilde{x} = \hat{x} - x$, and subtract (8) from (1) to yield the state estimation error dynamics:

$$\dot{\tilde{x}} = [A(t) - L(t)C(t)]\tilde{x}. \tag{9}$$

In order to transform the error dynamics (9) into a structure similar to that of the least squares equation in (5), pick a time span T, and on each time interval $[kT, (k+1)T)$, apply the following coordinate transformation to the error dynamics (9):

$$\tilde{x}(t) = \Phi(t, kT)\tilde{z}_k(t), \quad t \in [kT, (k+1)T), \quad k = 0, 1, 2, \cdots, \tag{10}$$

where the new coordinate $\tilde{z}_k(t)$ is defined only on the time interval $[kT, (k+1)T)$, and $\Phi(t, kT)$ is the state transition matrix of the open-loop system (1) defined by

$$\begin{aligned} \frac{\partial \Phi(t, kT)}{\partial t} &= A(t)\Phi(t, kT), \qquad t \in [kT, (k+1)T) \\ \Phi(kT, kT) &= I. \end{aligned} \tag{11}$$

From (9)–(11), the governing equation of $\tilde{z}_k(t)$ is given by

$$\dot{\tilde{z}}_k = -\Phi^{-1}(t, kT) L(t) w_o^T(t) \tilde{z}_k, \qquad w_o(t) = \Phi^T(t, kT) C^T(t), \quad t \in [kT, (k+1)T). \tag{12}$$

Comparison of (12) with the least squares equation (5) immediately suggests that the observer feedback gain $L(t)$ be chosen as

$$\begin{aligned} L(t) &= \gamma \Phi(t, kT) P_o(t) w_o(t) \\ &= \gamma \Phi(t, kT) P_o(t) \Phi^T(t, kT) C^T(t), \qquad t \in [kT, (k+1)T) \end{aligned} \tag{13}$$

where

$$\dot{P}_o(t) = -\gamma P_o(t) w_o(t) w_o^T(t) P_o(t), \quad P_o(kT^+) = p_0 I > 0, \quad t \in [kT, (k+1)T)$$

in which γ and p_0 can be any positive constants. The transformed state estimation error dynamics (12) then becomes

$$\dot{\tilde{z}}_k(t) = -\gamma P_o(t) w_o(t) w_o^T(t) \tilde{z}_k(t), \qquad t \in [kT, (k+1)T) \tag{14}$$

$$\dot{P}_o(t) = -\gamma P_o(t) w_o(t) w_o^T(t) P_o(t), \qquad P_o(kT^+) = p_0 I > 0 \tag{15}$$

which has exactly the same structure as the least squares algorithm in (5) and (6).

The following lemma shows that the *uniform observability* property of (1) guarantees that the regressor $w_o(t)$ in (14) is persistently exciting.

Lemma: If $(A(t), C(t))$ of (1) is uniformly observable as defined in (3), the regressor $w_o(t)$ in (14) is persistently exciting in the sense that

$$\alpha_1 m_1^2 I \le W_o^z(k) \triangleq \int_{kT-T}^{kT} w_o(\tau) w_o^T(\tau)\, d\tau \le \alpha_2 m_2^2 I, \quad k = 1, 2, \cdots \tag{16}$$

where $m_i's$ are two positive constants satisfying

$$m_1 \le \sigma_i[\Phi(kT, kT-T)] \le m_2, \qquad \forall k > 0. \tag{17}$$

Proof: See the Appendix. □

In the theorem below, the transformed state equations (14) and (15) will be used to analyze how $\|\tilde{z}_k(t)\|$ varies over the time interval $[kT, (k+1)T)$. Then, from the transformation relationship (10), the variation of $\|\tilde{x}(t)\|$ over the same time interval can be estimated for the purpose of stability analysis. One can thus establish the exponential stability property for the proposed observer.

Theorem: Consider (8) and (13). If (1) is uniformly observable, the state estimation error $\hat{x}(t) - x(t)$ converges to zero exponentially if the least squares gain γ in (14) and the reset initialization value p_0 are chosen large enough such that

$$\frac{\mu}{1 + \gamma p_0 \alpha_1 m_1^2} < 1,$$

where μ is as in (2) and $\alpha_1 m_1^2$ as in (16).

Proof: According to (15), the inverse of $P_o(t)$ satisfies

$$\dot{P}_o^{-1}(t) = \gamma w_o(t) w_o^T(t), \qquad \forall t \in [kT, (k+1)T).$$

Integrating the above equation from $t = kT^+$ to $t = (k+1)T^-$ gives

$$P_o^{-1}((k+1)T^-) - P_o^{-1}(kT^+) = \gamma \int_{kT}^{(k+1)T} w_o(\tau) w_o^T(\tau)\, d\tau.$$

It then follows from (16) in the lemma and $P_o(kT^+) = p_0 I$ that

$$\left(p_0^{-1} + \gamma \alpha_1 m_1^2\right) I \le P_o^{-1}((k+1)T^-) \le \left(p_0^{-1} + \gamma \alpha_2 m_2^2\right) I.$$

Equivalently

$$\frac{1}{p_0^{-1} + \gamma \alpha_2 m_2^2} I \le P_o((k+1)T^-) \le \frac{1}{p_0^{-1} + \gamma \alpha_1 m_1^2} I. \tag{18}$$

Using (14) and (15), one can verify that

$$\frac{d}{dt}\left(P_o^{-1}(t) \tilde{z}_k(t)\right) = 0, \qquad \forall t \in [kT, (k+1)T).$$

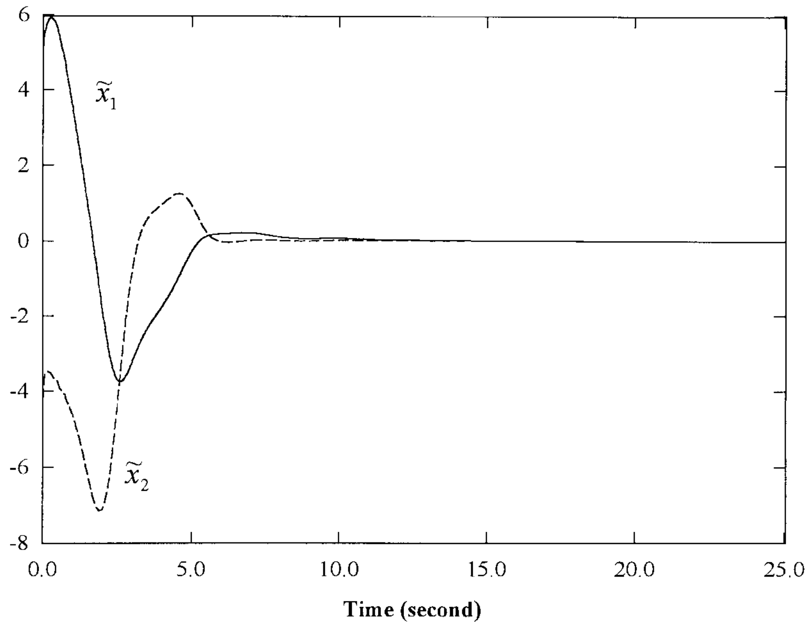

Fig. 1. State estimation error ($T = 0.25$).

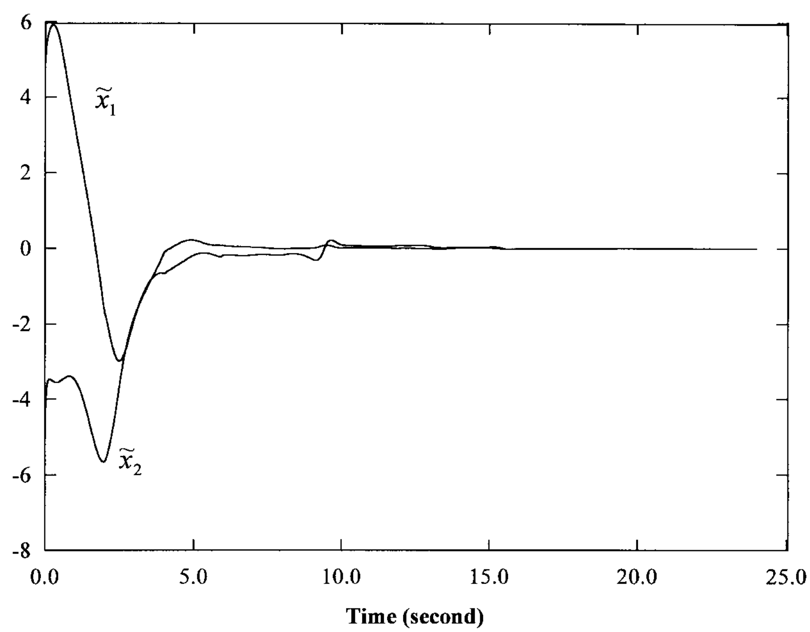

Fig. 2. State estimation error ($T = 2$).

Hence

$$P_o^{-1}(kT^+)\tilde{z}_k(kT^+) = P_o^{-1}((k+1)T^-)\tilde{z}_k((k+1)T^-).$$

Recalling that $P_o(kT^+) = p_0 I$, one can derive from the above equation

$$\begin{aligned}\|\tilde{z}_k((k+1)T^-)\| &\le \frac{1}{p_0}\|P_o((k+1)T^-)\| \cdot \|\tilde{z}_k(kT^+)\| \\ &\le \frac{1}{1+\gamma p_0 \alpha_1 m_1^2}\|\tilde{z}_k(kT^+)\| \end{aligned} \tag{19}$$

where the second inequality results from (18). In other words, the norm of $\tilde{z}_k(t)$ decreases by a factor of $1/(1+\gamma p_0 \alpha_1 m_1^2)$ over the time interval $[kT, kT+T)$.

Now, one can relate the variation of $\|\tilde{x}(t)\|$ with that of $\|\tilde{z}_k(t)\|$ through (10). At the beginning and the end of each time interval $[kT, (k+1)T)$, $\tilde{z}_k(t)$ and $\tilde{x}(t)$ are related by, according to (10)

$$\tilde{x}((k+1)T) = \Phi((k+1)T, kT)\tilde{z}_k((k+1)T^-) \tag{20}$$

$$\tilde{x}(kT) = \Phi(kT, kT)\tilde{z}_k(kT^+) = \tilde{z}_k(kT^+). \tag{21}$$

Taking the norm of $\tilde{x}((k+1)T)$ in (20) and using (2) yields

$$\begin{aligned}\|\tilde{x}((k+1)T)\| &\le \mu\|\tilde{z}_k((k+1)T^-)\| \\ &\le \frac{\mu}{1+\gamma p_0 \alpha_1 m_1^2}\|\tilde{z}_k(kT^+)\| \\ &= \frac{\mu}{1+\gamma p_0 \alpha_1 m_1^2}\|\tilde{x}(kT)\| \end{aligned}$$

where (19) and (21) have been used to obtain the above equations. Finally, one has

$$\|\tilde{x}(kT)\| \leq \left(\frac{\mu}{1+\gamma p_0 \alpha_1 m_1^2}\right)^k \|\tilde{x}(0)\|. \tag{22}$$

From the Hypothesis of the theorem $\mu/(1+\gamma p_0 \alpha_1 m_1^2) < 1$, one concludes from (22) that the state estimation error $\tilde{x}(kT)$ approaches zero exponentially when k approaches infinity. □

Remark 1: Notice that in (22), given any μ, m_1 and α_1, which characterize open-loop properties of the system (1), there always exist a least squares gain γ and a reset initialization value p_0 such that the number $\mu/(1+\gamma p_0 \alpha_1 m_1^2)$ is as small as possible. In other words, one can always pick either a large least squares gain γ *or* a large reset initialization value p_0 such that the decay rate of the observer is as fast as desired. Such a nice property cannot be achieved by a time-varying Kalman filter design since the relationship between the design parameters (the covariance matrices of the state and output noises) and the closed-loop decay rate is not clear in the time-varying case.

Remark 2: Another control design parameter other than γ and p_0 is the time interval length T in (13). In general, the open-loop properties such as μ, m_1, and α_1 in (22) depend on the value of T, but their relationships may vary widely for different system matrices. Nevertheless, if T is chosen too small, α_1 becomes almost zero [see (3) and (4)]. In other words, there is almost no obervability on such a small-length time interval. The observer will not be able to function effectively in this case. On the other hand, if T is chosen too large, the observer may lose its output injection at the end of each time interval because the output injection gain $L(t)$ may decrease to almost zero due to the decreasing nature of $P_0(t)$ [see (13) and (15)]. However, according to (22), for whatever value of T (which determines the values of μ, m_1, and α_1), there always exist γ and p_0 to ensure a desired convergence rate for the observer. Simulation experiences (see Figs. 1 and 2) do indicate that performance of the observer is mainly mandated by the least square gain γ and the reset initialization value p_0, and is relatively insensitive to the choice of T. For most values of T (except for very small or large values), proper tuning of γ and p_0 will always result in satisfactory performance of the observer.

A simulation example is given below to verify the proposed observer design.

Example: Consider the error dynamics (9), where the system matrices are given by

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1+1.5\sin t\cos t & -1+1.5\sin^2 t \end{bmatrix}$$
$$C(t) = [\cos\sqrt{t} + \sin\sqrt{t}, \quad 3\cos^2\sqrt{t}]$$

and the initial condition is $\tilde{x}^T(0) = [5, -5]$. Note that the system parameters vary nonperiodically due to the presence of $\sqrt{t}$. Simulation results indicate that the open-loop system is unstable. For the proposed observer design, the reset time interval T is first chosen to be 0.25 s, the reset initialization value $p_0 = 0.1$ in (15), and the least squares gain $\gamma = 20$ in (14). Fig. 1 shows the time history of the state estimation error, which converges exponentially to zero. An even faster response can be obtained if either a larger least squares gain γ or a larger reset initialization value p_0 is used. In the second simulation, the reset time interval T is changed to 2 s. It is seen from Fig. 2 that there has been no major change in the system performance.

IV. Conclusion

In this paper, the least squares algorithm with covariance reset is applied to the observer design for a general linear time-varying system. A unique feature of the new design is that the convergence rate of the observer can be effectively controlled by two scalar design parameters; the larger these two design parameters, the faster the convergence rate. Presently, the research is being conducted in extending the design method in this paper to the dual problem of observer design, i.e., to the problem of state feedback control design for a general linear time-varying system.

Appendix

Notice that $W_o^z(k)$ in the lemma is the observability grammian of the pair $(0, w_o^T(t))$, which is related to the system's observability grammian $W_o(k)$ in (4) by

$$W_o^z(k) = \Phi^T(kT, kT-T) W_o(k) \Phi(kT, kT-T).$$

Hence, given any constant vector v, one has

$$\alpha_1 \|\Phi(kT, kT-T)v\|^2 \leq v^T W_o^z(k) v \leq \alpha_2 \|\Phi(kT, kT-T)v\|^2,$$

due to (3). Further, using the following inequality from (17):

$$m_1\|v\| \leq \|\Phi(kT, kT-T)v\| \leq m_2\|v\|$$

one obtains

$$\alpha_1 m_1^2 \|v\|^2 \leq v^T W_c^z(k) v \leq \alpha_2 m_2^2 \|v\|^2$$

which leads to the claim of the lemma.

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