leads to the exponentially convergent observer dynamics

$$\begin{aligned} m\dot{\hat{v}} &= T - c_D\hat{v}|\hat{v}| - k_q(\hat{v} - \dot{q}) \\ I\dot{\hat{\omega}} &= -k_1\hat{\omega} + k_2U - Q - k_\alpha(\hat{\omega} - \dot{\alpha}) \end{aligned}$$

hence exploiting and augmenting the natural contraction properties of the system.

REFERENCES

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*. New York: Springer Verlag, 1978.
[2] R. Brockhaus, *Flight Control*. New York: Springer Verlag, 1994.
[3] R. Dawkins, *The Selfish Gene*. New York: Penguin, 1979.
[4] H. Goldstein, *Classical Mechanics*. New York: Addison-Wesley, 1980.
[5] W. Hahn, *Stability of Motion*. New York: Springer Verlag, 1967.
[6] P. Hartmann, *Ordinary Differential Equations*, 2nd ed. Boston, MA: Birkhauser, 1982.
[7] A. Isidori, *Nonlinear Control Systems*, 3rd ed: Springer Verlag, 1995.
[8] H. Khalil, *Nonlinear Systems*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[9] N. N. Krasovskii, *Problems of the Theory of Stability of Motion* (in English translation by Stanford Univ. Press, 1963). Moscow, Russia: Mir, 1959.
[10] W. Lohmiller, "Contraction analysis for nonlinear systems," Ph.D. dissertation, Dep. Mechanical Eng., M.I.T., 1998.
[11] W. Lohmiller and J. J. E. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, no. 6, 1998.
[12] ——, "Contraction analysis of mechanical systems," M.I.T. Nonlinear Systems Laboratory, Rep. NSL-980 501, 1998.
[13] D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles*. New York: Dover, 1989.
[14] D. G. Luenberger, *Introduction to Dynamic Systems*. New York: Wiley, 1979.
[15] R. Marino and T. Tomei, *Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[16] S. Massaquoi and J. J. E. Slotine, "The intermediate cerebellum may function as a wave variable processor," *Neuroscience Lett.*, vol. 215, 1996.
[17] G. Niemeyer and J. J. E. Slotine, "Stable adaptive teleoperation," *IEEE J. Oceanic Eng.*, vol. 16, pp. 152–162, Jan. 1991.
[18] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer Verlag, 1990.
[19] V. M. Popov, *Hyperstability of Control Systems*. New York: Springer-Verlag, 1973.
[20] L. Schwartz, *Analyze*. Paris, France: Hermann, 1993.
[21] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.
[22] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1992.
[23] L. Withcomb and D. Yoerger, "Preliminary experiments in the dynamical control of marine thrusters—Part 1: Dynamical modeling," *IEEE J. Oceanic Eng.*, vol. 23, 1998.

Exponential Stabilization of a Class of Unstable Bilinear Systems

Min-Shin Chen and Shia-Twu Tsao

***Abstract*—This paper considers the control design of bilinear systems with multiplicative control inputs. Previous control designs for such systems normally assume that the open-loop bilinear system is (neutrally) stable. In this paper, a new nonlinear control design is proposed for open-loop *unstable* bilinear systems. The new control stabilizes the bilinear system globally and exponentially *if* a sufficient stability condition, which can be checked by off-line computer simulations in advance of the control, is satisfied.**

***Index Terms*—Bilinear system, exponential stability, global stability, multiplicative control, nonlinear control.**

I. INTRODUCTION

Bilinear systems have been of great interest in recent years. This interest arises from the fact that many real-world systems can be adequately approximated by a bilinear model. Real-world examples include engineering applications in nuclear, thermal, and chemical processes, and nonengineering applications in biology, socio-economics, immunology, and so on. Detailed reviews of bilinear systems and their control designs can be found in [1] and [2]. For a bilinear system whose control input is both multiplicative and additive [2], one can use linear state feedback control [3] to obtain *local* asymptotical stability. Other control designs, such as the bang-bang control [4] or the optimal control [5], [6], obtain *global* asymptotic stability, but they all assume that the open-loop system is either stable or neutrally stable. When the open-loop system is unstable, it is difficult to obtain global asymptotical stability except when *independent* additive and multiplicative control inputs [7] exist.

This paper considers the control design for bilinear systems with multiplicative control inputs only. For such bilinear systems, it has been shown that *quadratic* state feedback control [8]–[10] can achieve global asymptotical stabilization, and *normalized* quadratic state feedback control [11] achieves global exponential stabilization. However, they also restrict the open-loop system to be stable or neutrally stable. In this paper, an attempt is made to find a nonlinear control, based on the normalized quadratic state feedback control design in [11], that can achieve global exponential stabilization for certain open-loop unstable bilinear systems. Our results show that the proposed new control will stabilize the system if a sufficient stability condition, which can be checked by off-line computer simulations in advance of the control, is satisfied.

II. NONLINEAR CONTROL

Consider bilinear systems with multiplicative control inputs

$$\dot{x}(t) = Ax(t) + u(t)Nx(t), \qquad x(0) = x_0 \tag{1}$$

where $x(t) \in R^n$ is the system state vector, $u(t)$ is a scalar control input, and $A \in R^{n\times n}$ and $N \in R^{n\times n}$ are constant square matrices. For simplicity, only the single-input case is treated; the results in this paper, however, can be easily extended to the multi-input case.

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Before introducing the control design, one first proposes the following coordinate transformation:

$$x(t) = e^{A(t-kT)} z_k(t), \qquad t \in [kT, kT+T) \tag{2}$$

where $T > 0$ is any chosen time interval length, $z_k(t)$ is the transformed state, and the transformation matrix is an open-loop fundamental matrix $\Phi(t, kT) = e^{A(t-kT)}$ [12]. With the transformation (2), the governing equation of the transformed state $z_k(t)$ becomes

$$\begin{aligned} \dot{z}_k(t) &= u(t) G_k(t) z_k(t) \\ G_k(t) &= e^{-A(t-kT)} N e^{A(t-kT)}, \qquad t \in [kT, kT+T). \end{aligned} \tag{3}$$

It should be noted that each transformed state $z_k(t)$ is defined only on a finite time interval $[kT, (k+1)T)$.

To stabilize the system (3), a nonlinear control is proposed as follows. If $e_{z_k}(t)$ denotes the *normalized* transformed state

$$e_{z_k}(t) = \frac{z_k(t)}{\|z_k(t)\|} = \frac{z_k(t)}{(z_k^T(t) z_k(t))^{1/2}} \tag{4}$$

the proposed control law is given by

$$u(t) = \begin{cases} -\alpha p_k(t) e_{z_k}^T(t) G_k(t) e_{z_k}(t), & z_k(t) \neq 0 \\ 0, & z_k(t) = 0, \end{cases} \quad t \in [kT, kT+T) \tag{5}$$

where $p_k(t) \in R^1$ is a positive time-varying, state-dependent gain defined by

$$\begin{aligned} \dot{p}_k(t) &= -\gamma p_k^2(t) \left(e_{z_k}^T(t) G_k(t) e_{z_k}(t) \right)^2 \\ p_k(kT) &= p_0 > 0, \qquad t \in [kT, kT+T) \end{aligned} \tag{6}$$

$G_k(t)$ is as in (3), α and γ are two positive control design parameters satisfying

$$\alpha \geq \gamma/2 > 0 \tag{7}$$

and $p_k(t)$ is reset to be p_0 at the beginning of every time interval $[kT, kT+T)$. Note that $p_k(t)$ is always positive and normally decreasing. In fact, one can also choose $\gamma = 0$; that is, the gain $p_k(t)$ becomes a positive constant p_0 for all time. In this case, however, one would need a different stability proof from the one given in this paper.

Solving the differential equation (6), one can express $p_k(t)$ as integration of $e_{z_k}(\tau), \tau \in [kT, t)$:

$$p_k(t) = \left[p_0^{-1} + \gamma \int_{kT}^{t} \left(e_{z_k}^T(\tau) G_k(\tau) e_{z_k}(\tau) \right)^2 d\tau \right]^{-1}, \quad t \in [kT, kT+T). \tag{8}$$

III. Stability Analysis

It is assumed that the system (1) satisfies the following rank condition [14]: an integer m exists such that

$$\text{span} \left\{ ad^k(A, N) x_0, k = 0, 1, 2, \cdots, m \right\} = R^n \tag{9}$$

for any nonzero x_0 in R^n, where $ad^k(A, N)$ is defined recursively as $ad^0(A, N) = N$ and $ad^{k+1}(A, N) = A \cdot ad^k(A, N) - ad^k(A, N) \cdot A$, $k = 0, 1, 2, \cdots$.

Lemma 1: If the system (1) satisfies the rank condition (9), and a constant vector e_0 exists such that

$$e_0^T G_k(t) e_0 \equiv 0, \qquad \forall t \in [kT, kT+T) \tag{10}$$

where $G_k(t)$ is defined in (3), then e_0 must be the null vector.

Proof: The proof is similar to that of Lemma 1 in [11]. □

Next, define a scalar function $\beta(\cdot)$: $S_n \to R^+ \cup \{0\}$, where S_n is the unit sphere in R^n, for the closed-loop system (1) and (5)

$$\beta(e_{z_k}(kT)) \triangleq \int_{kT}^{(k+1)T} \left| e_{z_k}^T(t) G_k(t) e_{z_k}(t) \right|^2 dt. \tag{11}$$

Given the matrix $G_k(t)$ in (3), once the initial condition of $e_{z_k}(t)$ at $t = kT$ is given, the whole trajectory $e_{z_k}(t)$, $t \in [kT, kT+T)$, is uniquely determined by the normalized closed-loop dynamics:

$$\begin{aligned} \dot{e}_{z_k}(t) &= \left(I - e_{z_k}(t) e_{z_k}^T(t) \right) (A + u(t) N) e_{z_k}(t) \\ u(t) &= -\alpha \left[p_0^{-1} + \gamma \int_{kT}^{t} \left(e_{z_k}^T(\tau) G_k(\tau) e_{z_k}(\tau) \right)^2 d\tau \right]^{-1} \\ &\quad \cdot e_{z_k}^T(t) G_k(t) e_{z_k}(t) \end{aligned} \tag{12}$$

according to (1), (5), and (8). Therefore, $\beta(\cdot)$ in (11), which consists of the integration of $e_{z_k}(\tau), \tau \in [kT, kT+T)$, is defined as a function of the initial condition $e_{z_k}(kT)$.

Lemma 2: A positive constant β^* exists such that

$$\inf_{e_{z_k}(kT) \in S_n} \beta(e_{z_k}(kT)) = \beta^* > 0 \tag{13}$$

where $\inf$ stands for the infimum taken over the unit sphere S_n.

Proof: Note that by its definition, $\beta(\cdot)$ must be nonnegative. Hence, to prove that $\beta(e_{z_k}(kT)) > 0$, one only needs to show that $\beta(e_{z_k}(kT))$ is nonzero if $e_{z_k}(kT) \in S_n$. A contradiction argument will be used to show this.

Assume that $\beta(e_{z_k}(kT)) = 0$ for some $e_{z_k}(kT) \in S_n$. Then, following (11), one has

$$e_{z_k}^T(t) G_k(t) e_{z_k}(t) \equiv 0, \qquad \forall t \in [kT, kT+T) \tag{14}$$

suggesting that the control input (5) is identically zero over the time interval $[kT, kT+T)$. From the transformed state (3), zero-control input implies that the transformed state will remain motionless during the entire time interval

$$z(t) = z(kT), \qquad \forall t \in [kT, kT+T)$$

and so does the normalized state

$$e_{z_k}(t) = e_{z_k}(kT), \qquad \forall t \in [kT, kT+T). \tag{15}$$

Substituting (15) into (14) gives

$$e_{z_k}^T(kT) G(t) e_{z_k}(kT) \equiv 0, \qquad \forall t \in [kT, kT+T).$$

Now, applying Lemma 1 to the above identity suggests that $e_{z_k}(kT) = 0$, contradicting the fact that $e_{z_k}(kT) \in S_n$. Therefore, $\beta(e_{z_k}(kT))$ must be nonzero and, hence, positive for any $e_{z_k}(kT) \in S_n$.

Finally, note that $\beta(e_{z_k}(kT))$ in (11) depends continuously on its argument $e_{z_k}(kT)$. Because the domain of $\beta(\cdot)$, the unit sphere S_n, is compact, it follows from [13, Theorem 4.4.1] that a positive constant β^* exists such that (13) holds. □

It will now be shown that under a sufficient stability condition, the controlled bilinear system is globally and exponentially stable.

Theorem: Consider the bilinear system (1) and the nonlinear control (5). If control design parameters α, γ, and p_0 exist such that

$$\frac{\|e^{AT}\|}{(1 + \gamma p_0 \beta^*)^{\alpha/\gamma}} < 1 \tag{16}$$

where β^* is given by (13), given any finite initial condition $x(0)$, the controlled system state $x(t)$ converges to zero exponentially.

Proof: First, from (6), one has

$$\frac{dp_k(t)}{p_k^2(t)} = -\gamma\left(e_{z_k}^T(t)G_k(t)e_{z_k}(t)\right)^2 dt, \quad \forall t \in [kT, (k+1)T).$$

Integrating this equation gives

$$\begin{aligned} p_k^{-1}(kT+T) - p_k^{-1}(kT) &= \gamma \int_{kT}^{kT+T} \left(e_{z_k}^T(t)G_k(t)e_{z_k}(t)\right)^2 dt \geq \gamma\beta^* \end{aligned}$$

where the inequality results from Lemma 2. Because $p_k(kT) = p_0$, one actually has

$$\frac{p_k(kT)}{p_k(kT+T)} \geq 1 + \gamma\beta^* p_0. \tag{17}$$

Second, define a positive scalar function

$$V_k(t) = p_k^{-1}(t)\|z_k(t)\|^2, \qquad t \in [kT, (k+1)T) \tag{18}$$

where $p_k(t) > 0$ is from (6). Taking the time derivative of $V_k(t)$ along (3), (5), and (6), one obtains

$$\begin{aligned} \dot{V}_k(t) &= -(2\alpha - \gamma)\left(e_{z_k}^T(t)G_k(t)e_{z_k}(t)\right)^2 \cdot \|z_k(t)\|^2 \\ &= -(2\alpha - \gamma)\left(e_{z_k}^T(t)G_k(t)e_{z_k}(t)\right)^2 \cdot p_k(t) \cdot V_k(t). \end{aligned}$$

Note that the choice $\alpha \geq \gamma/2$ makes $V_k(t)$ always nonincreasing. Substituting (6) into the above equation shows that

$$\frac{dV_k(t)}{V_k(t)} = \frac{2\alpha - \gamma}{\gamma} \cdot \frac{dp_k(t)}{p_k(t)}$$

and hence

$$V_k(kT+T) = \left[\frac{p_k(kT+T)}{p_k(kT)}\right]^{(2(\alpha/\gamma)-1)} \cdot V_k(kT). \tag{19}$$

Using the definition of $V_k(t)$ in (18), one can further deduce from (19) that

$$\begin{aligned} \|z_k((k+1)T)\| &\leq \left[\frac{p_k(kT+T)}{p_k(kT)}\right]^{\alpha/\gamma} \|z_k(kT)\| \\ &\leq \frac{1}{(1+\gamma p_0\beta^*)^{\alpha/\gamma}} \|z_k(kT)\| \end{aligned} \tag{20}$$

where the second inequality results from (17).

Finally, to check how $\|x(t)\|$ varies, note from (2) that

$$x(kT) = z_k(kT)$$

and

$$x((k+1)T) = e^{AT} z_k((k+1)T). \tag{21}$$

Taking the norm of $x((k+1)T)$ in (21), one obtains

$$\begin{aligned} \|x((k+1)T)\| &\leq \|e^{AT}\| \cdot \|z_k((k+1)T)\| \\ &\leq \frac{\|e^{AT}\|}{(1+\gamma p_0\beta^*)^{\alpha/\gamma}} \cdot \|z_k(kT)\| \\ &= \frac{\|e^{AT}\|}{(1+\gamma p_0\beta^*)^{\alpha/\gamma}} \cdot \|x(kT)\| \end{aligned} \tag{22}$$

where the second inequality results from (20) and the last equality from the first identity in (21). One can thus infer from the hypothesis (16) and the contraction mapping theorem [14] that given any finite $x(0)$, the system state $x(kT)$ converges to zero exponentially. □

Remark 1: When the bilinear system (1) is open-loop stable or neutrally stable, T always exists such that $\|e^{AT}\| \leq 1$. In these cases, the stability condition (16) is trivially satisfied. In other words, the proposed control is guaranteed to stabilize an open-loop (neutrally) stable bilinear system (1).

Remark 2: When the bilinear system is open-loop unstable, one would have to check the condition (16) for global and exponential stability. Note that in (16), the infimum β^* depends on various control design parameters: α, γ, p_0 and T; that is, $\beta^* = \beta^*(\alpha, \gamma, p_0, T)$. The simulation experiences indicate that the normalized β^*/T is very much insensitive to the choice of T except for very small T. Hence, different choices of T do not significantly affect the performance of the controlled system.

The relationships between β^* and design parameters α, γ, and p_0 are difficult to obtain *analytically* because of the complicated nonlinearity in the closed-loop dynamics. Their relationships, however, can still be found by *off-line finite time computer simulations*. The following procedure demonstrates how to obtain the relationship between β^* and p_0 (or other design parameters).

1) Pick an initial design parameters p_0, and fix two other design parameters α and γ.
2) Pick an initial condition $z_0(0)$ on the unit sphere S_n, and simulate the closed-loop dynamics (3) and (5) with $k = 0$ to obtain $z_0(\tau), \tau \in [0, T)$. Calculate the normalized state $e_{z_0}(\tau), \tau \in [0, T)$, and the integration (11) to find out $\beta(e_{z_0}(0))$.
3) Repeat the above step for sufficiently many initial conditions $z_0(0)$'s on the unit sphere S_n so that an approximate of the infimum β^* in Lemma 2 can be obtained.
4) Increase the design parameter p_0 by a small amount, and then repeat Steps 2) and 3) to calculate the new β^* so a plot of β^* versus p_0 can be obtained.
5) Create a plot of $(1+\gamma p_0\beta^*)^{\alpha/\gamma}$ versus p_0 from the plot of β^* versus p_0 and, finally, examine from the plot if the stability condition (16) is satisfied for certain ranges of p_0.

A simulation example of the controlled bilinear system is presented below.

Example: Consider an open-loop unstable bilinear system (1) with

$$A = \begin{bmatrix} 3 & -5 \\ 4 & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and the initial condition $x^T(0) = [5, -4]$. The open-loop system matrix has eigenvalues $2 \pm \sqrt{2}i$.

For the proposed control with $\gamma = 0.2$, $p_0 = 2$, and $T = 1$ s, Fig. 1 shows the plot of $(1+\gamma p_0\beta^*)^{\alpha/\gamma}$ versus α, which indicates the stability condition (16) is satisfied as long as $1.5 < \alpha < 3.6$. When α is chosen to be 2, the system state converges to zero in about 5 s, as is shown in Fig. 2. It should be noted that the stability condition (16) derived in the theorem is sufficient only. In practice, the range of α, which results in a stable closed-loop system, is larger than as indicated by Fig. 1.

IV. Other Control Designs

It is possible to construct other control laws based on the design presented above. For example, let $F(\cdot): R \to R$ be any piecewise continuous function such that

$$yF(y) > 0, \qquad \forall y \neq 0.$$

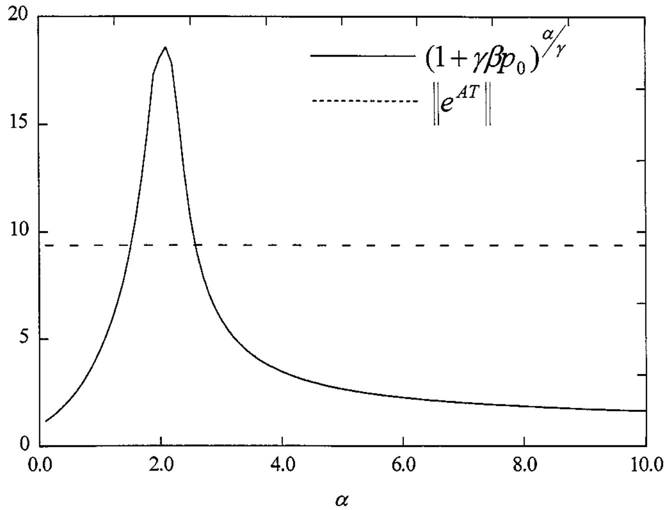

Fig. 1. Plot of $(1+\gamma\beta^* p_0)^{\alpha/\gamma}$ versus α.

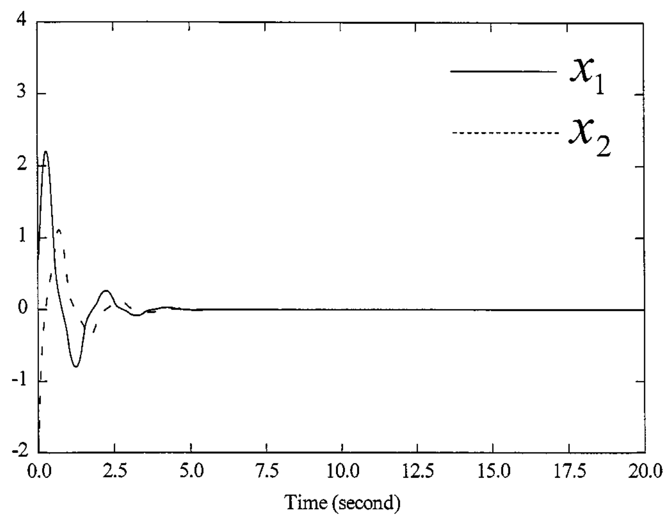

Fig. 2. Closed-loop state response.

The following control law then becomes a candidate to be considered for stabilization of the system (1):

$$u(t) = \begin{cases} -\alpha p_k(t) F\left(e_{z_k}^T(t) G_k(t) e_{z_k}(t)\right), & z_k(t) \neq 0 \\ 0, & z_k(t) = 0, \end{cases} \quad t \in [kT, kT+T). \tag{23}$$

where $p_k(t) \in R^1$ is modified as

$$\dot{p}_k(t) = -\gamma p_k^2(t) e_{z_k}^T(t) G_k(t) e_{z_k}(t) \cdot F\left[e_{z_k}^T(t) G_k(t) e_{z_k}(t)\right], \qquad t \in [kT, kT+T)$$

in which all parameters are as in (5) and (6).

A list of commonly seen choices of $F(\cdot)$ is given below.

1) $F(y) = y$, which corresponds to the *normalized quadratic* control (5) in Section II;
2) $F(y) = \text{sign}(y)$, which corresponds to the *switching* control;
3) $F(y) = y/(\epsilon + |y|)$, with ϵ being a small positive number, which corresponds to the *boundary-layer* control [15];
4) $F(y) = y/(\epsilon + y^2)$, with ϵ being a small positive number, which corresponds to the *smooth division* control because $F(y) \sim 1/y$ if $|y| \gg \epsilon$.

The class of control in (23) will achieve the same goal as stated in the theorem in Section III: it will exponentially stabilize open-loop (neutrally) stable bilinear system (1), and under the same condition (16) as in the theorem, stabilize open-loop unstable system (1). The only exception is that β^* in the stability condition (16) is now different. For the control in (23), β^* is the infimum over the unit sphere S_n of the following function $\beta(\cdot): S_n \to R^+$,

$$\beta(e_{z_k}(kT)) \triangleq \int_{kT}^{(k+1)T} e_{z_k}^T(t) G_k(t) e_{z_k}(t) \cdot F\left[e_{z_k}^T(t) G_k(t) e_{z_k}(t)\right] dt.$$

Because different choices of the function $F(\cdot)$ produce different values of the infimum β^*, control laws with different $F(\cdot)$ may result in different exponential convergent rates for the transformed state $z_k(t)$ in (3) (see (20)). Hence, the choice of $F(\cdot)$ will affect the stabilizing ability of the particular control design.

V. Conclusions

In this paper, a nonlinear control design is proposed for bilinear systems with multiplicative control inputs. The bilinear system, if it is to be stabilized, must satisfy a closed-loop stability condition (16), which can be checked by off-line finite time computer simulations before the control is applied.

References

[1] C. Bruni, G. Dipillo, and G. Koch, "Bilinear systems: An appealing class of nearly linear systems in theory and applications," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 334–348, 1974.
[2] R. R. Mohler, *Bilinear Systems, Volume II, Applications to Bilinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.
[3] I. Derese and E. Noldus, "Design of linear feedback laws for bilinear systems," *Int. J. Contr.*, vol. 31, no. 2, pp. 219–237, 1980.
[4] R. Longchamp, "Controller design for bilinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 547–548, 1980.
[5] G. S. Tzafestas, K. E. Anagnostou, and T. G. Pimenides, "Stabilizing optimal control of bilinear systems with a generalized cost," *Optimal Contr. Applicat. Methods*, vol. 5, pp. 111–117, 1984.
[6] A. Benallou, D. A. Mellichamp, and D. E. Seborg, "Optimal stabilizing controllers for bilinear systems," *Int. J. Contr.*, vol. 48, no. 4, pp. 1487–1501, 1988.
[7] I. Derese and E. Noldus, "Optimization of bilinear control systems," *Int. J. Syst. Sci.*, vol. 13, no. 3, pp. 237–246, 1982.
[8] M. Slemrod, "Stabilization of bilinear control systems with applications to nonconservative problems in elasticity," *SIAM J. Contr. Optim.*, vol. 16, pp. 131–141, 1978.
[9] E. P. Ryan and N. J. Buckingham, "On asymptotically stabilizing feedback control of bilinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 863–864, 1983.
[10] S. N. Singh, "Stabilizing feedback controls for nonlinear hamiltonian systems and nonconservative bilinear systems in elasticity," *J. Dyn. Syst. Meas. Contr.*, vol. 104, pp. 27–32, 1982.
[11] M. S. Chen, "Exponential stabilization of a constrained bilinear system," *Automatica*, vol. 34, no. 8, pp. 989–992, 1998.
[12] F. Callier and C. A. Desoer, *Linear System Theory*, Hong Kong: Springer-Verlag, 1992.
[13] J. E. Marsden and M. J. Hoffman, *Elementary Classical Analysis*. San Francisco, CA: Freeman, 1993.
[14] M. Vidyasagar, *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
[15] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.