旋轉圓盤受週期性外緣力時之參數共振研究

Parametric Resonance of a Spinning Disk Under Pulsalting Edge Loads

計畫編號: NSC 87-2212-E-002-042

執行期限: 86年8月1日至87年7月31日

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摘要

本計畫以解析的方法研究旋轉圓盤受外緣週期性外力時之參數共振現象。我們假設徑向外緣力的分佈可以傅立葉級數的方式展開。利用圓盤特徵函數的正交特性，我們將偏微分方程式離散化成一組一階的希爾方程式，再利用多重尺度法決定組合式參數共振發生的條件。

關鍵詞:**參數共振，旋轉圓盤，外緣力**

Abstract

The parametric resonance of a spinning disk under a space-fixed pulsating edge load is investigated analytically. We assume that the radial edge load can be expanded in a Fourier series. With use of the orthogonality properties among the eigenfunctions of a gyroscopic system, the partial differential equation of motion is discretized into a system of generalized Hill's equations in the first order form. The method of multiple scale is employed to determine the conditions for combination resonances to occur.

Key Words: Parametric Resonance, spinning disk, edge load

Introduction

The vibration analysis of a spinning disk under a space-fixed edge load attracts attention because of its possible application in such fields as circular saw cutting and grind wheel operation. Recently Chen (1994) formulated this problem with emphasis on the effects of relative motion between the disk and the edge load on the stability and natural frequencies of the loaded disk. The edge loads in this work are assumed to be independent of time. In the case when the edge loads are periodically varying, parametric resonance may be induced. Tani and Nakamura (1978; 1980) studied the dynamic instability of an annular disk under periodically varying in-plane edge traction. The disk considered in their papers is clamped on both the inner and outer radii, and the edge traction is uniform in the circumferential direction. Zajaczkowski (1983) investigated the parametric resonance of a clamped-free disk under both uniform and concentrated pulsating torques. In these papers both the annular disk and the periodic loading are fixed in space.

A natural extension of these previous analyses is to study the dynamic instability of a spinning annular disk under periodically varying edge load which is fixed in space. This investigation may find application in the wood cutting industry as it represents a more general model for the cutting process by a circular saw.

Equation of Motion

We assume the disk is clamped at the inner radius and subjected to a periodic radial traction at the outer radius. The edge in-plane traction can be expanded in a Fourier series $\pounds \cos ¥t \sum_{k=0}^{\infty} f_k \cos k¯$. The disk is rotating with constant speed Ω, while the edge load is fixed in space. The dimensionless equation of motion of the system, in terms of the transverse displacement w and with respect to the stationary coordinate system $(r,¯)$, is

$$M\frac{\partial^2 w}{\partial t^2} + G\frac{\partial w}{\partial t} + (K + \hat{K})w = 0 \qquad (1)$$

where

$$M \equiv 1, G \equiv 2\Omega \frac{\partial}{\partial \theta}, K \equiv \Omega^2 \frac{\partial^2}{\partial \theta^2} + \nabla^4 + L,$$

$$\bar{K} \equiv \varepsilon \cos \omega t \sum_{k=0}^{\infty} f_k \left[\bar{E}_k + ¢(r-1)\cos k\theta \frac{\partial}{\partial r} \right]$$

Equation (1) can also be cast in the first-order operator form

$$\left(\mathbf{A} + \bar{\mathbf{A}}\right) \frac{\partial \mathbf{x}}{\partial t} - \left(\mathbf{B} + \bar{\mathbf{B}}\right)\mathbf{x} = 0 \qquad (2)$$

by defining the state vector

$$\mathbf{x} \equiv \begin{Bmatrix} \dfrac{\partial w}{\partial t} \\ w \end{Bmatrix}$$

and the matrix differential operators

$$\mathbf{A} \equiv \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}, \quad \bar{\mathbf{A}} \equiv \begin{bmatrix} 0 & 0 \\ 0 & \bar{K} \end{bmatrix}, \quad \mathbf{B} \equiv \begin{bmatrix} -G & -K \\ K & 0 \end{bmatrix}, \quad \bar{\mathbf{B}} \equiv \begin{bmatrix} 0 & -\bar{K} \\ \bar{K} & 0 \end{bmatrix}$$

Figure 1 shows the dimensionless natural frequencies μ_{mn} of a freely spinning disk as functions of dimensionless rotation speed Ω.

Discretization

Since a closed form solution for Eq.(1) does not exist in general, we use an expansion in terms of finite number of eigenfunctions of the freely spinning disk to approximate the true solution of Eq.(1),

$$\mathbf{x}(r, \theta; t) \approx \sum_{p=0}^{N_1} \sum_{q=-N_2}^{N_2} c_{pq}(t) \mathbf{x}_{pq}(r, \theta) \qquad (3)$$

N_1 and N_2 are the maximum numbers of nodal circles and nodal diameters, respectively, of the modes used in the expansion. Substituting Eq.(3) into (11) and taking the inner product between $\mathbf{x}_{mn}$ and both sides of Eq. (1), we obtain a system of generalized Hill equations in the first order form,

$$\frac{dc_{mn}}{dt} - i\mu_{mn} c_{mn} + \varepsilon \cos \omega t \sum_{k=0}^{\infty} \sum_{p=0}^{N_1} \sum_{q=-N_2}^{N_2} \left[\frac{\bar{A}^{mn}_{pq(k)}}{A^{mn}_{mn}} \frac{dc_{pq}}{dt} - \frac{\bar{B}^{mn}_{pq(k)}}{A^{mn}_{mn}} c_{pq} \right] = 0 \qquad (4)$$

where

$$A^{mn}_{mn} = 4®\mu_{mn}(\mu_{mn} + n\Omega) \int_{\prime}^{1} R^2_{mn}(r) r dr$$

When $n - q \neq \pm k$, $\hat{A}^{mn}_{pq(k)} = \hat{B}^{mn}_{pq(k)} = 0$. On the other hand, when $n - q = \pm k$, we obtain

$$\hat{A}^{mn}_{pq(k)} = \hat{A}^{pq}_{mn(k)} =$$

$$= \sim_k \int_{\prime}^{1} [r\tilde{\pm}_{rk} \frac{dR_{mn}}{dr} \frac{dR_{pq}}{dr} + q(n \frac{\tilde{\pm}_{\bar{}k}}{r} - \frac{d\tilde{\pm}_{r\bar{}k}}{dr}) R_{mn} R_{pq} - (n+q)\tilde{\pm}_{r\bar{}k} R_{mn} \frac{dR_{pq}}{dr}] dr$$

$$\hat{B}^{mn}_{pq(k)} = i(\mu_{mn} + \mu_{pq}) \hat{A}^{mn}_{pq(k)}$$

where $\sim_0 = 2®f_0$, and $\sim_k = ®f_k$ when $k \neq 0$. .

Perturbation Technique

The method of multiple scale assumes an expansion of the form

$$c_{mn}(t) = c^{(0)}_{mn}(t, T_1) + \varepsilon c^{(1)}_{mn}(t, T_1) + O(\varepsilon^2) \qquad (5)$$

where $T_1 \equiv \varepsilon t$. Substituting (5) into (4) and equating coefficients of like powers of ε yield

$$\varepsilon^0: \quad D_0 c^{(0)}_{mn} - i\mu_{mn} c^{(0)}_{mn} = 0 \qquad (6)$$

ε^1:

$$D_0 c^{(1)}_{mn} - i\mu_{mn} c^{(1)}_{mn} = -D_1 c^{(0)}_{mn} - \cos \omega t \sum_{k=0}^{\infty} \sum_{p=0}^{N_1} \sum_{q=-N_2}^{N_2} \frac{\hat{A}^{mn}_{pq(k)}}{A^{mn}_{mn}} D_0 c^{(0)}_{pq} - \frac{\hat{B}^{mn}_{pq(k)}}{A^{mn}_{mn}} c^{(0)}_{pq} \qquad (7)$$

where $D_0 \equiv \frac{\partial}{\partial t}$, and $D_1 \equiv \frac{\partial}{\partial T_1}$. The general solution of Eq.(6) can be written in the form

$$c^{(0)}_{mn} = H_{mn}(T_1) e^{j\mu_{mn} T_0} \qquad (8)$$

Substituting (8) into (7) yields

$$D_0 c^{(1)}_{mn} - i\mu_{mn} c^{(1)}_{mn} = -D_1 H_{mn} e^{j\mu_{mn} T_0}$$
$$+ \frac{\mu_{mn}}{2} \sum_{k=0}^{\infty} \sum_{p=0}^{N_1} \sum_{q=-N_2}^{N_2} \frac{\hat{A}^{mn}_{pq(k)}}{A^{mn}_{mn}} H_{pq} \left[e^{j(\mu_{pq}+\omega)T_0} + e^{j(\mu_{pq}-\omega)T_0} \right] \qquad (9)$$

In the case when ω is near $\mu_{rs} + \mu_{uv}$, we assume that $\omega = \mu_{rs} + \mu_{uv} + \varepsilon s$, where s is a detuning parameter. The secular terms in Eq.(9) are eliminated if

$$D_1 H_{rs} - i\mu_{rs} \frac{\sum_{k=0}^{\infty} A_{u,-v(k)}}{2A_{rs}^{rs}} \overline{H}_{uv} e^{js T_1} = 0 \qquad (10)$$

$$D_1 H_{uv} - i\mu_{uv} \frac{\sum_{k=0}^{\infty} A_{r,-s(k)}}{2A_{uv}^{uv}} \overline{H}_{rs} e^{js T_1} = 0 \qquad (11)$$

It follows that the solution is bounded if and only if

$$s^2 \geq \Lambda_{uv}^{rs}$$

where

$$\Lambda_{uv}^{rs} = \frac{\mu_{rs}\mu_{uv}\left[\sum_{k=0}^{\infty} \overline{A}_{u,-v(k)}^{rs}\right]^2}{A_{rs}^{rs} A_{uv}^{uv}} \qquad (12)$$

On the other hand, combination resonance may occur when $s^2 < \Lambda_{uv}^{rs}$. Frequency $\mu_{rs}+\mu_{uv}$ is called the center frequency and $\sqrt{\Lambda_{uv}^{rs}}$ is called the width parameter of the parametric resonance.

Concentrated Edge Loading

We consider the case when the spinning disk is subjected to a space-fixed concentrated edge load. In this case the Fourier decomposition of the edge load would include all the harmonics of $\cos k\theta$, and the summation $\sum_{k=0}^{\infty} A_{u,-v(k)}$ in Eq.(12) reduces to a single non-zero term $A_{u,-v(k)}$, where $s+v=\pm k$. In other words, for any two modes (*r,s*) and (*u,v*), there always exists a specific Fourier component which renders the summation $\sum_{k=0}^{\infty} A_{u,-v(k)}$ non-zero. Bearing the relation $\mu_{mn}=-\mu_{m,-n}$ in mind and speaking of only positive natural frequencies, we can conclude that combination resonance of the sum type occurs when both modes are non-reflected or both modes are reflected. On the other hand, combination resonance of the difference type can occur only when one mode is reflected and the other is non-reflected. Single mode parametric resonance can occur for any mode, reflected or non-reflected.

The center frequency and the width parameter of a spinning disk under a pulsating concentrated edge load are shown in Fig.2. Solid lines represent the single mode parametric resonance, while dashed lines and dotted lines represent the cases of combination resonance of the sum type and difference type respectively. In the frequency range of Fig.3 only the modes with zero nodal circle contribute to the parametric resonance. The mode labels in Fig.2 are simplified by neglecting the number of nodal circle. For instance, $0+2_f$ represents the combination resonance of the sum type involving modes $(0,0)$ and $(0,2)_f$, and 2_b-3_r represents the combination resonance of the difference type involving modes $(0,2)_b$ and $(0,3)_r$. It can be seen from Fig.2 that the rotation speed tends to squeeze the width of parametric resonance region.

In the special case when the excitation frequency γ approaches zero, which corresponds to the case of constant concentrated edge load, we can see from Fig.2 that there exist two rotation speeds at which parametric resonance can occur. The rotation speed at which the solid line labeled 3_b+3_b intersects the horizontal line $\mu_{rs}+\mu_{uv}=0$ corresponds to the first critical speed. Therefore, the divergence instability reported by Chen (1994) can be considered as a limit case of single mode parametric resonance. On the other hand, the rotation speed at which the dotted line labeled 2_b-3_r intersects the line $\mu_{rs}+\mu_{uv}=0$ is a speed at which the backward wave $(0,2)_b$ and reflected wave $(0,3)_r$ become degenerate. Therefore, the flutter instability reported by Chen (1994) can be considered as a limit case of combination resonance of the difference type.

Conclusions

Dynamic stability of a spinning annular disk under periodically varying in-plane loading on the outer rim is studied analytically. The results show that combination resonance is possible only when there exists a specific Fourier component $\cos k^{-}$ in the edge load, where k equals the sum of the number of nodal diameters of these two modes. Sum type resonance occurs when both modes are non-reflected or both modes are reflected. On the other hand, difference type resonance occurs when one mode is reflected and the other is non-reflected.

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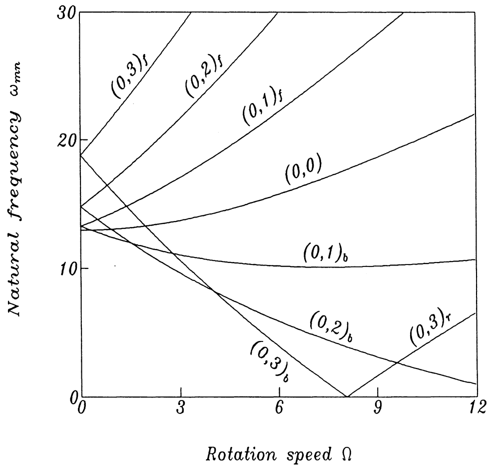

Fig.1 Dimensionless natural frequency versus dimensionless rotation speed for a freely spinning disk.

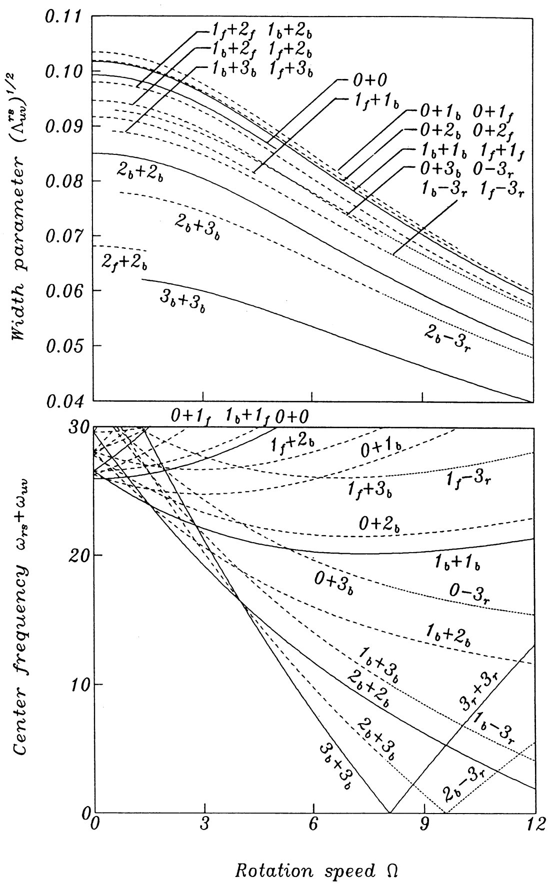

Fig.2 Center frequencies and width parameters of a spinning disk under a space-fixed pulsating concentrated edge load.