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Short Paper

KINEMATIC ANALYSIS OF MECHANISMS WITH ROLLING PAIRS USING MATRIX TRANSFORMATION METHOD

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ABSTRACT

An efficient procedure is presented for the kinematic analysis of mechanisms with multiple Holonomic pairs. The matrix transformation method is used for modeling the high pair in the mechanism. Then, displacement constraints for two links in rolling contact are derived using the line integral of the link contour. Finally, the displacement constraints associated with the loop closure equations from the overall transformation matrix can be obtained as a set of simultaneous equations for the analysis of the mechanism. These equations can be solved via the numerical method. The procedure is also illustrated by an example with parabolic and circular link shapes.

Key Words: kinematic analysis, matrix transformation, rolling contact, loop closure equation.

I. INTRODUCTION

The analysis of mechanisms with higher pairs is not a new topic. Much attention on this subject has been paid to the analysis of cam-linkages, geared-linkages, and linkages with ball-and-socket joints. Traditionally, a linkage containing higher pairs is always analyzed by a reduction to an equivalent mechanism with only lower pairs. Then, the vector loop method is applied to find equations required for the kinematic analysis. This is a useful approach for mechanisms with kinematic pairs made up of regular geometry. However, for kinematic pairs of irregular geometry, the equivalent mechanism changes its proportions as the configuration varies. The equivalent mechanism approach may not be conveniently applicable in solving those kinds of problems. Therefore, various approaches have been developed, among which the matrix method is most frequently used. Denavit and Hartenberg (1955) were the first to introduce the matrix method to the kinematic analysis of mechanisms. Sheth and Uicker (1971) used the matrix method to analyze the mechanism, in which dealing with the high pairs was separated from the geometry of the link shape. Sheth *et al.* (1990) further developed a generalized transformation matrix for kinematic pairs of point contacts. Gutkowski (1990) used a similar approach in Sheth *et al.* (1990) to analyze one-degree-of-freedom spatial mechanisms with higher pairs. It can be noted that compared with the vector loop approach, the matrix method has the advantage of prescribing the geometry of two paired bodies in analytical form or discretized geometric model where link shapes are described by surface patches such as Bezier surfaces. Such mechanisms can be efficiently formulated in a form that is ready for computer-assisted analysis.

In analyzing planar mechanisms with higher pairs, Paul (1979) used a parametric equation to describe the cam link profile and derive an additional equation besides the loop closure equations. This additional equation, namely "the fundamental auxiliary equation," in fact is the angular displacement constraint for the vectors in the loop closure equation. In addition to cam pairs, for mechanisms with gear or rolling pairs, the loop closure equations are

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supplemented by auxiliary equations that express the no-slip condition at the contact point.

In this work, a different approach has been taken. We employ the matrix method for the analysis of mechanisms with higher pairs; especially, a mechanism with rolling pairs. It is shown that by suitably defining the coordinate systems on the links, joint parameters as well as parameters for rolling constraints can be identified clearly and systematically. Subsequently, the overall loop closure matrix and rolling constraints can be obtained in the position-level form. These simultaneous equations can then be solved via the analytical or numerical method. It can be noted that the fundamental auxiliary equation in Paul's method is the equation derived from the orientation submatrix in the overall loop-closure matrix. This paper is arranged as follows. We will first review the method for establishing the transformation matrix for kinematic pairs in point contact. Then, the constraint for two bodies in planar rolling contact will be established in position form rather than in velocity form. Finally, an example will be used to illustrate the procedure for the position analysis of mechanisms with rolling pairs.

II. REVIEW OF THE TRANSFORMATION MATRIX FOR A KINEMATIC PAIR IN POINT CONTACT

In Sheth *et al.* (1990), the transformation for two objects in point contact can be established as follows. As shown in Fig. 1, two links, $i-1$ and i, are in contact at a point. Let the contact point on link $i-1$ be A and on link i be B. A coordinate system $(xyz)_{i-1}$ is defined and attached to link $i-1$ while a different coordinate systems $(uvw)_i$ is defined and attached to link i. Meanwhile, two intermediate moving coordinate systems are defined at the position of the contact point, one denoted by $[\boldsymbol{t}_{i-1}, (\boldsymbol{n}_{i-1}\times\boldsymbol{t}_{i-1}), \boldsymbol{n}_{i-1}]$, is placed at point A and the other denoted by $[\boldsymbol{t}_i, (\boldsymbol{n}_i\times\boldsymbol{t}_i), \boldsymbol{n}_i]$, is placed at point B. Therefore, the transformation pair matrix regarding the point contact from system $(uvw)_i$ to system $(xyz)_{i-1}$ can be written as

$$^{i-1}\boldsymbol{P}_i={}^{i-1}\boldsymbol{T}_A{}^A\boldsymbol{T}_B{}^B\boldsymbol{T}_i \tag{1}$$

where $^{i-1}\boldsymbol{T}_A$ is the transformation matrix from the moving coordinate system at A to the system $(xyz)_{i-1}$; $^B\boldsymbol{T}_i$ the transformation matrix from the system $(uvw)_i$ to the moving coordinate system at B; and $^A\boldsymbol{T}_B$ the transformation matrix from the moving coordinate system at B to the moving coordinate system at A. The moving coordinate system can be set up according to the geometry of the contact surface via the differential geometry method in Thomas (1969). Details of derivation can be referred to Sheth *et al.* (1990) and Gutkowski (1990). As can be seen in the matrix method, the parameters involved are the link parameters and joint variables. Thus, these parameters will be further cross related when other criteria regarding the operating conditions of the mechanism are introduced; for example, if two links are in pure rolling and/or sliding contact. In what follows, we shall consider the condition for two links in pure rolling contact.

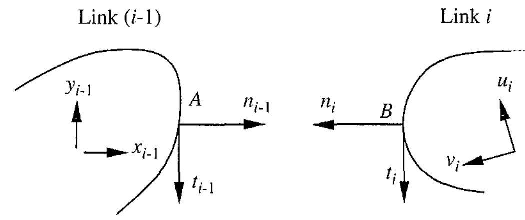

Fig. 1 Choice of coordinate systems for two links in rolling contact

III. TREATMENT OF THE KINEMATICS OF ROLLING PAIRS

It is well known that if two bodies are in rolling contact, the constraint imposed upon the two bodies is non-holonomic. Many studies of the kinematics of two bodies in rolling contact can be found in the field of robotic grasping and fine motion control (Agrawal and Pandravada, 1993; Bottema and Roth, 1979; Cai and Roth, 1986; Chen and Kumar, 1995; Cole *et al.*, 1988; Kerr and Roth, 1986; Li and Canny, 1990; Montana, 1988). Generally, the rolling constraints are expressed in velocity form and cannot be integrated. However, in the case of planar contact, the velocity level constraints can be integrated and yield position level constraints. The problem becomes holonomic. Cai and Roth (1986) studied the instantaneous kinematics for a general planar motion between two bodies in point contact. The contact velocities of two bodies are expressed in terms of surface curvatures of the two bodies. It can be noted that if the curvatures are not constants, then the equations cannot explicitly be integrated, either. Chen and Kumar (1995) performed the analysis of two planar robots with rolling contact. Agrawal and Pandravada (1993) studied rolling contact kinematics similarly using a line and a circle. They both limited the study to simple geometry as a line contacting a circle. In this study, complex geometry of the objects in rolling contact is considered. The parameter of the constraint for rolling contact is expressed in terms of the parameter used in the transformation matrix. As a result, a set of position equations consisting of the rolling constraints and the loop-closure equations are

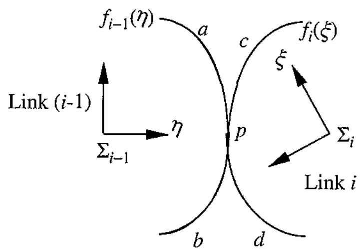

Fig. 2 Path traveled by the contact point on each link

obtained.

As shown in Fig. 2, two links, i–1 and i are in rolling contact at point P. Let Σ_{i-1} and Σ_i be the coordinate system fixed on the link i–1 and i, respectively. It can be noted that the real expression of velocity constraint for the two objects in rolling contact may vary in form depending on the parameter and coordinate system defined. Let the contact point be expressed in arc length coordinates and the magnitude of the contact speed along the surface of body i–1 and i be denoted as $\dot{D}_{i-1}$ and $\dot{D}_i$, respectively. Then, the velocity constraint for the rolling contact becomes:

$$\dot{D}_i = \dot{D}_{i-1} \tag{2}$$

Integrating each term in the above equation along the contour of each individual link, we can have the constraint in position form as

$$D_i = D_{i-1} \tag{3}$$

Eq. (3) is a general form for the displacement constraint of rolling contact. It implies that for two links in rolling contact, the length of the path of contact point travelling along each individual body will remain the same. Hence, we can alternatively rewrite the form as the line integral expressed in terms of the surface parameter of the link contour. This yields:

$$D_{i-1} = \int_a^b \sqrt{1 + [f_{i-1}'(\zeta)]^2}\, d\zeta \tag{4a}$$

and

$$D_i = \int_c^d \sqrt{1 + [f_i'(\zeta)]^2}\, d\zeta \tag{4b}$$

where f_* is the equation of the link contour (*); f' is the derivative of f with respect to the surface parameter; and (a, b), and (c, d) are respectively the set of initial and final positions of contact point on link i–1 and i. Eqs. (4.a and b) also represent the length of the path of the contact point traveling along each individual body. Results of the equations can be obtained analytically for a simple geometry condition or numerically for a complex one. It can be seen later, since link parameters have been defined for the transformation matrix usage, parameters for rolling constraint as expressed by Eq. (4) can be associated with the link parameters. Thus, combining Eq. (4) and the loop closure equations from the transformation matrix method, we can obtain sufficient displacement equations for the analysis of mechanism with rolling pairs.

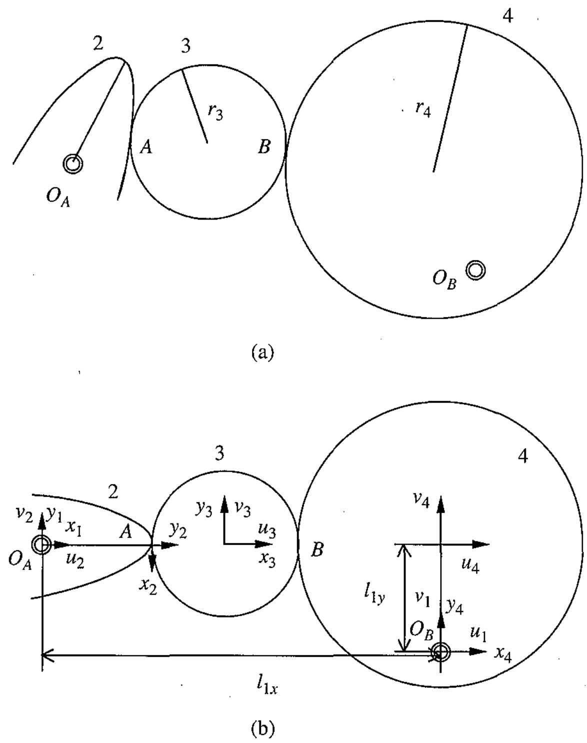

Fig. 3 (a) A four-link mechanism having parabolic and circular link shapes; (b) Coordinate systems and initial position defined on the mechanism

IV. KINEMATICS OF MECHANISM HAVING ROLLING PAIRS

In this section, we assume the shapes of the links in rolling contact are parabolic and circular curves. As shown in Fig. 3(a), a four-link mechanism has two rolling pairs at A and B. The profile of link 2 is parabolic while the shape of link 3 and 4 are both circular. The coordinate systems are defined as shown in Fig. 3(b). Denote l_{1x} as the horizontal distance between O_A and O_B; l_{1y} as the vertical distance between O_A and O_B; l_2 as the distance between two origins of $(xyz)_2$ and $(uvw)_2$; and l_4 as the distance between two origins of $(xyz)_4$ and $(uvw)_4$. For the system presented, the shape matrices are

$$S_1=\begin{bmatrix}1&0&0&-l_{1x}\\0&1&0&l_{1y}\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad S_2=\begin{bmatrix}0&1&0&-l_2\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$S_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\text{ and } S_4=\begin{bmatrix}1&0&0&0\\0&1&0&-l_4\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Assume that the surface equation of link 2 with respect to the coordinate system $(xyz)_2$ is $y=ax^2$. The pair matrices from $(uvw)_3$ to $(xyz)_2$ and from $(uvw)_4$ to $(xyz)_3$ can be established according to the method described in Sheth *et al.* (1990). The results are as follows.

$${}^2P_3=\begin{bmatrix}1/\sqrt{1+4a^2x^2}&0&-2ax/\sqrt{1+4a^2x^2}&x\\2ax/\sqrt{1+4a^2x^2}&1&1/\sqrt{1+4a^2x^2}&ax^2\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\cdot\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-\sin\theta_{23}&\cos\theta_{23}&0&0\\0&0&1&0\\\cos\theta_{23}&\sin\theta_{23}&0&-r_3\\0&0&0&1\end{bmatrix} \tag{5a}$$

and

$${}^3P_4=\begin{bmatrix}-\sin\theta_{34}&0&\cos\theta_{34}&r_3\cos\theta_{34}\\\cos\theta_{34}&0&\sin\theta_{34}&r_3\sin\theta_{34}\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$$\cdot\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-\sin\theta_4&\cos\theta_4&0&0\\0&0&1&0\\\cos\theta_4&\sin\theta_4&0&-r_4\\0&0&0&1\end{bmatrix} \tag{5b}$$

where θ_{23} is the angle measured from axis $\boldsymbol{u}_3$ to the contact point A, θ_{34} the angle measured from axis $\boldsymbol{x}_3$ to the contact point B, θ_4 the angle measured from axis $\boldsymbol{u}_4$ to the contact point B, r_3 the radius of link 3 and r_4 the radius of link 4. After the shape matrices

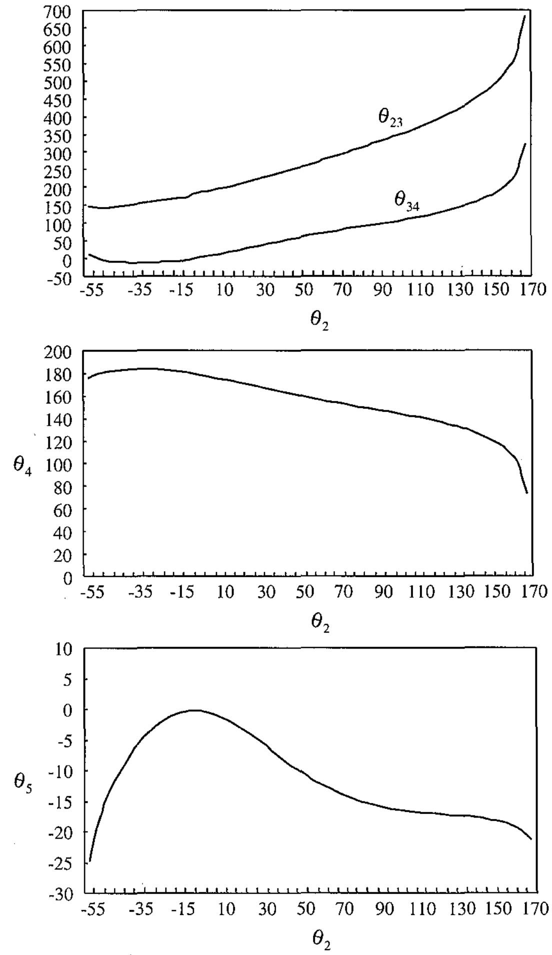

Fig. 4 Plot of joint variables versus input θ_2

and pair matrices are established, the loop closure equation can be written as

$$[S_1\,{}^1P_2(\theta_2)][S_2\,{}^2P_3(x,\theta_{23})][S_3\,{}^3P_4(\theta_{34},\theta_4)]$$

$$\cdot[S_4\,{}^4P_1(\theta_5)]=\boldsymbol{I} \tag{6}$$

where θ_5 is the joint variable at O_B. On the other hand, by using Eq. (4), the rolling constraints for the pairs between link (2, 3), and (3, 4) are,

$$\int_{x_i}^{x}\sqrt{1+(2at)^2}\,dt-r_3(\theta_{23}-\theta_{23,i})=0 \tag{7a}$$

$$r_3(\theta_{34}-\theta_{34,i})-r_4(\theta_4-\theta_{4,i})=0 \tag{7b}$$

where the subscript in $(\#)_{,i}$ represents the initial value of (#). Thus, combining the three equations from the loop closure matrix and two rolling constraints, it is possible to solve for (x, θ_{23}, θ_{34}, θ_4, θ_5). A set of numerical data are given as follow: l_{1x}=7, l_{1y}=2, l_2= l_4=2, r_3=1, r_4=4 and a=-0.5; initial values x_i=0, $\theta_{23,i}=\pi$, $\theta_{34,i}$=0, and $\theta_{4,i}=\pi$. The numerical analysis is performed via the Newton-Cotes method. Results of the analysis as variables (θ_{23}, θ_{34}, θ_4, θ_5) versus input θ_2 are shown in Fig. 4.

V. CONCLUSION

This paper presents a systematic method for the kinematic analysis of planar mechanisms with rolling pairs. It is shown that by using the matrix method, the link parameters and joint variables can be clearly identified and modeled. The rolling constraints can also be obtained easily using the links parameters. Thus, combining the loop closure equations with the rolling constraints, the set of simultaneous equations required for the kinematic analysis is readily achieved. These displacement equations can be solved via the numerical method. It is hoped that this work can be helpful in the analysis of mechanisms with rolling pairs.

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NOMENCLATURE

D_i length of the path of the contact point traveling along the surface of two bodies in rolling contact

$\dot{D}_{i-1}$ magnitude of the contact speed along the surface of two bodies in rolling contact

${}^{i-1}\boldsymbol{P}_i$ transformation pair matrix regarding the point contact from system i to system $i-1$

${}^{A}\boldsymbol{T}_B$ transformation matrix from the coordinate system B to the coordinate system A

θ joint angle

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