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Disturbance response decoupling and achievable performance with application to vehicle active suspension

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This paper derives a structural condition on the controller for a given (stable) plant which guarantees that some pre-specified closed-loop transfer function is the same as in the open loop. We also present conditions to test whether the achievable dynamic response of other transmission paths remains effectively the same if the controller is so restricted. The results are applied to simple quarter- and half-car vehicle models, and illustrated numerically for a double-wishbone half-car model.

1. Introduction

This paper continues the work of Smith and Wang (2002) to derive a new condition on the controller structure for a given (stable) plant which guarantees that some pre-specified closed-loop transfer function is fixed. The controller consists of two blocks in series, the first of which is a left annihilator of a certain open-loop plant transfer function and the second is arbitrary. As such, the condition takes a simpler form than that of Smith and Wang (2002), by assuming that the plant is open-loop stable and the number of outputs to be left invariant is no smaller than the number of actuators and that this transmission path has full normalrank.

The motivation for this problem comes from vehicle active suspension control (Thompson 1971, Pitcher *et al.* 1977, Hrovat and Hubbard 1981, Wright and Williams 1984, Sharp and Hassan 1986, Smith 1995) where there is a need to insulate the vehicle body from both road irregularities and load disturbances (e.g. inertial loads induced by braking and cornering). It is frequently the case that the hardware structure ensures that some disturbance paths are satisfactory in the open-loop, e.g. the response to road disturbances (Pitcher *et al.* 1977, Williams *et al.* 1993, Williams and Best 1994). In Smith and Wang (2002) and Wang and Smith (2001) the authors introduced the idea of parametrizing all stabilizing controllers to leave some transmission path invariant and derived some general conditions to achieve this.

This paper also considers a new issue related to our disturbance response decoupling problem, namely whether the *achievable performance* in other transmission paths is reduced when the controller is restricted in form. We will derive conditions to determine whether the achievable performance remains effectively the same.

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The paper is outlined as follows. Section 2 derives the main results of the paper. Theorem 1 sets up the controller structure as a left annihilator for a given plant. Theorem 2 derives the conditions under which the achievable performance of other transmission paths remains effectively unchanged. Section 3 applies the results of § 2 to the standard quarter- and half-car models employing a 'Sharp' actuator with various choices of measured variables. Section 4 presents a numerical example which derives the required controller structure for a half-car double-wishbone model. The multi-body simulation package *AutoSim* was employed to derive the linearized model and to perform the non-linear simulations.

2. Controller structure for disturbance response decoupling

We consider the LFT (linear fractional transformation) model in figure 1, where the Laplace transfer function of the generalised plant P is partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and further partitioned conformably with the disturbance signals as

$$\begin{bmatrix} \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} P_{11,11} & P_{11,12} \\ P_{11,21} & P_{11,22} \end{bmatrix} & \begin{bmatrix} P_{12,1} \\ P_{12,2} \end{bmatrix} \\ \begin{bmatrix} P_{21,1} & P_{21,2} \end{bmatrix} & P_{22} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} \\ \hat{u} \end{bmatrix} \quad (1)$$

where $w_1 \in \mathbb{R}^{m_1}$, $w_2 \in \mathbb{R}^{m_2}$, $u \in \mathbb{R}^{m_3}$, $z_1 \in \mathbb{R}^{p_1}$, $z_2 \in \mathbb{R}^{p_2}$, $y \in \mathbb{R}^{p_3}$ at any time instant and $\hat{u}$ denotes the Laplace transform of $u(t)$, etc.

We consider the problem of parametrizing all stabilizing controllers which leave $T_{\hat{w}_1 \to \hat{z}_1}$ (the transfer function from $\hat{w}_1$ to $\hat{z}_1$) the same as in the open-loop. To make sense, this problem requires that P is stable, which will indeed be satisfied in our application. In this

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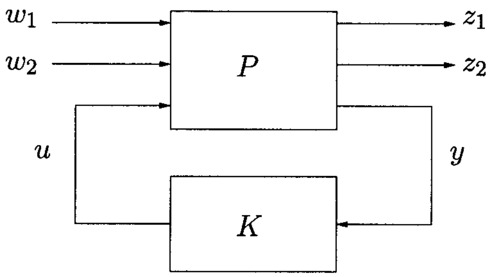

Figure 1. Generalized model in LFT form.

case, the set of *all* stabilizing controllers can be parametrized by

$$K = -(I - QP_{22})^{-1}Q \tag{2}$$

for $Q \in \mathbb{RH}_\infty^{m_3 \times p_3}$. The closed loop transfer function in figure 1 can be expressed as

$$T_{\binom{\hat{w}_1}{\hat{w}_2} \to \binom{\hat{z}_1}{\hat{z}_2}} = P_{11} - P_{12}QP_{21} \tag{3}$$

Thus the problem reduces to parametrizing the subset of all stabilizing controllers which leave $T_{\hat{w}_1 \to \hat{z}_1} = (P)_{11,11}$. To facilitate the derivation, we denote the normalrank of $P_{12,1}$ and $P_{21,1}$ as r_2 and r_3 respectively. In Smith and Wang (2002), the authors developed a general theorem for the above problem. Now we will explore the problem from a different angle in the following theorem.

Theorem 1: *Let P be (open-loop) stable. Consider the controller structure of figure 2 with P given by* (1). *Then*

(i) $T_{\hat{w}_1 \to \hat{z}_1} = P_{11,11}$ *if* $\tilde{U}_2$ *is a left annihilator of* $P_{21,1}$.

(ii) *If* $P_{12,1}$ *has full column normalrank and* $T_{\hat{w}_1 \to \hat{z}_1} = P_{11,11}$, *then* K_1U_2 *is a left annihilator of* $P_{21,1}$.

(iii) *Assume* $m_3 = r_2$ *and let* $\tilde{U}_2 \in \mathbb{RH}_\infty^{(p_3 - r_3) \times p_3}$ *be a part of a unimodular matrix and be a left annihilator of* $P_{21,1}$. *Then* $K = K_1U_2$, *where* $K_1 = -(I - Q_1\tilde{U}_2P_{22})^{-1}Q_1$ *and* $Q_1 \in \mathbb{RH}_\infty^{m_3 \times (p_3 - r_3)}$, *parametrizes all stabilizing controllers of P for which* $T_{\hat{w}_1 \to \hat{z}_1} = P_{11,11}$.

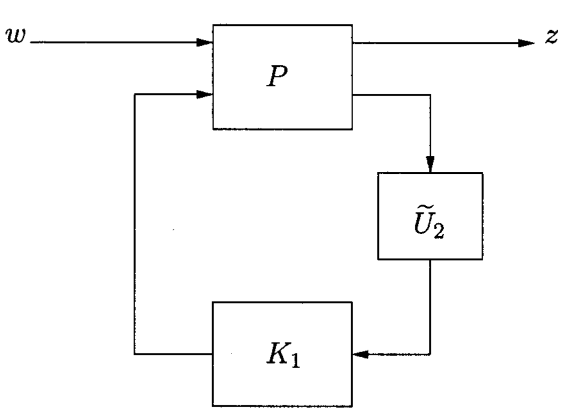

Figure 2. Controller structure as a left annihilator.

Proof:

(i) The closed-loop response of the system is $T_{\hat{w} \to \hat{z}} = P_{11} + P_{12}(I - K_1\tilde{U}_2P_{22})^{-1}K_1\tilde{U}_2P_{21}$. If $\tilde{U}_2$ is a left annihilator of $P_{21,1}$, i.e. $\tilde{U}_2P_{21,1} = 0$, then

$$\begin{aligned} T_{\hat{w}_1 \to \hat{z}_1} &= P_{11,11} + P_{12,1}(I - K_1\tilde{U}_2P_{22})^{-1}K_1\tilde{U}_2P_{21,1} \\ &= P_{11,11} \end{aligned} \tag{4}$$

which is invariant of the feedback control.

(ii) If $P_{12,1}$ has full column normalrank and (4) holds for some controller K_1, then $K_1U_2P_{21,1} = 0$. That is K_1U_2 is a left annihilator of $P_{21,1}$.

(iii) Suppose $\tilde{U}_1 \in \mathbb{RH}_\infty^{r_3 \times p_3}$ is a completion of $\tilde{U}_2$ such that $\tilde{U} = (\tilde{U}_1', \tilde{U}_2')'$ is unimodular, and $\tilde{U}^{-1} = U = (U_1, U_2)$. Since $\tilde{U}_2$ is a left annihilator of $P_{21,1}$, we can write

$$\tilde{U}P_{21,1} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix} P_{21,1} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

for some $F \in \mathbb{RH}_\infty^{r_3 \times m_1}$, which gives

$$P_{21,1} = [U_1, \quad U_2] \begin{bmatrix} F \\ 0 \end{bmatrix} = U_1F$$

where $U_1 \in \mathbb{RH}_\infty^{p_3 \times r_3}$. Note that F has normalrank r_3 because $P_{21,1}$ has normalrank r_3 and the multiplication of a unimodular matrix to obtain F does not change the normalrank. Hence, F has full row normalrank.

A stabilizing controller in the form (2) leaves $T_{\hat{w}_1 \to \hat{z}_1} = P_{11,11}$ *if and only if* $(P_{12}QP_{21})_{1,1} = P_{12,1}QU_1F = 0$. This is equivalent to $QU_1 = 0$ since $P_{12,1}$ (resp. F) has full column (resp. row) normalrank. We now show that this requires $Q = Q_1U_2$ for some Q_1. Clearly

$$Q(U_1, \ U_2) = (0_{r_2 \times r_3}, \ Q_1) \tag{5}$$

for some $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$. This gives

$$Q = (0, \ Q_1)U^{-1} = Q_1\tilde{U}_2 \tag{6}$$

Conversely, if $Q = Q_1\tilde{U}_2$ holds for some $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$, then we have $Q = (0, \ Q_1)U^{-1}$ which again implies $QU_1 = 0$. Hence $T_{\hat{w}_1 \to \hat{z}_1} = P_{11,11}$ □

Remark 1: The idea of Theorem 1(iii) is to 'shrink' the parametrization set of all stabilizing controllers from $Q \in \mathbb{RH}_\infty^{m_3 \times p_3}$ to Q in the form of (6). The form of the result in Theorem 1 is more explicit than Smith and Wang (2002), though less general, and emphasizes the left annihilator property. It is implicit in the proof of Theorem 1(iii), and explicit in Smith and Wang (2002), that a left annihilator of $P_{21,1}$ can be found by constructing a 'left normalrank factorization' (Smith and Wang 2002) $P_{21,1} = U_1 F$ where F has full row normalrank and U_1 is part of a unimodular matrix over $\mathbb{RH}_\infty$ (see Vidyasagar 1985). It is shown constructively in Smith and Wang (2002) that any matrix with elements in $\mathbb{RH}_\infty$ has a left normalrank factorization.

A question which was not addressed in Smith and Wang (2002) is whether the parametrization of Theorem 1 introduces performance limitations on other transmission paths. We will now give a result which allows us to test if the achievable performance is significantly reduced. To do this, we will work directly with the Q-parameters in the Youla parametrization. A matrix is said to be proper (resp. strictly proper) if all its elements are proper (resp. strictly proper).

Theorem 2: *Given two sets of proper, stable transfer matrices, $A = \{G_1 Q H_1: Q$ is a matrix with elements in $\mathbb{RH}_\infty\}$ and $B = \{G_2 Q H_2: Q$ is a matrix with elements in $\mathbb{RH}_\infty\}$, where $G_2 \underline{J} = G_1$ for some $\underline{J}$ and $\underline{K} H_2 = H_1$ for some $\underline{K}$:*

(i) *Suppose $G_1 Q H_1$ is strictly proper in $\mathbb{RH}_\infty$ for any Q which has elements in $\mathbb{RH}_\infty$, $\underline{J}Q\underline{K}(1/s\tau + 1)^r$ is a matrix with elements in $\mathbb{RH}_\infty$ for some $r > 0$ and any $\tau > 0$, then for any $R_1 \in A$ we can find a $R_2 \in B$ such that $\|R_1 - R_2\|_\infty < \epsilon$ for any $\epsilon > 0$.*

(ii) *If $\underline{J}Q\underline{K}$ is a proper matrix for any Q which has elements in $\mathbb{RH}_\infty$, then for any $R_1 \in A$ we can find a $R_2 \in B$ such that $R_1 - R_2 = 0$.*

Proof:

(i) For any $R_1 \in A$, we take the following candidate for R_2

$$R_2 = R_1 \left(\frac{1}{s\tau + 1} \right)^r$$

for some $\tau > 0$. Note that $R_2 \in B$ since by definition $R_1 = G_1 Q H_1$ for some Q which has elements $\mathbb{RH}_\infty$ and so

$$R_2 = G_2(\underline{J}Q\underline{K}(1/(s\tau + 1))^r) H_2$$

where

$$(\underline{J}Q\underline{K}(1/(s\tau + 1))^r) H_2$$

has elements in $\mathbb{RH}_\infty$. Hence

$$\|R_1 - R_2\|_\infty = \left\| R_1 \left(1 - \left(\frac{1}{s\tau + 1} \right)^r \right) \right\|_\infty \qquad (7)$$

We now wish to show that the right-hand side of (7) can be made arbitrary small by taking τ sufficiently small. Firstly, since R_1 is strictly proper, given any $\epsilon > 0$ we can find an $\omega_0(\epsilon)$ such that $\bar{\sigma}(R_1(j\omega)) < \epsilon/2$ for $\omega > \omega_0$, and

$$\bar{\sigma}_{\omega > \omega_0} \left(R_1(j\omega) \left(1 - \left(\frac{1}{j\omega\tau + 1} \right)^r \right) \right) \leq 2\bar{\sigma}_{\omega > \omega_0}(R_1(j\omega)) < \epsilon$$

Secondly, for $\omega \in [0, \omega_0]$, $(1 - 1/(j\omega\tau + 1))^r$ can be made as small as we like by choosing a sufficiently small τ. Hence we can choose τ sufficiently small so that $\bar{\sigma}_{\omega \in [0,\omega_0]}(R_1(1 - (1/j\omega\tau + 1)^r)) < \epsilon$ for any $\epsilon > 0$. Combining these together, we can ensure that $\bar{\sigma}(R_1(1 - 1/(j\omega\tau + 1)^r)) < \epsilon$ for all $\omega \geq 0$, and therefore, $\|(R_1 - R_2)(j\omega)\|_\infty < \epsilon$ by choosing τ sufficiently small.

(ii) If $\underline{J}Q\underline{K}$ is a proper matrix for any Q which has elements in $\mathbb{RH}_\infty$, then for any $R_1 \in A$ we can set $R_2 = R_1$ since $R_1 = G_1 Q H_1 = G_2(\underline{J}Q\underline{K}) H_2 \in B$ because $\underline{J}Q\underline{K}$ has elements in $\mathbb{RH}_\infty$. □

Corollary 1: *Given two sets of scalar, strictly proper transfer functions, $A = \{G_1 Q: Q \in \mathbb{RH}_\infty\}$ and $B = \{G_2 Q: Q \in \mathbb{RH}_\infty\}$, where G_1 is strictly proper in $\mathbb{RH}_\infty$ and $G_2 = G_1 \times (n_1/m_1)$ in which m_1, n_1 are Hurwitz polynomials with $\deg(n_1) < \deg(m_1)$, for any $R_1 \in A$ we can find a $R_2 \in B$ such that $\|R_1 - R_2\|_\infty < \epsilon$ for any $\epsilon > 0$. If $\deg(n_1) = \deg(m_1)$ then we can choose $R_2 = R_1$.*

3. Application to vehicle models

In this section, we will apply the disturbance response decoupling theorems to quarter- and half-car models.

3.1. *The quarter-car model*

We begin with the quarter-car model of figure 3 where the sprung and unsprung masses are m_s and m_u and the tyre is modelled as a linear spring with constant k_t. The suspension consists of a passive damper of constant c_s in parallel with a series combination of an actuator A and a spring of constant k_s (sometimes referred to as a 'Sharp' actuator (Wright and Williams 1989)). Following Sharp and Hassan (1987) the actuator is modelled so that the relative displacement across the actuator will be a low-pass filtered version of the actuator's command signal, i.e.

$$\hat{z}_s - \hat{z}_a = \gamma(s)\hat{u} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\, \hat{u} \qquad (8)$$

The external disturbances are taken to be a load F_s and a road displacement z_r. The dynamic equations of the

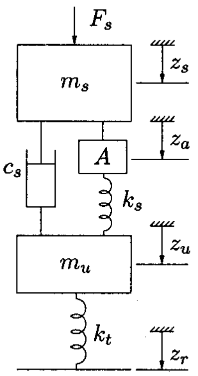

Figure 3. The quarter-car model.

model are given by

$$m_s \ddot{z}_s = F_s - u_p \tag{9}$$

$$m_u \ddot{z}_u = u_p + F_r \tag{10}$$

where

$$u_p = c_s(\dot{z}_s - \dot{z}_u) + k_s(z_a - z_u) \tag{11}$$

$$F_r = k_t(z_r - z_u) \tag{12}$$

We assume that k_s and c_s are chosen to give satisfactory responses for the transmission path from z_r to z_s and z_u. We therefore seek to parametrize all controllers which leave this transmission path the same as in the open-loop.

3.1.1. *Two measurement case*. We assume that the measurements $\ddot{z}_s$ and $z_s - z_u$ are available. We now write the system in the form of figure 1 with $z_1 = [z_s,\ z_u]'$, $w_1 = z_r$, $w_2 = F_s$, $y = [\ddot{z}_s,\ z_s - z_u]'$, u equals the actuator command signal as in (8) and z_2 omitted. The corresponding dimensions are $m_1 = m_2 = m_3 = 1$, $p_1 = 2$, $p_2 = 0$, $p_3 = 2$. We find that

$$P_{21} = \frac{1}{d(s)} \begin{bmatrix} k_t(c_s s + k_s)s^2 & (m_u s^2 + c_s s + k_s + k_t)s^2 \\ -k_t(m_s s^2) & m_u s^2 + k_t \end{bmatrix}$$

where

$$d(s) = (m_s m_u)s^4 + c_s(m_s + m_u)s^3 + (m_s(k_s + k_t) + m_u k_s)s^2 + c_s k_t s + k_s k_t \tag{13}$$

As expected, all roots of $d(s)$ are in the left-half plane, which can be confirmed by the Routh–Hurwitz Criterion. It can be checked that $r_2 = r_3 = 1$. Following Remark 1 we can construct a left annihilator $\widetilde{U}_2 \in \mathbb{RH}_\infty^{(p_3 - r_3)\times p_3}$ of $P_{21,1}$

$$\widetilde{U}_2 = \begin{bmatrix} \dfrac{m_s}{c_s s + k_s} & 1 \end{bmatrix} \tag{14}$$

which satisfies all the conditions of Theorem 1(iii).

We now discuss the achievable performance after disturbance response decoupling. The set of all stabilizing controllers is parametrized by (2) with $Q = [Q_\alpha,\ Q_\beta]$, and we can check that the load responses are given by

$$\begin{bmatrix} \hat{z}_s \\ \hat{z}_u \end{bmatrix} = \left(P_{11,12} - L_1(s) \frac{1}{d(s)} \begin{bmatrix} k_s(m_u s^2 + k_t) \\ -k_s m_s s^2 \end{bmatrix} \right) \hat{F}_s \tag{15}$$

where

$$L_1(s) = \gamma(s)((m_u s^2 + c_s s + k_s + k_t)s^2 Q_\alpha + (m_u s^2 + k_t)Q_\beta)/d(s)$$

and $d(s)$ is given in (13). After disturbance response decoupling with the controller structure U_2 given in (14), we can check that the load responses are given by

$$\begin{bmatrix} \hat{z}_s \\ \hat{z}_u \end{bmatrix} = \left(P_{11,12} - L_2(s) \frac{1}{d(s)} \begin{bmatrix} k_s(m_u s^2 + k_t) \\ -k_s m_s s^2 \end{bmatrix} \right) \hat{F}_s \tag{16}$$

where $L_2(s) = \gamma(s)Q_1/(c_s s + k_s)$. To compare the achievable performance, set

$$Q_1 = ((m_u s^2 + c_s s + k_s + k_t)s^2 Q_\alpha + (m_u s^2 + k_t)Q_\beta)/d(s)$$

and note that $L_2 = L_1/(c_s s + k_s)$. Since $L_1(s)$ is strictly proper, it follows from Corollary 1 that $L_2(s)$ can approximate $L_1(s)$ to an error of ϵ in the H_∞ norm. Hence the freedom in the achievable load responses after disturbance response decoupling is effectively unchanged.

3.1.2. *Three measurement case*. We continue to illustrate our theory by considering the quarter-car model with the additional measurement $\ddot{z}_u$. We now write the system in the form of figure 1 with $y = [\ddot{z}_s,\ z_s - z_u,\ \ddot{z}_u]'$ and all other variables the same as in the two-measurement case. We find that

$$P_{21} = \frac{1}{d(s)} \begin{bmatrix} k_t(c_s s + k_s)s^2 & (m_u s^2 + c_s s + k_s + k_t)s^2 \\ -k_t(m_s s^2) & (m_u s^2 + k_t) \\ k_t(m_s s^2 + c_s s + k_s)s^2 & (c_s s + k_s)s^2 \end{bmatrix}$$

where $d(s)$ is given by (13). It can be checked that $p_3 = 3$, $r_2 = r_3 = 1$. Following Remark 1 we can construct a left annihilator $\widetilde{U}_2 \in \mathbb{RH}_\infty^{(p_3 - r_3)\times p_3}$ of $P_{21,1}$

$$\widetilde{U}_2 = \begin{bmatrix} 1 & 0 & \dfrac{-(c_s s + k_s)}{m_s s^2 + c_s s + k_s} \\ 0 & 1 & \dfrac{m_s}{m_s s^2 + c_s s + k_s} \end{bmatrix} \tag{17}$$

which satisfies all the conditions of Theorem 1(iii).

To discuss the achievable performance, the set of all stabilizing controllers is parametrized by (2) with

$Q = [Q_\gamma,\ Q_\delta,\ Q_\zeta]$, and the load responses are then given by

$$\begin{bmatrix} \hat{z}_s \\ \hat{z}_u \end{bmatrix} = \left(P_{11,12} - L_3(s) \frac{1}{d(s)} \begin{bmatrix} k_s(m_u s^2 + k_t) \\ -k_s m_s s^2 \end{bmatrix} \right) \hat{F}_s \quad (18)$$

where

$$L_3(s) = \gamma(s)((m_u s^2 + c_s s + k_s + k_t) s^2 Q_\gamma + (m_u s^2 + k_t) Q_\delta + (c_s s + k_s) s^2 Q_\zeta)/d(s).$$

We observe (as in Smith 1995, Theorem 7) that any achievable performance with the three measurements can also be achieved by two measurement $[\ddot{z}_s,\ z_s - z_u]'$ in that for any L_3 we can achieve a $L_1 = L_3$ by setting $Q_\alpha = Q_\gamma + Q_\zeta(c_s s + k_s)/(m_u s^2 + c_s s + k_s + k_t)$ and $Q_\beta = Q_\delta$ in $L_1(s)$. After disturbance response decoupling with the controller structure U_2 given in (17), and with the parameter $Q_1 = [Q_{1,1},\ Q_{1,2}]$, the load responses are given by

$$\begin{bmatrix} \hat{z}_s \\ \hat{z}_u \end{bmatrix} = \left(P_{11,12} - L_4(s) \frac{1}{d(s)} \begin{bmatrix} k_s(m_u s^2 + k_t) \\ -k_s m_s s^2 \end{bmatrix} \right) \hat{F}_s \quad (19)$$

where

$$L_4(s) = \gamma(s)(s^2 Q_{1,1} + Q_{1,2})/(m_s s^2 + c_s s + k_s)$$

Therefore, for any L_3 we can choose $Q_{1,1}$, $Q_{1,2}$ as follows

$$Q_{1,1} = \frac{(m_u s^2 + c_s s + k_s + k_t)(m_s s^2 + c_s s + k_s)}{d(s)} \times \left(Q_\gamma + \frac{c_s s + k_s}{m_u s^2 + c_s s + k_s + k_t} Q_\zeta \right)$$

$$Q_{1,2} = \frac{(m_u s^2 + k_t)(m_s s^2 + c_s s + k_s)}{d(s)} Q_\delta$$

so that $L_4 = L_3$. Hence the achievable load responses are the same after the disturbance response decoupling step.

We can also check this same fact on the achievable load performance by directly using the matrix formulation in Theorem 2. The original achievable load responses are $P_{11,12} - P_{12} Q P_{21,2}$ with $Q \in \mathbb{RH}_\infty^{1\times 3}$, and after disturbance response decoupling the achievable load responses are $P_{11,12} - P_{12} Q_1 U_2 P_{21,2}$ with $Q_1 \in \mathbb{RH}_\infty^{1\times 2}$. We can check that P_{12} is strictly proper and $P_{21,2}$ is proper. Hence we can set $A = \{P_{12} Q P_{21,2}\}$ and $B = \{P_{12} Q_1 U_2 P_{21,2}\}$ in Theorem 2 and assign $G_1 = G_2 = P_{12}, H_1 = P_{21,2}$ and $H_2 = U_2 P_{21,2}$, so that $\underset{\sim}{J} = I$ and $\underset{\sim}{K}$ is found as

$$\underset{\sim}{K} = \frac{m_s s^2 + c_s s + k_s}{d(s)} \begin{bmatrix} m_u s^2 + c_s s + k_s + k_t & 0 \\ 0 & m_u s^2 + k_t \\ c_s s + k_s & 0 \end{bmatrix}$$

Accordingly, the achievable load responses remain the same after disturbance response decoupling (Theorem 2(ii)).

3.2. *The half-car model*

In this section, we shall apply the controller parametrization method to the half-car model shown in figure 4. As in the quarter-car model, the actuators A_1 and A_2 are modelled so that the relative displacement across each is equal to a low-pass filtered version of the actuator's command signal, i.e.

$$\hat{z}_s + l_1 \hat{z}_\psi - \hat{z}_{a_1} = \gamma(s) \hat{u}_1 \quad (20)$$

$$\hat{z}_s - l_2 \hat{z}_\psi - \hat{z}_{a_2} = \gamma(s) \hat{u}_2 \quad (21)$$

where $\gamma(s)$ is defined as in (8). The linearized dynamic equations can be expressed as

$$m_s \ddot{z}_s = F_s - u_{p_1} - u_{p_2} \quad (22)$$

$$I_\psi \ddot{z}_\psi = F_\psi - u_{p_1} l_1 + u_{p_2} l_2 \quad (23)$$

$$m_1 \ddot{z}_{u_1} = u_{p_1} + F_{r_1} \quad (24)$$

$$m_2 \ddot{z}_{u_2} = u_{p_2} + F_{r_2} \quad (25)$$

where the passive suspension forces u_{p_1}, u_{p_2}, and the tyre forces F_{r_1}, F_{r_2} are given by

$$u_{p_1} = c_1(\dot{z}_s + l_1 \dot{z}_\psi - \dot{z}_{u_1}) + k_1(z_{a_1} - z_{u_1})$$

$$u_{p_2} = c_2(\dot{z}_s - l_2 \dot{z}_\psi - \dot{z}_{u_2}) + k_2(z_{a_2} - z_{u_2})$$

$$F_{r_1} = k_{t_1}(z_{r_1} - z_{u_1})$$

$$F_{r_2} = k_{t_2}(z_{r_2} - z_{u_2})$$

We now write the system in the form of figure 1 with $z_1 = [z_s,\ z_\psi,\ z_{u_1},\ z_{u_2}]'$, $w_1 = [z_{r_1},\ z_{r_2}]'$, $w_2 = [F_s,\ F_\psi]'$,

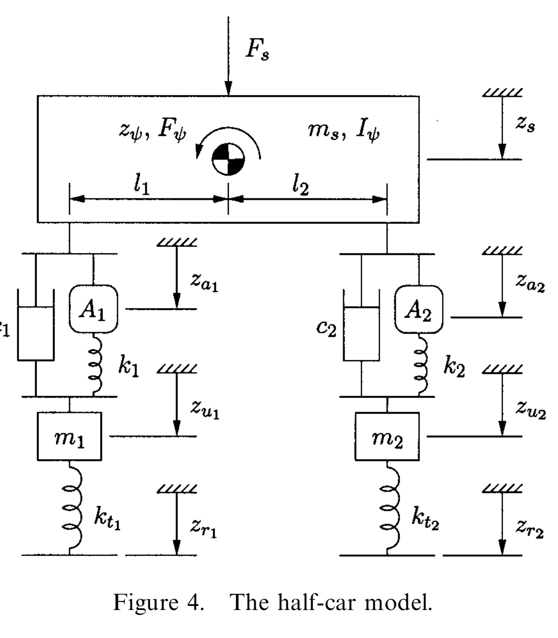

Figure 4. The half-car model.

$y = [\ddot{z}_s, \ddot{z}_\psi, D_1, D_2]'$ where $D_1 = z_s + l_1 z_\psi - z_{u1}$, $D_2 = z_s - l_2 z_\psi - z_{u2}$ are strut deflections, $u = [u_1, u_2]'$ as in (20), (21) and z_2 omitted. It can be checked that $p_3 = 4$, $r_2 = r_3 = 2$. The matrix $U_2 \in \mathbb{RH}_\infty^{(p_3 - r_3)\times p_3}$ given by

$$\tilde{U}_2 = \begin{bmatrix} \frac{m_s l_2}{(c_1 s + k_1)(l_1 + l_2)} & \frac{I_\psi}{(c_1 s + k_1)(l_1 + l_2)} & 1 & 0 \\ \frac{m_s l_1}{(c_2 s + k_2)(l_1 + l_2)} & \frac{-I_\psi}{(c_2 s + k_2)(l_1 + l_2)} & 0 & 1 \end{bmatrix} \quad (26)$$

satisfies $\tilde{U}_2 P_{21,1} = 0$. Thus the conditions of Theorem 1(iii) hold.

Now we discuss the achievable load responses of the half-car model after disturbance response decoupling. All stabilizing controllers a reparametrized by (2) and the load responses are $P_{11,12} - P_{12} Q P_{21,2}$ with $Q \in \mathbb{RH}_\infty^{2\times 4}$. After the disturbance response decoupling procedure, the load responses become $P_{11,12} - P_{12} Q_1 U_2 P_{21,2}$, where U_2 is given by (26) and $Q_1 \in \mathbb{RH}_\infty^{2\times 2}$. Therefore, we consider two sets of strictly proper matrices: $\mathcal{A} = \{P_{12} Q P_{21,2}\}$ and $\mathcal{B} = \{P_{12} Q_1 \tilde{U}_2 P_{21,2}\}$ in Theorem 2, which gives $G_1 = G_2 = P_{12}$, $H_1 = P_{21,2}$ and $H_2 = U_2 P_{21,2}$. We find $\underline{L} = I$ and $\underline{K}$ as

$$\underline{K} = H_1 H_2^{-1} = \frac{1}{p_8(s)} \begin{bmatrix} o(s^9) & o(s^9) \\ o(s^9) & o(s^9) \\ o(s^7) & o(s^7) \\ o(s^7) & o(s^7) \end{bmatrix} \quad (27)$$

where $p_8(s)$ is an eighth order Hurwitz polynomial and $o(s^i)$ indicates an ith order polynomial. Thus, from Theorem 2, the achievable load responses after disturbance response decoupling can approximate the original load responses to an error of ϵ in the H_∞ norm.

4. Numerical example

4.1. *Calculation of the controller structure*

A key step in the disturbance response decoupling designs is the computation of the matrix U_2 which determines the required controller structure. Throughout the application of vehicle active suspension design in Smith and Wang (2002), it was always possible to calculate U_2 symbolically using *Maple*. However, for more complicated vehicle models this may not be feasible. A basis for a numerical approach, which was found to be tractable in a variety of examples (which all satisfied $r_3 = m_1$), used the following sequence of steps

(1) find a factorization $P_{21,1} = U_1 F$ where U_1, F are stable, F is square, and $U_1 = [M', N']'$ with $M(\infty)$ being square and non-singular;
(2) find a minimal realization of $L := NM^{-1}$;
(3) find a left coprime factorization $L = \tilde{M}^{-1}\tilde{N}$ and set $U_2 = [-\tilde{N}, \tilde{M}]$ (e.g., see Zhou *et al.* 1996, Theorem 12.19)).

In our examples it was always possible to carry out (1) in an *ad hoc* manner by considering the relative degrees of the elements of $P_{21,1}$. Indeed, in the example below it was possible to choose $F = I$. We note the connection with Theorem 1 that U_2 constructed in (3) is a left annihilator of $P_{21,1}$ and is a part of a unimodular matrix.

4.2. *Double-wishbone model*

We now consider the half-car double-wishbone model, as shown in figure 5. The left (right) suspension strut consists of a passive damper of constant c_1 (c_2) in parallel with a series combination of actuators A_1 (A_2) and a spring of constant k_1 (k_2) (this is a 'Sharp' actuator as introduced in § 3). The actuators are modelled in such a way that the relative displacement across each is equal to a low-pass filtered version of the actuator's

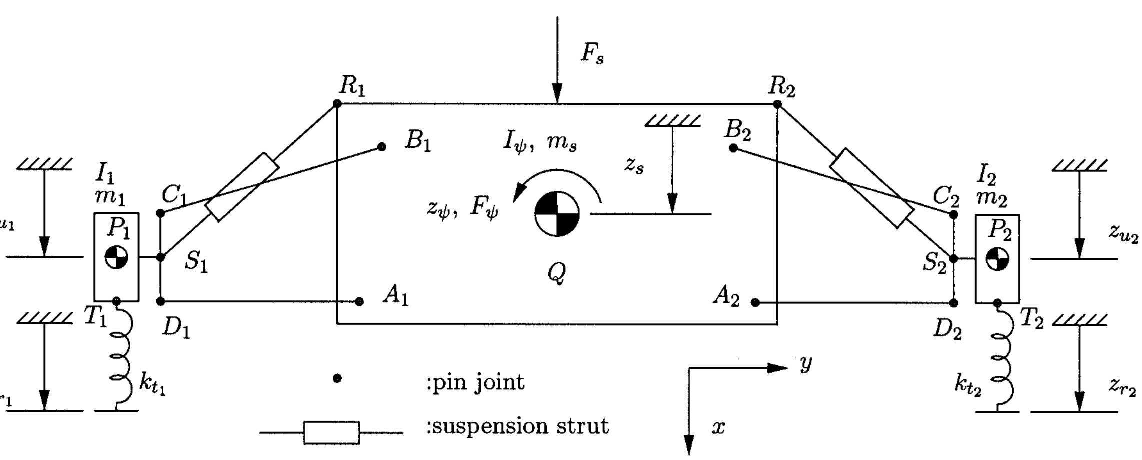

Figure 5. The half-car double-wishbone model.

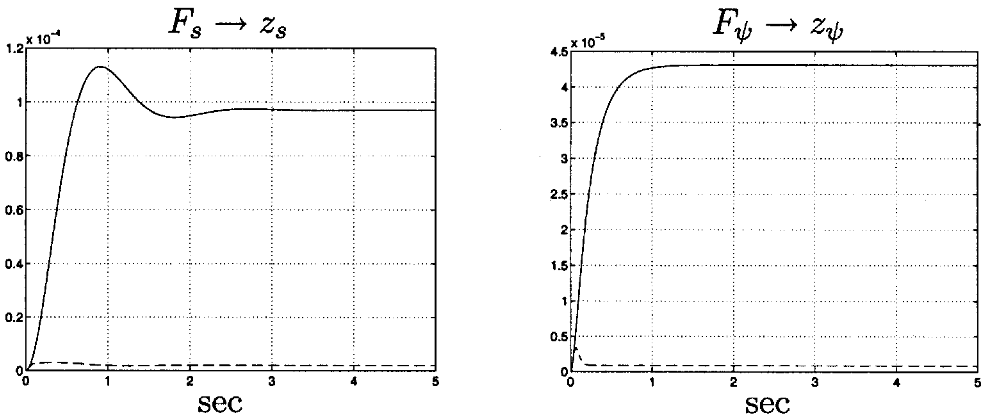

Figure 6. Step responses for linearlized model of $T_{\hat{F}_s \to \hat{z}_s}$ and $T_{\hat{F}_\psi \to \hat{z}_\psi}$: passive (solid) and active control (dashed)

command signal, as in (20) and (21). The tyre forces are modelled as being always vertical. This model was built and linearized by *AutoSim*. We will analyse the linearized model and find the controller structure which improves the load responses and keeps the road responses the same as in the passive case.

We now write the system in the form of figure 1 with $z_1 = [z_s, z_\psi, z_{u_1}, z_{u_2}]'$, $w_1 = [z_{r_1}, z_{r_2}]'$, $w_2 = [F_s, F_\psi]$, $y = [\ddot{z}_s, \ddot{z}_\psi, D_1, D_2]'$ where D_1 and D_2 are the strut deflections and z_2 omitted. The following parameters will be used for the model: $m_s = 625\,\text{kg}$, $I_\psi = 170\,\text{kg m}^2$, $m_1 = m_2 = 50\,\text{kg}$, $I_1 = I_2 = 0.1\,\text{kg m}^2$, $c_1 = c_2 = 1\,\text{kNs/m}$, $k_1 = k_2 = 4\,\text{kN/m}$, $k_{t_1} = k_{t_2} = 250\,\text{kN/m}$. Those parameters are chosen to give satisfactory (i.e. soft) road responses, so that we can apply the disturbance response decoupling design for the load responses. The actuator dynamics are modelled as in (8) with $\omega_n = 100$ rad/sec and $\delta = 0.707$. The geometric layout of the model in the nominal configuration is given by the following coordinates

$Q = (0, 0)$	$A_1 = (0.25, -1)$
$B_1 = (-0.05, -0.94)$	$C_1 = (0.15, -1.4)$
$D_1 = (0.25, -1.4)$	$R_1 = (-0.25, -1.04)$
$S_1 = (0.2, -1.4)$	$P_1 = (0.2, -1.5)$
$T_1 = (0.35, -1.5)$	$A_2 = (0.25, 1)$
$B_2 = (-0.05, 0.94)$	$C_2 = (0.15, 1.4)$
$D_2 = (0.25, 1.4)$	$R_2 = (-0.25, 1.04)$
$S_2 = (0.2, 1.4)$	$P_2 = (0.2, 1.5)$
$P_2 = (0.35, 1.5)$	

in units of meters. We find the $\widetilde{U}_2$ matrix, after model reduction (using balanced truncation from 8th to 5th order), to be as shown in equation (28) below. (The L_∞-norm error in $\widetilde{U}_2$ due to model reduction was 4.1243×10^{-7}. The H_∞ norm of $\widetilde{U}_2 P_{21,1}$ was equal to 5.73×10^{-14} and 5.42×10^{-12} before and after model reduction, so the reduced order $\widetilde{U}_2$ was considered as still preserving disturbance response decoupling property.) Using the H_∞ loop shaping method (Zhou *et al.* 1996) with the weighting function

$$W_1 = \frac{8(s+80)}{s+2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we obtain a K_1 (after model reduction using balanced truncation from 12th to 5th order) as

$$K_1 = \begin{bmatrix} K_{1,11} & K_{1,12} \\ K_{1,21} & K_{1,22} \end{bmatrix}$$

where $K_{1,12} = K_{1,21} = 0$ and

$$K_{1,11} = K_{1,22} = \frac{-26.05(s+21.80)(s+54.48)(s+114.34)(s+70.11 \pm 69.98j)}{(s+2)(s+55.22)(s+165.48)(s+66.23 \pm 120.51j)}$$

The controller gives desirable load responses, as shown in figure 6. In addition it reduces $T_{\hat{F}_s \to \hat{z}_s}(0)$ from 9.70×10^{-5} to 1.95×10^{-6} and $T_{\hat{F}_\psi \to \hat{z}_\psi}(0)$ from 4.31×10^{-5} to 8.65×10^{-7}, when compared to the passive case (i.e. no feedback). The simulation results show that the road disturbance responses are the same as the open loop.

$$\widetilde{U}_2 = \begin{bmatrix} \dfrac{26.79}{(s+4.20)(s+93.65)} & \dfrac{5.96}{(s+4.14)(s+118.79)} & \dfrac{-(s+4.16)}{s+4.17} & \dfrac{9.4 \times 10^{-3}}{s+4.17} \\[2ex] \dfrac{26.79}{(s+4.20)(s+93.65)} & \dfrac{-5.96}{(s+4.14)(s+118.79)} & \dfrac{9.4 \times 10^{-3}}{s+4.17} & \dfrac{-(s+4.16)}{s+4.17} \end{bmatrix} \quad (28)$$

5. Concluding remarks

This paper has presented two new results on disturbance response decoupling. The first (Theorem 1) established the required control structure in terms of a left annihilator of a plant open-loop transfer function. The second (Theorem 2) gave conditions to determine if the achievable performance for other disturbance paths would be reduced by the parametrization. We remark that Theorem 2 can also be applied to study achievable performance for the general case of Smith and Wang (2002).

The approach was illustrated for standard quarter- and half-car models with various different choices of measurements. The required control structures were derived in parametric form. A controller structure was calculated numerically for a half-car double-wishbone model.

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