旋轉圓盤於臨界轉速時之非線性反應

Steady State Deflection of a Circular Plate Rotating Near Its Critical Speed

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摘要

本計畫利用馮卡門的非線性平板模型分析一個圓盤在臨界轉速附近旋轉，並且受一空間中靜止側向力作用時之穩態響應。我們發現當圓盤轉速超過臨界轉速時，存在有三個穩態解，其中只有一個是穩定的。

關鍵詞:旋轉圓盤，馮卡門平板，臨界轉速

Abstract

The steady state response of a disk spinning near its critical speed and under space-fixed time-invariant load is analyzed by using von Karman's nonlinear plate model. It is found that as the disk rotates beyond a modified critical speed there exist three steady state deflections, among which only one is in the same direction as the applied load and is stable in the presence of space-fixed damping.

Keywords: spinning disk, von Kaeman's plate model, critical speed

Introduction

Conventional linearized plate theory predicts that the steady state deflection of a spinning disk under space-fixed time-invariant load approaches infinity as the rotation speed approaches the critical speed. This conclusion contradicts experimental result that shows the existence of finite steady state deflection even at the critical speed (Tobias and Arnold, 1957). In order to capture the physical essence of critical speed resonance, Raman and Mote (1999) recently adopted von Karman's plate model (Nowinski, 1964) to study the nonlinear oscillations of a disk spinning near its critical speed and subject to rotating damping. Since their analysis is performed in a rotating frame, the forcing terms are time- dependent and averaging technique has to be used. In many spinning disk applications, however, the external damping is space-fixed, such as the damping in circular saw guides and disk drive head-suspensions. In this Note we study critical speed resonance in a space-fixed frame with emphasis on the effects of space-fixed damping on the stability of steady state deflections. In the present formulation the forcing terms are time-invariant and no perturbation techniques are needed.

Equations of Motion

The dimensionless equations of motion of a disk spinning with constant speed Ω with respect to a space-fixed frame $(r, ¯)$ can be written as

$$w_{,tt} + 2\Omega w_{,t¯} + \Omega^2 w_{,¯¯} + c_f w_{,t} + \nabla^4 w - r^{-1}\left(±_r r w_{,r}\right)_{,r} - r^{-2} ±_¯ w_{,¯¯} =$$

$$w_{,rr}\left(r^{-1}¤_{,r} + r^{-2}¤_{,¯¯}\right) + \left(r^{-1}w_{,r} + r^{-2}w_{,¯¯}\right)¤_{,rr} - 2\left(r^{-1}w_{,¯}\right)_{,r}\left(r^{-1}¤_{,¯}\right)_{,r} + q(r, ¯) \quad (1)$$

$$\nabla^4 ¤ = -£\left[w_{,rr}\left(r^{-1}w_{,r} + r^{-2}w_{,¯¯}\right) + 2r^{-3}w_{,r¯}w_{,¯} - r^{-2}\left(w_{,r¯}\right)^2 - r^{-4}\left(w_{,¯}\right)^2\right] \quad (2)$$

w and $¤$ are transverse displacement and stress function, respectively. The relations between dimensionless quantities and physical quantities (with asterisks) are

$$t = \frac{t^*}{b^2}\sqrt{\frac{D}{°h}}, \quad \Omega = \Omega^* b^2 \sqrt{\frac{°h}{D}}, \quad r = \frac{r^*}{b},$$

$$w = w^*\sqrt{\frac{b}{h^3}}, \quad ¤ = ¤^*\frac{h}{D}, \quad q = q^*\sqrt{\frac{b^9}{D^2 h^3}},$$

$$c_f = c_f^*\frac{b^2}{\sqrt{°hD}}, £ = 12(1-¬^2)\frac{h}{b}, \quad ¦ = \frac{a}{b}$$

The parameters ˘, h, E, ¬, and D are the mass density, thickness, Young's modulus, Poisson ratio, and flexural rigidity of the disk, respectively. c_f^* represents a space-fixed homogeneous damping. $q^*(r^*, ¯)$ is the space-fixed time-invariant loading. The disk is assumed to be "partially clamped" at the inner radius $r^*=a$, and is free at the outer radius $r^*=b$ (Benson and Bogy, 1978). $\pm_r$ and $\pm_{¯}$ in Eq.(1) are dimensionless stresses due to centrifugal effect. In the special case when $£ = 0$, the solution ¤ in Eq.(2) is identically zero, and as a consequence Eq.(1) reduces to

$$w_{,tt} + 2\Omega w_{,t¯} + \Omega^2 w_{,¯¯} + c_f w_{,t} +$$

$$\nabla^4 w - r^{-1}\left(\pm_r r w_{,r}\right)_{,r} - r^{-2} \pm_{¯} w_{,¯¯} = q \tag{3}$$

Equation (3) is the conventional equation used in the literature without considering von Karman's effect. The natural frequency and the orthonormal eigenfunction of a freely spinning disk (i.e., $c_f = 0$, q=0) are denoted by μ_{mn} and $w_{mn} = R_{mn}(r)e^{in¯}$, respectively.

Steady State Deflection Near Critical Speed

We assume that when the disk rotates near its critical speed Ω_c of mode w_{mn}, the solution w of Eqs.(1) and (2) can be approximated by a two-mode expansion,

$$w(r,¯,t) = c_{mn}(t)w_{mn} + \bar{c}_{mn}(t)\bar{w}_{mn} \tag{4}$$

Both $c_{mn}(t)$ and $w_{mn}(r,¯)$ in Eq.(4) are complex functions. In order to solve ¤ in Eq.(2) we introduce a set of eigenfunctions $¤_{mn}$ satisfying the following differential equation,

$$\nabla^4 ¤_{mn} - {}^4_{mn} ¤_{mn} = 0 \tag{5}$$

$¤_{mn}$ satisfy the same boundary conditions as ¤ does. After expressing ¤ in terms of eigenfunctions series $¤_{mn}$ and following Galerkin's procedure, we can discretize Eqs.(1) and (2) into

$$\&\&_{mn} + \left(2in\Omega + c_f\right)\&_{mn} + ©_{mn} c_{mn}$$

$$£¥\left|c_{mn}\right|^2 c_{mn} - q_{mn} = 0 \tag{6}$$

where

$$q_{mn} = \int_{¯=0}^{2@} \int_{r=¦}^{1} q(r,¯) R_{mn}(r) e^{-in¯} r\,dr\,d¯ \tag{7}$$

$$©_{mn} = \mu_{mn}(\mu_{mn} + 2n\Omega) \tag{8}$$

Constant ¥ can be obtained via numerical integration involving eigenfunctions w_{mn} and $¤_{mn}$. It is noted that for a reflected wave the integer n is considered as positive, while the natural frequency μ_{mn} is considered as negative. Therefore $©_{mn}$ is positive in the sub-critical speed range, and is negative in the super-critical speed range. $\left|c_{mn}\right|$ represents the absolute value of complex number c_{mn}. The steady state solutions $c_{mn}^{(s)}$ satisfy the equation,

$$©_{mn} c_{mn}^{(s)} + £¥\left|c_{mn}^{(s)}\right|^2 c_{mn}^{(s)} - q_{mn} = 0 \tag{9}$$

It is noted that this cubic equation allows only real roots. In the special case of a freely spinning disk when $q_{mn} = 0$, there is one trivial steady state solution $c_{mn}^{(s)} = 0$ in the sub-critical speed range $\Omega < \Omega_c$. On the other hand, in the super-critical speed range $\Omega > \Omega_c$, there are three real roots, i.e., $c_{mn}^{(s)} = 0$ and $c_{mn}^{(s)} = \pm\sqrt{\frac{-©_{mn}}{£¥}}$. The dashed lines in Fig.1 represent the steady state deflections $c_{03}^{(s)}$ of a disk with clamping ratio ¦ =0.5 and Poisson ratio ¬ =0.27. The critical speed Ω_c of mode w_{03} is 8.75. The dimensionless thickness £ is taken to be 10^{-6} and the constant ¥ is calculated as 0.393.

In the case of a loaded disk with $q_{mn} > 0$, we can show that there is a modified critical speed $\Omega_c^* > \Omega_c$,

$$\Omega_c^* = \Omega_c + \frac{3}{2n^2\Omega_c}\left(\frac{£¥q_{mn}^2}{4}\right)^{1/3} \tag{10}$$

When $\Omega < \Omega_c^*$ there is only one positive real root. On the other hand, there are three distinct real roots as $\Omega > \Omega_c^*$. The solid lines in Fig.1 represent the steady state solution $c_{03}^{(s)}$ when $q_{03} = 1$. The deflection A is always in the same direction as the applied load, while the deflections B and C are in the opposite direction. The small arrows in Fig.1 indicate the deflection change from freely spinning disk to loaded disk.

Stability Analysis of Steady State Solutions

In order to investigate the stability of the steady state solutions, we express the solution c_{mn} in Eq.(6) as

$$c_{mn}(t) = c_{mn}^{(s)} + \hat{c}(t) \tag{11}$$

After Substituting Eq.(11) into Eq.(6), using Eq.(9) and linearizing with respect to $\hat{c}$, we obtain the following equation,

$$\ddot{\hat{c}} + \left(2in\Omega + c_f\right)\dot{\hat{c}} + ©_{mn}\hat{c} + £¥\left(c_{mn}^{(s)}\right)^2\left(2\hat{c} + \bar{\hat{c}}\right) = 0 \tag{12}$$

The eigenvalue a of Eq.(12) can be obtained by solving the following quartic equation,

$$a^4 + 2c_f a^3 + \left[2©_{mn} + 4£¥\left(c_{mn}^{(s)}\right)^2 + c_f{}^2 + 4n^2\Omega^2\right]a^2 + 2c_f\left[©_{mn} + 2£¥\left(c_{mn}^{(s)}\right)^2\right]a + \left[©_{mn} + 3£¥\left(c_{mn}^{(s)}\right)^2\right]\left[©_{mn} + £¥\left(c_{mn}^{(s)}\right)^2\right] = 0 \tag{13}$$

The steady state solution is unstable when the real part of any of the four eigenvalues a's is positive.

First of all we consider the deflections of the freely spinning disk with $c_f = 0$. For the trivial deflection $c_{mn}^{(s)}$=0, the square of the eigenvalues solved from Eq.(13) are $a^2 = -\mu_{mn}^2$ and $-(\mu_{mn} + 2n\Omega)^2$. For the nontrivial solutions $c_{mn}^{(s)} = \pm\sqrt{\frac{-©_{mn}}{£¥}}$, $a^2 = 0$ and $2©_{mn} - 4n^2\Omega^2$. Therefore, the steady state solutions of a freely spinning disk are neutrally stable to the first order of the stability analysis.

For the case of a loaded disk, it is difficult to express a^2 in terms of physical parameters explicitly. However, we can study how a^2 varies as q_{mn} increases from zero by differentiating Eq.(13) with respect to q_{mn} to obtain the first order derivative $\left.\frac{\partial\left(a^2\right)}{\partial q_{mn}}\right|_{q_{mn}=0}$. For the trivial solution $c_{mn}^{(s)}$=0, we can show that the derivatives for $a^2 = -\mu_{mn}^2$ and $-(\mu_{mn} + 2n\Omega)^2$ are zero. Therefore, deflection B remains neutrally stable when the space-fixed load is present.

For the nontrivial deflections $c_{mn}^{(s)} = \pm\sqrt{\frac{-©_{mn}}{£¥}}$ in the super-critical speed range, the derivative for a^2=0 is

$$\left.\frac{\partial\left(a^2\right)}{\partial q_{mn}}\right|_{q_{mn}=0,\ a^2=0} = \mu\frac{1}{2n^2\Omega^2 - ©_{mn}}\sqrt{\frac{-©_{mn}}{£¥}} \tag{14}$$

From Eq.(14) we can predict that a^2 is negative for deflection A, and positive for deflection C. As a consequence, the steady state deflection C of the loaded disk is unstable, and deflection A remains neutrally stable.

Space-Fixed Damping Effects

We next study the behavior of the eigenvalues when space-fixed damping is present. To do so, we differentiate Eq.(13) with respect to c_f and calculate the derivative at $c_f = 0$ and $q_{mn} = 0$. For the trivial solution $c_{mn}^{(s)}$=0 the derivative of eigenvalue $\pm\mu_{mn}$ is

$$\left.\frac{\partial a}{\partial c_f}\right|_{c_f=0,\ a=\pm i\mu_{mn}} = \frac{-\mu_{mn}}{4n\Omega} \tag{15}$$

In the super-critical speed range, the right hand side of Eq.(15) is positive real. We therefore conclude that deflection B of the loaded disk is unstable in the presence of c_f.

For the negative deflection of the freely spinning disk, we first observe that c_f has no effect on the degenerate eigenvalues $a = 0$. However, we have shown in the preceding section that applied load tends to drive one of these two degenerate eigenvalues to positive real for deflection C. Therefore, we conclude that deflection C of the loaded disk is also unstable when external damping c_f is present.

For the positive deflection of the freely spinning disk, applied load tends to drive the degenerate eigenvalues $a = 0$ to purely imaginary, while c_f has no effect on these two eigenvalues. Therefore, deflection A is neutrally stable to the first order of the stability analysis. However, if we approximate a of the undamped loaded disk by Eq.(14), then we obtain an estimate of the eigenvalue change of the loaded disk as

$$\left.\frac{\partial a}{\partial c_f}\right|_{c_f=0,\ a^2=\frac{-q_{mn}}{2n^2\Omega^2-\text{©}_{mn}}\sqrt{\frac{-\text{©}_{mn}}{\text{£¥}}}} = \frac{\text{©}_{mn}\sqrt{\text{£¥}}(2n^2\Omega^2-\text{©}_{mn})-q_{mn}\sqrt{-\text{©}_{mn}}}{2\sqrt{\text{£¥}}(2n^2\Omega^2-\text{©}_{mn})^2-2q_{mn}\sqrt{-\text{©}_{mn}}} \tag{16}$$

The right hand side of Eq.(16) is negative for small q_{mn}. Therefore, c_f drives the eigenvalues from 0 to negative real. On the other hand, the derivative of the eigenvalues $a = \pm i\sqrt{4n^2\Omega^2 - 2\text{©}_{mn}}$ can be calculated as

$$\left.\frac{\partial a}{\partial c_f}\right|_{c_f=0,\ a=\pm i\sqrt{4n^2\Omega^2-2\text{©}_{mn}}} = -1 + \frac{\text{©}_{mn}}{4n^2\Omega^2 - 2\text{©}_{mn}} \tag{17}$$

The right hand side of Eq.(17) is always negative. Therefore, we conclude that deflection A of the loaded disk is stabilized by the external damping.

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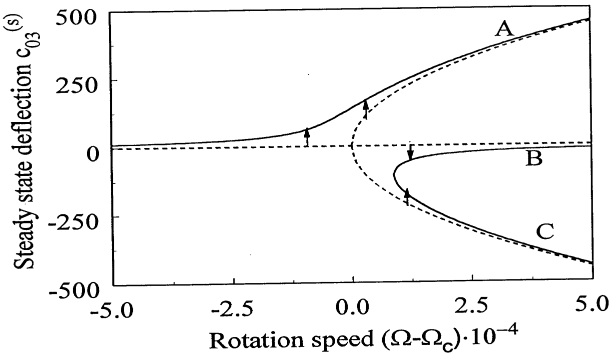

Figure 1 Steady state deflection $c_{03}^{(s)}$ near the critical speed.